COMPONENTS OF SYMMETRIC WIDE-MATRIX VARIETIES

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Abstract. We show that if X_n is a variety of $c \times n$-matrices that is stable under the group $\mathrm{Sym}([n])$ of column permutations and if forgetting the last column maps X_n into X_{n-1}, then the number of $\mathrm{Sym}([n])$-orbits on irreducible components of X_n is a quasipolynomial in n for all sufficiently large n. To this end, we introduce the category of affine $\mathbf{FI^{op}}$-schemes of width one, review existing literature on such schemes, and establish several new structural results about them. In particular, we show that under a shift and a localisation, any width-one $\mathbf{FI^{op}}$-scheme becomes of product form, where $X_n = Y^n$ for some scheme Y in affine c-space. Furthermore, to any $\mathbf{FI^{op}}$-scheme of width one we associate a *component functor* from the category $\mathbf{FI}$ of finite sets with injections to the category $\mathbf{PF}$ of finite sets with partially defined maps. We present a combinatorial model for these functors and use this model to prove that $\mathrm{Sym}([n])$-orbits of components of X_n, for all n, correspond bijectively to orbits of a groupoid acting on the integral points in certain rational polyhedral cones. Using the orbit-counting lemma for groupoids and theorems on quasipolynomiality of lattice point counts, this yields our Main Theorem.

1. The main result and background

1.1. Main result. For a nonnegative integer n we define $[n] := \{1, \ldots, n\}$.

Let K be a Noetherian ring (commutative with 1), let $c \in \mathbb{Z}_{\geq 0}$, and, for all $n \in \mathbb{Z}_{\geq 0}$, let I_n be an ideal in the polynomial ring $A_n := K[x_{i,j} \mid i \in [c], j \in [n]]$ such that the following two conditions are satisfied:

(1) I_n is preserved by the (left) action of the symmetric group $\mathrm{Sym}([n])$ on A_n via K-algebra automorphisms determined by $\pi x_{i,j} = x_{i,\pi(j)}$; and
(2) $I_n \subseteq I_{n+1}$.

Dually, let X_n be the prime spectrum of A_n/I_n, a closed subscheme of $\mathrm{Spec}(A_n)$. Then the two conditions above express that

(1) X_n is preserved by the induced action of $\mathrm{Sym}([n])$ on $\mathrm{Spec}(A_n)$; and
(2) the projection $\mathrm{Spec}(A_{n+1}) \to \mathrm{Spec}(A_n)$ dual to the inclusion $A_n \to A_{n+1}$ maps X_{n+1} into X_n.

Such a sequence $(X_n)_n$ of schemes of matrices is called a *width-one* $\mathbf{FI^{op}}$*-scheme* of finite type over K (see Section 2 for a more convenient, functorial definition) or, more informally, a *symmetric wide-matrix scheme*, where the adjective *wide* refers to the fact that c is constant and we are interested in the case where $n \gg 0$; for brevity, we will usually drop the adjective *symmetric*.

Recall that a *quasipolynomial* is a function $f : \mathbb{Z} \to \mathbb{R}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$$

where each $c_i : \mathbb{Z} \to \mathbb{R}$ is periodic with integral period. Equivalently, f is a quasipolynomial if and only if there exist an N and polynomials $f_0, \ldots, f_{N-1}$ such that $f(n) = f_i(n)$ whenever $n \equiv i$ modulo N.

Theorem 1.1.1 (Main Theorem)**.** *Let $(X_n)_n$ be a width-one $\mathbf{FI^{op}}$-scheme of finite type over a Noetherian ring K. Then the action of $\mathrm{Sym}([n])$ on X_n induces an action of $\mathrm{Sym}([n])$ on the set $\mathcal{C}(X_n)$ of irreducible components of X_n, and there exists a quasipolynomial $f : \mathbb{Z} \to \mathbb{R}$ and a natural number $n_0 \in \mathbb{Z}_{\geq 0}$ such that the number $|\mathcal{C}(X_n)/\mathrm{Sym}([n])|$ of $\mathrm{Sym}([n])$-orbits on $\mathcal{C}(X_n)$ equals $f(n)$ for all $n \geq n_0$.*

1.2. **Examples.** We illustrate the Main Theorem by a number of examples. We recall that the irreducible components of X_n are in one-to-one correspondence with the inclusion-wise minimal prime ideals A_n that contain I_n.

Example 1.2.1. Let K be a domain, take $c = 1$, write x_j instead of $x_{1,j}$, and let I_n be the ideal generated by all monomials $x_i x_j x_k$ with $i, j, k \in [n]$ distinct. Clearly, the sequence $(I_n)_n$ satisfies the conditions (1) and (2) above. A prime ideal containing I_n contains at least one variable from each triple of distinct variables. Hence the minimal prime ideals containing I_n are the ideals $I_S := (\{x_i \mid i \in S\})$ where $S \subseteq [n]$ is a set of cardinality $n - 2$; the corresponding subscheme is the coordinate plane corresponding to the coordinates not labelled by S. Hence X_n has $\binom{n}{n-2} = \binom{n}{2}$ irreducible components, which form a single orbit under the symmetric group $\mathrm{Sym}([n])$. The quasipolynomial from the Main Theorem is 1. ♣

Example 1.2.2. Set $K := \mathbb{C}$, let $d \in \mathbb{Z}_{\geq 0}$, take $c = 1$, and let I_n be the ideal generated by all polynomials $x_i^d - 1$ with $i \in [n]$. The irreducible components of X_n are the points $(\zeta_1, \ldots, \zeta_n)$ where each ζ_i is an d-th root of unity. Thus X_n has d^n irreducible components, and these form $\binom{n+d-1}{d-1}$ orbits under the group $\mathrm{Sym}([n])$, each of which has a unique representative of the form

$$(1, \ldots, 1, e^{2\pi i/d}, \ldots, e^{2\pi i/d}, e^{2\cdot 2\pi i/d}, \ldots, e^{2\cdot 2\pi i/d}, \ldots, e^{(d-1)\cdot 2\pi i/d}, \ldots, e^{(d-1)\cdot 2\pi i/d}),$$

where the numbers of occurrences of $e^{j\cdot 2\pi i/d}$, $j = 0, \ldots, d-1$ are arbitrary non-negative integers whose sum is n. ♣

Example 1.2.3. Set $K := \mathbb{C}(t)$, where t is a variable, let $c = 1$, and let I_n be the ideal generated by all polynomials of the form $x_i^2 - t$ with $i \in [n]$. Then I_1 is a prime ideal, but for $n \geq 2$ and two distinct $i, j \in [n]$ any prime ideal containing I_n also contains $(x_i^2 - t) - (x_j^2 - t) = (x_i - x_j)(x_i + x_j)$ and hence either $x_i - x_j$ or $x_i + x_j$. The minimal prime ideals containing I_n correspond bijectively to unordered partitions of $[n]$ into two (potentially empty) parts, i.e., unordered pairs $\{A, B\}$ where A, B are disjoint subsets of $[n]$ whose union is $[n]$, via the map that sends the prime ideal P to

$$\{\{j \mid x_1 - x_j \in P\}, \{j \mid x_1 + x_j \in P\}\}.$$

The number of unordered partitions of $[n]$ is 2^{n-1}, and the number of $\mathrm{Sym}([n])$ orbits on these is $\lceil n/2 \rceil$—indeed, $\{A, B\}, \{\tilde{A}, \tilde{B}\}$ are in the same $\mathrm{Sym}([n])$-orbit if and only if $\min\{|A|, |B|\} = \min\{|\tilde{A}|, |\tilde{B}|\}$, and this number takes any of the values in $\{0, \ldots, \lfloor n/2 \rfloor\}$. Here the quasipolynomial is the function $f(n) = \lceil n/2 \rceil$, and it holds only for $n \geq 1$—indeed, $X_0 = \mathrm{Spec}(\mathbb{C}(t))$ has one irreducible component, rather than 0. ♣

Example 1.2.4. Set $K := \mathbb{C}$, $c := 1$, let $d \in \mathbb{Z}_{\geq 1}$, and let I_n be the ideal generated by all differences $x_i^d - x_j^d$ with $i, j \in [n]$. In this case, each prime ideal containing I_n also contains, for each $i \neq j$, a polynomial of the form $x_i - \zeta x_j$, where ζ is a d-th root of unity. Via the map that sends P to $(A_0, A_1, \ldots, A_{d-1})$ with

$$A_b = \{j \in [n] \mid x_j - e^{2b\pi i/n} x_1 \in P\},$$

the irreducible components of the scheme defined by I_n correspond bijectively to orbits of *ordered* partitions of $[n]$ into d parts (some of which may be empty) under the action of $\mathbb{Z}/d\mathbb{Z}$ by rotation of the parts. The number of $\mathrm{Sym}([n])$-orbits on such components is therefore equal to the number of $(\mathbb{Z}/d\mathbb{Z})\times\mathrm{Sym}([n])$-orbits on ordered partitions. Modding out $\mathrm{Sym}([n])$ first, what remains is to count the $\mathbb{Z}/d\mathbb{Z}$-orbits on ordered *integer* partitions of n into d nonnegative parts. This is done using the orbit-counting lemma (due to Cauchy, Frobenius, and not Burnside) for $\mathbb{Z}/d\mathbb{Z}$: for $e \in \mathbb{Z}/d\mathbb{Z}$ define $f(e) := \gcd(d,e) \in \{1,\ldots,d\}$. Then rotation by e on such integer partitions has the same fixed points as rotation by $f(e)$. This number is 0 if n is not divisible by $d/f(e)$, and equal to $\binom{n/(d/f(e))+f(e)-1}{f(e)-1}$ otherwise: the first $f(e)$ positions in the partition can be filled arbitrarily with nonnegative integers whose sum is $n/(d/(f(e)))$, and this determines the partition fixed under rotating over $f(e)$. Thus the number of $\mathrm{Sym}([n])$-orbits on components equals

$$\frac{1}{d}\sum_{e\in\mathbb{Z}/d\mathbb{Z}:(d/f(e))|n}\binom{n/(d/f(e))+f(e)-1}{f(e)-1},$$

which is, indeed, a quasipolynomial in n. ♣

These examples illustrate different aspects of the proof of Theorem 1.1.1. First, the projection $X_{n+1} \to X_n$ maps each irreducible component of X_{n+1} into some component of X_n, but not necessarily *onto* some such component: in Example 1.2.1, the coordinate planes involving the variable x_{n+1} are mapped onto coordinate *lines* rather than planes. However, "most" coordinate planes are mapped onto coordinate planes. We will capture these relations between components of the X_n as n varies by the so-called *component functor* (see Section 4), which is a contravariant functor from **FI** to the category **PF** of finite sets with partially defined maps. This functor plays a fundamental role in the proof of Theorem 1.1.1, and also yields a more detailed picture of the components of the X_n for varying n.

Second, Example 1.2.2 illustrates that, while the number of components of X_n can grow exponentially with n, the number of orbits is upper-bounded by a polynomial.

Third, in Example 1.2.3 we see that, if we adjoin $\sqrt{t}$ to the ground field K, then the example reduces to a variation on Example 1.2.2: there are 2^n components and $n+1$ orbits on components. This suggests that the quasipolynomiality in the Main Theorem is due to the action of a Galois group. We will see that this is, indeed, the case when the wide-matrix scheme is of *product type*; see §5.4.

Finally, Example 1.2.4 shows that even when K is an algebraically closed field, quasipolynomiality (rather than polynomiality) occurs. In part, this is because we will have to work over larger base fields that are transcendental extensions of the ground field K; and in part, it is because Galois groups are not the sole reason for quasipolynomiality: in §5.7, we will replace the orbit-counting via Galois groups to orbit-counting via certain groupoids.

We conclude this subsection with an interesting application of the Main Theorem.

Corollary 1.2.5. *Let K be a field, let $S \subseteq K$ be a finite subset, and let k be a natural number. For every $n \in \mathbb{Z}_{\geq 0}$, define*

$$M_n := \{A \in S^{n\times n} \subseteq K^{n\times n} \mid \mathrm{rk}(A) \leq k\},$$

the set of all rank-$\leq k$ matrices all of whose entries are in S. Let $\mathrm{Sym}([n])$ *act by simultaneous row and column permutations on M_n. Then $|M_n/\mathrm{Sym}([n])|$ is a quasipolynomial in n for $n \gg 0$.*

Proof. Consider the morphism $\varphi : \mathbb{A}_K^{k\times n} \times \mathbb{A}_K^{k\times n} \to \mathbb{A}_K^{n\times n}$ given by $(A, B) \mapsto A^T \cdot B$ with image Y_n, the subvariety of rank-$\leq k$ matrices. By classical invariant theory, φ is the quotient map for the action of the reductive K-group GL_k acting via $(g,(A, B)) \mapsto (g^{-T}A, gB)$ and hence, by properties of the quotient map, φ yields a bijection between closed subsets of Y_n and closed GL_k-stable subsets of $\mathbb{A}_K^{k\times n} \times \mathbb{A}_K^{k\times n} \cong \mathbb{A}_K^{2k\times n}$. Let $X_n = \varphi^{-1}(M_n) \subseteq \mathbb{A}_K^{2k\times n}$ correspond to M_n under this bijection. Then $(X_n)_n$ is a wide-matrix scheme, and the irreducible components of X_n are in bijection with the points of M_n. Hence the corollary follows from the Main Theorem. □

Remark 1.2.6. The same argument works for *symmetric* rank-k matrices and for *skew-symmetric* rank-k matrices. More generally, by the same argument: if $Y_n \subseteq \mathbb{A}_K^{n\times n}$ is a closed subvariety of the variety of rank-$\leq k$ matrices such that Y_n is preserved under the action of $\mathrm{Sym}([n])$ by conjugation and such that forgetting the last row and column maps Y_{n+1} to Y_n, then, too, the number of orbits of $\mathrm{Sym}([n])$ on irreducible components of Y_n is a quasipolynomial in n for $n \gg 0$.

Example 1.2.7. Consider $K = \mathbb{Q}$ and let $M_{k,n}$ be the set of *symmetric* $n \times n$-matrices with entries in $\{0, 1\}$ of rank pricesly k. Then:

- $M_{0,n}$ consists of the zero matrix only, so there is a single S_n-orbit.
- Each S_n-orbit in $M_{1,n}$ has a unique representaive of the form

$$\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$$

  where J is an $m \times m$-matrix ($1 \leq m \leq n$) with all ones, and the zeros are block matrices of appropriate sizes. Hence there are n orbits.
- There are three types of S_n-orbits in $M_{2,n}$, with representatives

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & J & 0 \\ J^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} J_1 & J_2 & 0 \\ J_2^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  where, in the first case, J_1, J_2 are all-one matrices of formats $n_1 \times n_1, n_2 \times n_2$ with $1 \leq n_1 \leq n_2$ and $n_1+n_2 \leq n$; in the second case, J is an all-one $n_1 \times n_2$-matrix with $1 \leq n_1 \leq n_2$ and $n_1 + n_2 \leq n$; and in the third case, J_1 is an all-one $n_1 \times n_1$-matrix and J_2 is an all-one $n_1 \times n_2$-matrix with $1 \leq n_1, n_2$ and $n_1 + n_2 \leq n$. It follows that the number of S_n-orbits on $M_{2,n}$ equals

$$2 \cdot \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil + \binom{n}{2},$$

  clearly a quasipolynomial in n. ♣

1.3. **Relations to existing literature.** The functorial viewpoint and the notions of **FI**-algebras and **FI$^{\mathbf{op}}$**-schemes that we will use are strongly influenced by the literature on **FI**-modules [CEF15, CEFN14]. Furthermore, the insight that counting combinatorial objects is best done through *species*—functors from the category of finite sets with bijections to itself—is due to [Joy81]. While we use neither nontrivial results about **FI**-modules nor nontrivial results about species, this paper could not have been written without this background.

Given a wide-matrix scheme $(X_n)_n$ over K, one can define the inverse limit $X_\infty := \lim_{\leftarrow n} X_n$. This limit is a variety in the space of $c\times\mathbb{N}$-matrices and preserved by the action of the symmetric group $\mathrm{Sym}(\mathbb{N})$ permuting columns. Furthermore, for $n \gg 0$, X_n is in fact the image of X_∞ under projection—this follows from Proposition 2.8.1—and so, as we are interested in properties of X_n for large n, X_∞ contains all relevant information. Much literature on wide matrix schemes uses this set-up—a $\mathrm{Sym}(\mathbb{N})$-invariant subvariety of an infinite-dimensional affine space—rather than the functorial set-up. But, as we will see, for counting purposes the functorial set-up is more convenient.

The descending chain property of wide matrix schemes (see Section 2.7) was first established in [Coh67, Coh87] and then rediscovered in [AH07, HS12], and used to prove the Independent Set Theorem in algebraic statistics in the latter paper. A further application to algebraic statistics is [Dra10]. Images of wide-matrix schemes under monomial maps also satisfy the descending chain property [DEKL16].

These Noetherianity results admit proofs using Gröbner methods in the spirit of [SS17], which can be turned into explicit algorithms. A special-purpose algorithm was used in [BD11] to find the defining equations for the Gaussian two-factor model, a general-purpose algorithm was implemented in Macaulay 2 [HKL13]. The results of the current paper are also effective: there exists an algorithm that, on input the finitely many equations defining a wide-matrix scheme, computes the quasi-polynomial from the Main Theorem. But we believe that it is unlikely that a general-purpose algorithm for this will ever be implemented.

The sequence of ideals $(I_n)_n$ defining a wide-matrix scheme has a Hilbert function in two variables: one for the degree and one for n. It turns out to be a rational function of a very specific form [GN18, KLS17, NR17].

Further commutative algebra for wide-matrix schemes was developed in [NR19], where Noetherianity of finitely generated modules over their coordinate rings was established; [VNNR20], where the codimension of X_n in its ambient space and the projective dimension of I_n in its ambient polynomial ring are studied; and [VNNR18], which concerns the Castelnuovo regularity of I_n. A beautiful, as yet open conjecture from the latter two papers, is that the projective dimension and the regularity both are precisely a linear function of n for $n \gg 0$. For codimension, this is established in [VNNR20, Theorem 3.8]; it also follows from our work (see Theorem 5.4.4).

Finally, Noetherianity implies, roughly speaking, that each wide-matrix scheme is a finite union of irreducible wide-matrix schemes. Here irreducibility does not mean that each individual X_n is irreducible, but rather that X_∞ is irreducible in the topology in which the closed subsets are $\mathrm{Sym}(\mathbb{N})$-stable closed subsets of the space of $c \times \mathbb{N}$-matrices. For instance, the wide-matrix scheme where $c = 1$ and $X_n = \{0,1\}^n$ turns out to be irreducible in this setting. In [NS20], these irreducible varieties are classified for $c = 1$.

1.4. **Organisation of this paper.** This paper is organized as follows. $\mathbf{FI}^{\mathbf{op}}$-schemes are introduced in Section 2. In particular, we define wide-matrix spaces in this context. Every width-one $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type is isomorphic to a closed $\mathbf{FI}^{\mathbf{op}}$-subscheme of a wide-matrix space (see Lemma 2.6.7).

In Section 3, we prove that any width-one $\mathbf{FI^{op}}$-scheme of finite type is of *product type* after a suitable shift and a suitable localisation (see the Shift Theorem 3.1.1 and Proposition 3.3.1). The shifting technique is also used by the first author in his work on polynomial functors [Dra19], and the Shift Theorem is reminiscent of, and was inspired by, the Shift Theorem in the upcoming paper [BDES21]. It is the strongest new structural result that we prove about wide-matrix schemes.

In Section 4, for an $\mathbf{FI^{op}}$-scheme X, we define the component functor $\mathcal{C}_X$ from **FI** to the category **PF** of finite sets with partially defined maps. The component functor is one of the most important notions of this paper, and in the remainder of the paper we obtain an almost complete combinatorial description of $\mathcal{C}_X$ in the case where X is a width-one $\mathbf{FI^{op}}$-scheme.

Section 5 is devoted to the proof of the Main Theorem. In three steps, we construct more and more refined combinatorial models for $\mathcal{C}_X$. The first ones, called *elementary model functors*, allow us to prove the Main Theorem when X is of product type (see §5.4). By the Shift Theorem this situation is always attained by a shift and a localisation, and to undo the simplifications caused by that shift and localisation, we need the two more complicated combinatorial models dubbed *model functors* (see §5.6) and *pre-component functors* (see §5.8). A major generalisation in the step from elementary model functors to model functors is that we pass from counting orbits under a finite group—in the application to $\mathcal{C}_X$, the image of a Galois group—*which is part of the defining data* of an elementary model functor, to counting orbits under a finite groupoid *which emerges by itself* from the defining data of a model functor. The proof of Theorem 5.7.1 that model functors have a quasipolynomial count is entirely elementary, but very subtle. In comparison, the step from model functors to pre-component functors is conceptually small. In §5.9 we prove that pre-component functors always have a quasipolynomial count, and in §5.10 we establish that the component functor of a wide-matrix scheme satisfies the axioms of a pre-component functor. This, then, completes the proof of the Main Theorem.

2. Width-1 $\mathbf{FI^{op}}$-schemes

In this section we collect fundamental facts about **FI**-algebras and $\mathbf{FI^{op}}$-schemes.

2.1. The category FI.

The category **FI** has as objects finite sets, and for $S, T \in$ **FI** the hom-set $\mathrm{Hom}_{\mathbf{FI}}(S,T)$ is the set of *injections* $S \to T$. The category $\mathbf{FI^{op}}$ is its opposite category.

2.2. FI-algebras and $\mathbf{FI^{op}}$-schemes.

Let K be a ring (commutative, with 1). All K-algebras A are required to be commutative, have a 1, and the homomorphism $K \to A$ is required to send 1 to 1. Homomorphisms of K-algebras are unital ring homomorphisms $A \to B$ compatible with the homomorphisms from K into them.

Definition 2.2.1. An **FI**-algebra B over K is a covariant functor from **FI** to the category of K-algebras with unital K-algebra homomorphisms. Dually, B gives rise to a *contravariant* functor X from **FI** to the category of affine schemes over K. To remind ourselves of the contravariance of this functor, we call such a functor an affine $\mathbf{FI^{op}}$-scheme over K.

A morphism $B \to A$ of **FI**-algebras over K is a natural transformation from B to A: it consists of a K-algebra homomorphism $\varphi(S) : B(S) \to A(S)$ for all S such that for all $S, T \in \mathbf{FI}$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ the following diagram commutes.

$$\begin{array}{ccc} B(S) & \xrightarrow{\varphi(S)} & A(S) \\ {\scriptstyle B(\pi)}\downarrow & & \downarrow{\scriptstyle A(\pi)} \\ B(T) & \xrightarrow[\varphi(T)]{} & A(T). \end{array}$$

Morphisms of affine $\mathbf{FI^{op}}$-schemes are defined dually. Since we will only consider affine schemes, we will sometimes drop the adjective "affine". ♢

Remark 2.2.2. If B is an $\mathbf{FI}$-algebra over K, then we write $B_n := B([n])$. In fact, B is then also an $\mathbf{FI}$-algebra over B_0: for each finite set S the unique inclusion $\emptyset \to S$ yields an algebra homomorphism $B_0 \to B(S)$, and functoriality implies that the K-algebra homomorphisms $B(S) \to B(T)$ corresponding to injections $S \to T$ are compatible with the B_0-algebra structure. ♠

The classical equivalences of categories $B \to \mathrm{Spec}(B)$ and $X \mapsto K[X]$ between K-algebras and affine schemes over K yield equivalences of categories between $\mathbf{FI}$-algebras and affine $\mathbf{FI^{op}}$-schemes. Given an $\mathbf{FI}$-algebra B over K, we write $\mathrm{Spec}(B)$ for the $\mathbf{FI^{op}}$-scheme $S \mapsto \mathrm{Spec}(B(S))$ and given an affine $\mathbf{FI^{op}}$-scheme X, we write $K[X]$ for the $\mathbf{FI}$-algebra $S \mapsto K[X(S)]$.

An *ideal* in an $\mathbf{FI}$-algebra B has the obvious definition: it consists of an ideal $I(S)$ for each $S \in \mathbf{FI}$ such that for each $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ the map $B(\pi) : B(S) \to B(T)$ maps $I(S)$ into $I(T)$.

Given an $\mathbf{FI}$-algebra B, for each $S \in \mathbf{FI}$ the symmetric group $\mathrm{Sym}(S)$ acts from the left on $B(S)$: indeed, $\mathrm{Sym}(S) = \mathrm{Hom}_{\mathbf{FI}}(S, S)$, and each π in the latter set yields a K-algebra homomorphism $B(\pi) : B(S) \to B(S)$. The axioms expressing that B is a functor imply that $(\pi, b) \mapsto B(\pi)(b)$ is a left action of $\mathrm{Sym}(S)$ by K-algebra automorphisms on $B(S)$.

Similarly, given an $\mathbf{FI^{op}}$-scheme X, for each $S \in \mathbf{FI}$ the symmetric group $\mathrm{Sym}(S)$ acts on $X(S)$ by automorphisms of affine K-schemes. When acting on points of $X(S)$ with values in a K-algebra L, i.e., on the set of K-algebra homomorphisms $K[X(S)] \to L$, this is a naturally a *right* action, reflecting the fact that X is a contravariant functor.

Tensor products of $\mathbf{FI}$-algebras over K are defined in the straightforward manner, and they correspond to products in the category of affine $\mathbf{FI^{op}}$-schemes over K.

Remark 2.2.3. The symmetric group $\mathrm{Sym}([n])$ acts on B_n by K-algebra automorphisms, and the map $B(\iota) : B_n \to B_{n+1}$, where $\iota : [n] \to [n+1]$ is the standard inclusion, is a $\mathrm{Sym}([n])$-equivariant K-algebra homomorphism, if $\mathrm{Sym}([n])$ is regarded as the subgroup of $\mathrm{Sym}([n+1])$ consisting of all permutations that fix $n+1$. Conversely, from the data (for all n) of B_n, the action of $\mathrm{Sym}([n])$ on B_n, and the $\mathrm{Sym}([n])$-equivariant map $B_n \to B_{n+1}$ the $\mathbf{FI}$-algebra B can be recovered up to isomorphism. This gives another, more concrete picture of $\mathbf{FI}$-algebras similar to that used in §1.1. However, the definition of $\mathbf{FI}$-algebras as a functor from $\mathbf{FI}$ to K-algebras is more elegant and, as we will see, often more convenient. ♠

2.3. **Base change.**

Definition 2.3.1. If B is an $\mathbf{FI}$-algebra over a ring K, and L is a K-algebra, then we obtain an $\mathbf{FI}$-algebra B_L over L by setting $S \mapsto L \otimes_K B(S)$. In the special case where L is the localisation $K[1/h]$ for some $h \in K$, we also write $B[1/h]$ for B_L.

Dually, if $X = \mathrm{Spec}(B)$ is the associated $\mathbf{FI^{op}}$-scheme, then we write $X_L = \mathrm{Spec}(B_L)$ for the base change, and $X[1/h]$ if $L = K[1/h]$. $\diamondsuit$

2.4. **Wide-matrix spaces.** In this paper, the following **FI**-algebras and $\mathbf{FI^{op}}$-schemes play a prominent role.

Example 2.4.1. Let A_K be the **FI**-algebra that maps S to $K[x_j \mid j \in S]$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ to the K-algebra homomorphism determined by $x_j \mapsto x_{\pi(j)}$. For each $c \in \mathbb{Z}_{\geq 0}$, $A_K^{\otimes c}$ (where the tensor product is over K) is isomorphic to, and will be identified with, the **FI**-algebra over K that maps S to $K[x_{i,j} \mid i \in [c], j \in S]$. Write $\mathrm{Mat}_{c,K} := \mathrm{Spec}(A_K^{\otimes c})$. If L is a K-algebra, then the *set of L-points* $\mathrm{Mat}_{c,K}(L)$ is the contravariant functor from **FI** to sets that assigns to S the set $L^{c \times S}$ of $c \times S$-matrices over L, and to a morphism $\pi : S \to T$ the map $L^{c\times T} = (L^c)^T \to (L^c)^S = L^{c \times S}$ where the middle map is composition with π. The $\mathbf{FI^{op}}$-scheme $\mathrm{Mat}_{c,K}$ is called the *wide-matrix space over K with c rows* or, less precisely, a *wide-matrix space.* $\clubsuit$

2.5. **Width.** Let B be an **FI**-algebra over K.

Definition 2.5.1. For $S \in \mathbf{FI}$ and $b \in B(S)$, we call the minimal n such that b lies in $\pi B([n])$ for some $\pi \in \mathrm{Hom}_{\mathbf{FI}}([n], S)$ the *width* of b, denoted $w(b)$. $\diamondsuit$

Example 2.5.2. Assuming that K is not the zero ring, the element $x_{1,1} + x_{2,4} + x_{3,4}^2 \in A_K^{\otimes 3}([4])$ has width 2: it is the image of $x_{1,1}+x_{2,2}+x_{3,2}^2$ under $A_K^{\otimes 3}(\pi)$ where $\pi : [2] \to [4]$ is defined by $1 \mapsto 1$ and $2 \mapsto 4$. $\clubsuit$

Note that the width satisfies $w(b_1 + b_2), w(b_1 \cdot b_2) \leq w(b_1) + w(b_2)$.

2.6. **FI-algebras finitely generated in width ≤ 1.**

Definition 2.6.1. Let B be an **FI**-algebra. Let $S_i, i \in I$ be a collection of objects in **FI** and for each $i \in I$ let b_i be an element of $B(S_i)$. There is a unique smallest **FI**-subalgebra R of B such that $R(S_i) \ni b_i$ for all i. This is called the ***FI**-algebra generated by the b_i*. Concretely, $R(S)$ is the K-subalgebra of $B(S)$ generated by all elements $\pi(b_i)$ for all $i \in I$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S_i, S)$. $\diamondsuit$

Definition 2.6.2. An **FI**-algebra B over K is *finitely generated* if there exists a finite collection $(b_i \in B(S_i))_{i\in I}$ of elements that generates B. $\diamondsuit$

Definition 2.6.3. An **FI**-algebra B over K is *generated in width $\leq w$* if there exists a collection $(b_i \in B(S_i))_{i\in I}$ of elements of width $\leq w$ that generates B. $\diamondsuit$

We will be mostly interested in **FI** algebras that are finitely generated in width ≤ 1 in the following sense.

Definition 2.6.4. An **FI**-algebra B over K is *finitely generated in width $\leq w$* if B is finitely generated and generated in width $\leq w$. It is straightforward to see that this is equivalent to the condition that B is generated by a finite collection of elements $b_i \in B(S_i)$ of width $\leq w$. $\diamondsuit$

Put differently yet, given an **FI**-algebra, recall that B_n is defined as $B([n])$. Then B is finitely generated in width $\leq w$ if and only if $B_0, B_1, \ldots, B_w$ are finitely generated K-algebras and B is generated by these as an **FI**-algebra over K.

Example 2.6.5. The algebra A_K and its tensor power $A^{\otimes c}$ are finitely generated in width ≤ 1, namely, by the elements $x_{i,1} \in A^{\otimes c}([1])$ with $i = 1, \ldots, c$. $\clubsuit$

We now introduce the main characters of our paper.

Definition 2.6.6. An $\mathbf{FI^{op}}$*-scheme of width one*, or *width-one* $\mathbf{FI^{op}}$*-scheme*, of finite type over K is the spectrum of an $\mathbf{FI}$-algebra over K finitely generated in width ≤ 1. ♢

The class of $\mathbf{FI}$-algebras finitely generated in width at most 1 is closed under taking finite direct sums and tensor products over K. Dually, the corresponding class of schemes is closed under disjoint unions and Cartesian products.

The following lemma will be useful later.

Lemma 2.6.7. *Let B an $\mathbf{FI}$-algebra over K finitely generated in width ≤ 1, let $X = \operatorname{Spec}(B)$ be the corresponding width-one $\mathbf{FI^{op}}$-scheme of finite type over K, and set $Z := X([1])$. Then for each $S \in \mathbf{FI}$ the map $X(S) \to \prod_{j \in S} X(\{j\}) \cong Z^S$, where the product is over $\operatorname{Spec}(B_0)$, is a closed embedding. Furthermore, the $\mathbf{FI^{op}}$-scheme Z^S is isomorphic to a closed $\mathbf{FI^{op}}$-subscheme of $\operatorname{Mat}_{c,B_0}$ for some c.*

Proof. Dually, we need to show that the map $\bigotimes_{j \in S} B(\{j\}) \to B(S)$, where the tensor product is over B_0 and where $B\{j\} \to B(S)$ comes from the inclusion $\{j\} \to S$, is surjective. This follows from the fact that B is generated in width at most 1. For the last statement, note that B_1 is finitely generated as a K-algebra, hence *a fortiori* as a B_0-algebra. If B_1 is generated by c elements over B_0, then B is a quotient of $A_{B_0}^{\otimes c}$, Z is a closed subscheme of the c-dimensional affine space $\mathbb{A}^c_{B_0}$ over B_0, and $S \mapsto Z^S$ a closed $\mathbf{FI^{op}}$-subscheme of $\operatorname{Mat}_{c,B_0}$. □

2.7. **Noetherianity.** The following result is by now classical, and the starting point of a growing body of literature on $\mathbf{FI}$-algebras.

Theorem 2.7.1. *Let K be a Noetherian ring. Then every $\mathbf{FI}$-algebra B over K that is finitely generated in width ≤ 1 is Noetherian, i.e., if $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in B, then $I_n = I_{n+1}$ for all $n \gg 0$.*

A Gröbner basis proof of a closely related theorem—formulated for an infinite symmetric group acting on an infinite-dimensional polynomial ring—first appeared in [Coh67, Coh87] and was rediscovered in [AH07] (for $c = 1$) and [HS12] (for general c). In the current set-up, the result follows from the reference in the following remark.

Remark 2.7.2. In fact, B is Noetherian in a stronger sense: any finitely generated B-module satisfies the ascending chain condition on submodules [NR19, Theorem 6.15]. ♠

2.8. **Nice width-one $\mathbf{FI^{op}}$-schemes.** The following consequence of Noetherianity will be useful to us: it implies that when we are interested in the tail of a width-one $\mathbf{FI^{op}}$-scheme X of finite type over a Noetherian ring K, i.e., in $X([n])$ for $n \gg 0$, then we may without loss of generality assume that the map $X([n+1]) \to X([n])$ dual to the inclusion $[n] \to [n+1]$ is dominant for all n.

Proposition 2.8.1. *Let K be a Noetherian ring and let $X = \operatorname{Spec}(B)$ be an affine width-one scheme of finite type over K. Then there exists an $n_0 \in \mathbb{Z}_{\geq 0}$ such that for all $S, T \in \mathbf{FI}$ with $n_0 \leq |S| \leq |T|$ and all $\pi \in \operatorname{Hom}_{\mathbf{FI}}(S, T)$, the homomorphism $B(\pi) : B(S) \to B(T)$ is injective. Define*

$$B'(S) := \begin{cases} B(S) & \text{if } |S| \geq n_0, \text{ and} \\ B(S)/\ker(B(\sigma)) & \text{if } |S| \leq n_0 \end{cases}$$

where σ is any chosen element of $\mathrm{Hom}_{\mathbf{FI}}(S,[n_0])$ *(the result doesn't depend on σ). For any $S,T \in \mathbf{FI}$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the K-algebra homomorphism $B(\pi) : B(S) \to B(T)$ induces a well-defined K-algebra homomorphism $B'(\pi) : B'(S) \to B'(T)$, and thus B' becomes an $\mathbf{FI}$-algebra over K, finitely generated in width ≤ 1, with the property that $B'(\pi)$ is injective for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$. Set $X' := \mathrm{Spec}(B')$; then $X'(\pi) : X'(T) \to X'(S)$ is dominant for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$.*

Proof. First we show that for all $S,T \in \mathbf{FI}$ we have $\ker B(\pi) = \ker B(\sigma)$ for all $\pi,\sigma \in \mathrm{Hom}_{FI}(S,T)$. For any two injections $\pi,\sigma : S \to T$ there exists a permutation τ of T such that $\pi = \tau \circ \sigma$. For $f \in \ker B(\sigma)$ we have $B(\pi)(f) = B(\tau \circ \sigma)(f) = B(\tau)(B(\sigma)(f)) = B(\tau)(0) = 0$. This implies that $\ker B(\sigma) \subset \ker B(\pi)$. By symmetry, also the reverse inclusion holds. In particular this shows that $\ker B(\sigma)$ is independent of the choice of σ and it is stable under the action of the group $\mathrm{Sym}(S)$.

Now suppose that the first claim of the proposition is not true, that is, there does not exist such an n_0. Then there exists a strictly increasing sequence of positive integers $(m_i)_i$ and injections $\pi_i : [m_i] \to [m_i+1]$ such that for all i, $\ker B(\pi_i)$ is not trivial. Let I_i be the $\mathbf{FI}$-ideal in B generated by $\bigcup_{j=1}^{i} \ker B(\pi_j)$; by the first paragraph, $I_i(T) = \{0\}$ for all T with $|T| > m_i$. Hence the sequence $(I_i)_i$ is a strictly increasing chain of $\mathbf{FI}$-ideals of B; this is a contradiction to the fact that B is Noetherian.

Let $S,T \in \mathbf{FI}$ and let $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$. If $|S| \geq n_0$, then $B'(S) = B(S)$ and it is immediate that $B(\pi) : B(S) \to B(T)$ induces a K-algebra homomorphism $B'(S) \to B'(T)$. Otherwise, let $\sigma : S \to [n_0]$ be an injection, so that $B'(S) = B(S)/\ker B(\sigma)$. If $|T| \geq n_0$, then π factors via σ and it follows that $\ker B(\pi) \supseteq \ker B(\sigma)$; again, $B(\pi)$ induces a map $B'(S) \to B'(T) = B(T)$. Finally, if also $|T| \leq n_0$, then let $\iota : T \to [n_0]$ be an injection. Replace σ by $\iota \circ \pi$, another injection $S \to [n_0]$. Then $\ker B(\sigma) = \ker(B(\iota) \circ B(\pi))$ by the first paragraph, and hence $B(\pi)$ maps $\ker B(\sigma)$ into $\ker B(\iota)$, so that, once more, it induces a map $B'(S) \to B'(T)$.

The check that B' is an $\mathbf{FI}$-algebra over K finitely generated in width ≤ 1 is straightforward, and the check that each $B'(\pi)$ is injective follows from a similar analysis to that in the previous paragraph. The final statement is standard: injective K-algebra homomorphisms yield dominant morphisms. □

Definition 2.8.2. We call an $\mathbf{FI}$-algebra B over a ring K *nice* if for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the map $B(\pi) : B(S) \to B(T)$ is injective; also its spectrum is then called nice. Proposition 2.8.1 says that if K is Noetherian, then any width-one affine $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over K agrees with a nice scheme for sufficiently large S. ♢

Lemma 2.8.3. *Let B be a nice* $\mathbf{FI}$*-algebra over K and let $h \in K$. Then $B[1/h]$ is a nice* $\mathbf{FI}$*-algebra over $K[1/h]$.*

Proof. For each $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$, $B[1/h](\pi)$ is the $K[1/h]$-algebra homomorphism $B(S)[1/h] \to B(T)[1/h]$ obtained by the K-algebra homomorphism $B(S) \to B(T)$ by localisation. By assumption, the latter is injective. Hence, since localisation is an exact functor from K-modules to $K[1/h]$-modules, so is the former. □

2.9. Reduced $\mathbf{FI}^{\mathbf{op}}$-schemes.

Definition 2.9.1. The $\mathbf{FI}$-algebra B over K is called *reduced* if $B(S)$ has no nonzero nilpotent elements for any $S \in \mathbf{FI}$. Then also $X = \mathrm{Spec}(B)$ is called reduced. ♢

The following lemma is immediate.

Lemma 2.9.2. *Let B be an* **FI**-*algebra over K and for each $S \in$* **FI** *let $B^{\mathrm{red}}(S)$ be the quotient of $B(S)$ by the ideal of nilpotent elements. Then for $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the homomorphism $B(\pi)$ induces a homomorphism $B^{\mathrm{red}}(S) \to B^{\mathrm{red}}(T)$, and this makes B^{red} into a reduced* **FI**-*algebra over K. Furthermore, if B is finitely generated in width ≤ 1, then so is B^{red}.* □

It follows that, to prove our Main Theorem, we may always assume that X is reduced.

2.10. **Shifting.** The idea of shifting goes back to the first author's work on topological Noetherianity of polynomial functors [Dra19], except that here, we shift over a finite set rather than over a vector space.

Definition 2.10.1. Let S_0 be a finite set. Then $\mathrm{Sh}_{S_0} : \mathbf{FI} \to \mathbf{FI}$ is the functor that sends S to the disjoint union $S_0 \sqcup S$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ to $\mathrm{Sh}_{S_0} \pi : S_0 \sqcup S \to S_0 \sqcup T$ that is the identity on S_0 and equal to π on S. For an **FI**-algebra B over K we write $\mathrm{Sh}_{S_0} B := B \circ \mathrm{Sh}_{S_0}$ and for the affine $\mathbf{FI}^{\mathbf{op}}$-scheme $X = \mathrm{Spec}(B)$ over K we write $\mathrm{Sh}_{S_0} X := X \circ \mathrm{Sh}_{S_0} = \mathrm{Spec}(\mathrm{Sh}_{S_0} B)$. Furthermore, for a homomorphism $\varphi : B \to R$ of **FI**-algebras over K, we write $\mathrm{Sh}_{S_0} \varphi$ for the morphism $\mathrm{Sh}_{S_0} B \to \mathrm{Sh}_{S_0} R$ that sends S to $\varphi(S_0 \sqcup S)$, and similarly for morphisms of affine $\mathbf{FI}^{\mathbf{op}}$-schemes. A straightforward check shows that Sh_{S_0} is a covariant functor from **FI**-algebras over K into itself and from affine $\mathbf{FI}^{\mathbf{op}}$-schemes over K into itself. ♢

If B is finitely generated in width ≤ 1, then so is $\mathrm{Sh}_{S_0} B$; and hence, if X is a width-one $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over K, then so is $\mathrm{Sh}_{S_0} X$.

Remark 2.10.2. If $B' := \mathrm{Sh}_{S_0} B$, then B' is naturally an **FI**-algebra over $B'_0 = B(S_0)$ (see Remark 2.2.2). Thus shifting naturally leads to a change of base ring—informally, by shifting we "move some functions into the constants". For an $f \in B(S_0)$, its image in $B(S_0 \sqcup S)$ under $B(\iota)$, where ι is the natural injection $S_0 \to S_0 \cup S$, will also be denoted simply by f. This is slight abuse of notation, especially as $B(\iota)$ needs not be an injection if B is not nice, but this will not lead to confusion.

In the interpretation from Remark 2.2.3 of **FI**-algebras consisting of algebras acted upon by $\mathrm{Sym}([n])$ with suitable maps between them, one may model shifting by restricting the action to the subgroup of $\mathrm{Sym}([n])$ that fixes the numbers 1 up to $n_0 := |S_0|$. We will, however, not explicitly use this model. ♠

For future use, we note that $\mathrm{Sh}_{S_0}(\mathrm{Sh}_{S_1} B)$ is canonically isomorphic to $\mathrm{Sh}_{S_0 \sqcup S_1} B$, and similarly for $\mathbf{FI}^{\mathbf{op}}$-schemes. Furthermore, shifting preserves reducedness and niceness.

3. The Shift Theorem

3.1. **Formulation of the Shift Theorem.** Recall from Lemma 2.6.7 that a width-one $\mathbf{FI}^{\mathbf{op}}$-scheme $X = \mathrm{Spec}(B)$ of finite type over a ring K is a closed $\mathbf{FI}^{\mathbf{op}}$-subscheme of $S \mapsto Z^S$, where $Z = X([1])$ and where the product is over B_0. In this section we establish the fundamental result that in fact, after a suitable shift and localisation, X becomes *equal* to such a product.

Theorem 3.1.1 (Shift Theorem)**.** *Let B be a reduced and nice* **FI**-*algebra that is finitely generated in width ≤ 1 over a ring K, assume that $1 \neq 0$ in B_0, and set*

$X := \mathrm{Spec}(B)$. *Then there exists an* $S_0 \in \mathbf{FI}$ *and a nonzero element* $h \in B(S_0)$ *such that* $X' := (\mathrm{Sh}_{S_0} X)[1/h]$ *is isomorphic to* $S \mapsto Z^S$*, where* $Z = X'([1])$ *and where the product is over* $B(S_0)[1/h]$.

3.2. **Shift-and-localise.** Before proving the Shift Theorem, we establish that shifting and localisation commute in a suitable sense.

Lemma 3.2.1. *Let* B *be an* $\mathbf{FI}$*-algebra over* K*,* $S_0, S_1 \in \mathbf{FI}$*,* $h_0 \in B(S_0)$ *nonzero,* $B' := (\mathrm{Sh}_{S_0} B)[1/h_0]$*,* $h_1 \in B'(S_1)$ *nonzero, and* $B'' := (\mathrm{Sh}_{S_1} B')[1/h_1]$*. Then there exists a nonzero* $h \in B(S_0 \sqcup S_1)$ *such that* $(\mathrm{Sh}_{S_0 \sqcup S_1} B)[1/h] \cong B''$ *as* $\mathbf{FI}$*-algebras over* K*.*

Proof. By multiplying h_1 with a suitable power of the image of h_0 in $B'(S_1)$, we achieve that h_1 lies in the image of $B(S_0 \sqcup S_1)$ in $B(S_0 \sqcup S_1)[1/h_0] = B'(S_1)$. Let $\tilde{h}_1$ be an element of $B(S_0 \sqcup S_1)$ mapping to h_1. Then, by a straightforward computation, $h := h_0\tilde{h}_1 \in B(S_0 \sqcup S_1)$ does the trick. □

3.3. **Proof of the Shift Theorem.**

Proof. By Lemma 2.6.7, X is (isomorphic to) a closed $\mathbf{FI}^{\mathbf{op}}$-subscheme of Mat_{c,B_0} for some c. Let $R : S \to B_0[x_{ij} \mid i \in [c], j \in S]$ be the coordinate ring of the latter wide-matrix space, and let I be the ideal of X in R.

Fix any monomial order on $\mathbb{Z}^c_{\geq 0}$. We will use this order to compare monomials in the variables $x_{1j}, \ldots, x_{cj}$ for any j.

Elements of $I([1])$ are B_0-linear combinations of monomials $x_{1,1}^{\alpha_1} \cdots x_{c,1}^{\alpha_c}$ with $\alpha \in \mathbb{Z}^c_{\geq 0}$. Let $M \subseteq \mathbb{Z}^c_{\geq 0}$ be the set of (exponent vectors of) leading monomials of *monic* elements of $I([1])$. By Dickson's lemma, there exist finitely many monic elements $f_1, \ldots, f_k \in I([1])$ whose leading monomials $u_1, \ldots, u_k$ generate M.

Now there are two possibilities. Either for every $n \in \mathbb{Z}_{\geq 1}$ and every nonzero $f \in I([n]) \subseteq R([n])$, some monomial in f is divisible by $R(\pi)u_i$ for some $i \in [k]$ and some $\pi \in \mathrm{Hom}_{\mathbf{FI}}([1], [n])$—or not. In the former case, using that the f_i are monic, we can do division with remainder by the $R(\pi)f_i$ until the remainder is zero, and it follows that $f_1, \ldots, f_k \in I([1])$ generate the $\mathbf{FI}$-ideal I. Then X itself is a product as desired, and we can take $S_0 := \emptyset$ and $h := 1 \neq 0 \in B_0$.

In the latter case, let n_0 be minimal such that there exists a nonzero $f \in I([n_0])$ none of whose terms are divisible by any $R(\pi)u_i$. Regard f as a polynomial in $x_{1,n_0}, \ldots, x_{c,n_0}$ with coefficients in $R([n_0 - 1])$, let $u = x_{1,n_0}^{\alpha_1} \cdots x_{c,n_0}^{\alpha_c}$ be the leading monomial of f, and let $\tilde{h} \in R([n_0-1])$ be the coefficient of u in f. Now $\tilde{h} \notin I([n_0-1])$ by minimality of n_0 and the fact that no term in $\tilde{h}$ is divisible by any $R(\pi)u_i$ with $i \in [k]$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}([1], [n-1])$—indeed, such a term, multiplied with u, would yield a term in f with the same property. Set $S_0 := [n_0 - 1]$ and let h be the image of $\tilde{h}$ in $B(S_0)$; this is nonzero by construction.

Now set $B' := (\mathrm{Sh}_{S_0} B)[1/h]$ and $X' := \mathrm{Spec}(B')$. Then X' is a closed $\mathbf{FI}^{\mathbf{op}}$-subscheme of Mat_{c,B'_0}, and we claim that if we construct $M' \subseteq \mathbb{Z}^c_{\geq 0}$ for X' in the same manner as we constructed M for X, then $M' \supsetneq M$. Indeed, if $\iota : [1] \to S_0 \sqcup [1]$ is the natural inclusion, then $R(\iota)$ maps f_i to an element in the ideal of $X(S_0 \sqcup [1])$ with the same leading monomial u_i, and this maps to an element of the ideal of $X'([1])$ with that same leading monomial. This shows that $M' \supseteq M$. Furthermore, via the bijection $\tau : [n_0] \to S_0 \sqcup [1]$ that is the identity on $S_0 = [n_0 - 1]$ and sends n_0 to 1 we obtain another element $R(\tau)f$ in the ideal of $X(S_0 \sqcup [1])$, whose image in the ideal of $X'([1])$ has an invertible leading coefficient (namely, h) and leading monomial $x_{1,1}^{\alpha_1} \cdots x_{c,1}^{\alpha_c}$. We thus find that $\alpha \in M'$, while $\alpha \notin M$ by construction.

The fact that B is nice and reduced implies that so is B'. Hence we can continue in the same manner with B'. By Dickson's lemma, the set M can strictly increase only finitely many times. Hence after finitely many shift-and-localise steps, we reach the former case, where we know that X is a product.

Finally, we invoke Lemma 3.2.1 to conclude that this finite sequence of shift-and-localise steps can be turned into a single shift followed by a single localisation inverting a nonzero element. □

We will use the following strengthening of the Shift Theorem in the case where K is Noetherian.

Proposition 3.3.1. *In the setting of the Shift Theorem, if we further assume that K is Noetherian, then there exists a nonzero $h' \in B_0'$ such that $B'' := B'[1/h']$ and $X'' := \mathrm{Spec}(B'')$ have the following properties:*

1. *like in the Shift Theorem, X'' is isomorphic to $S \mapsto V^S$ where $V := X''([1])$ and where the product is over B_0'';*
2. *B_0'' is a domain; and*
3. *for each $S \in \mathbf{FI}$, every irreducible component of V^S maps dominantly into $\mathrm{Spec}(B_0'')$.*

Proof. The $\mathbf{FI^{op}}$-scheme $X' = \mathrm{Spec}(B')$ from the Shift Theorem maps S to Z^S, where $Z = X'([1])$ and where the product is over B_0'. By construction, B is reduced, nice, and $1 \neq 0$ in B_0'. Any localisation by a nonzero $h' \in B_0'$ satisfies (1). We will now construct h' so as to satisfy (2) and (3).

As K is Noetherian and B_0' is a finitely generated K-algebra, B_0' is Noetherian. Hence $\mathrm{Spec}(B_0')$ is the union of finitely many irreducible components; let C be one of them. Then there exists a nonzero $h_1 \in B_0'$ that vanishes identically on all other irreducible components of $\mathrm{Spec}(B_0')$. Now $B_0'[1/h_1]$ is a domain, namely, the coordinate ring of $C[1/h_1]$.

Furthermore, $B_1'[1/h_1]$ is a finitely generated $B_0'[1/h_1]$-algebra and by generic freeness, there exists a nonzero $h_2 \in B_0'[1/h_1]$ such that $B_1'[1/h_1][1/h_2]$ is a free $B_0'[1/h_1][1/h_2]$-module. After multiplying with a power of (the image of) h_1, we may assume that h_2 the image of some $\tilde{h}_2 \in B_0$. Then set $h' := h_1\tilde{h}_2$.

Set $B'' := B'[1/h']$ and $X'' := \mathrm{Spec}(B'') = X'[1/h']$. Now B_0'' is a localisation of the domain $B'[1/h_1]$, hence a domain, so (2) holds.

Furthermore, for every $S \in \mathbf{FI}$, $X''(S)$ is the product over $B_0'' = B'[1/h']$ of $|S|$ copies of $V := X''([1])$. Its coordinate ring $B''(S)$ is then a tensor product over B_0'' of $|S|$ copies of the free B_0''-module B_1'', and hence $B''(S)$ is itself a free B_0''-module. Furthermore, again since niceness is preserved, the map $B_0'' \to B''(S)$ is injective. Then, by the going-down theorem for flat extensions, every minimal prime ideal of $B''(S)$ intersects B_0'' in the zero ideal, so that every irreducible component of $X''(S)$ maps onto $\mathrm{Spec}(B_0'')$, as desired. □

Definition 3.3.2. Let L be a Noetherian domain, $Q \supseteq L$ a ring extension such that Q is a finitely generated L-algebra and free as an L-module. Set $Z := \mathrm{Spec}(Q)$. Then the $\mathbf{FI^{op}}$-scheme over L defined by $S \mapsto Z^S$, where the product is over L, is said to be *of product type*. As we have seen above, each irreducible component of Z^S then maps dominantly into $\mathrm{Spec}\, L$. ♢

In Section 5 we will establish our Main Theorem for $\mathbf{FI^{op}}$-schemes of product type and then relate the general case to the product case via the Shift Theorem.

Example 3.3.3. To illustrate the Shift Theorem and Proposition 3.3.1 we analyse an example due to Mario Kummer: let $X_d(S)$ be the reduced, closed subvariety of $\mathrm{Mat}_{2,\mathbb{C}}(S)$ sonsisting of all S-tuples of points (x_i, y_i) for which there exists a nonzero degree-$\leq d$ polynomial $p \in \mathbb{C}[x, y]$ with $p(x_i, y_i) = 0$ for all $i \in S$. Kummer showed that $X_d(S)$ is an irreducible variety for all $d \geq 1$ and all S, so it is not particularly interesting from the perspective of counting components. However, it *is* interesting from the perspective of the Shift Theorem. Take $n_0 := \dim \mathbb{C}[x, y]_{\leq d} - 1$, so that through n_0 points (x_i, y_i), $i = 1, \ldots, n_0$ in general position goes a unique plane curve C of degree $\leq d$. The coefficients of the corresponding polynomial p are rational functions of the (x_i, y_i) with $i \in [n_0]$. Take for h a common multiple of the denominators of these rational functions, so that C is a curve over the ring $\mathbb{C}[x_1, y_1, \ldots, x_{n_0}, y_{n_0}][1/h] =: B_0'$. Then $X' = (\mathrm{Sh}_{[n_0]} X)[1/h]$ is the $\mathbf{FI^{op}}$-variety that maps S to C^S, where the product is over B_0'. ♣

4. The component functor

To establish the Main Theorem, we will analyse the functor that assigns to a finite set S the set of components of $X(S)$. This functor takes values in another category called **PF**.

4.1. Contravariant functors $\mathbf{FI} \to \mathbf{PF}$.

Definition 4.1.1. Let **PF** be the category whose objects are finite sets and whose morphisms $T \to S$ are partially defined maps from T to S. If $\pi : T \to S$ and $\sigma : S \to U$ are morphisms in this category, then $\sigma \circ \pi$ is defined precisely at those $i \in T$ for which π is defined at i *and* σ is defined at $\pi(i)$; and $\sigma \circ \pi$ takes the value $\sigma(\pi(i))$ there. ♢

We will be interested in contravariant functors $\mathcal{F} : \mathbf{FI} \to \mathbf{PF}$ and morphisms between these.

Definition 4.1.2. A *morphism* from a contravariant functor $\mathcal{F} : \mathbf{FI} \to \mathbf{PF}$ to another such functor $\mathcal{F}'$ is a collection of everywhere defined maps $(\Psi(S) : \mathcal{F}(S) \to \mathcal{F}'(S))_{S \in \mathbf{FI}}$ such that for all $S, T \in \mathbf{FI}$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(T) & \xrightarrow{\Psi(T)} & \mathcal{F}'(T) \\ {\scriptstyle \mathcal{F}(\pi)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(\pi)} \\ \mathcal{F}(S) & \xrightarrow[\Psi(S)]{} & \mathcal{F}'(S) \end{array}$$

commutes in the following sense: if the leftmost map $\mathcal{F}(\pi)$ is defined at some $f \in \mathcal{F}(T)$, then the rightmost map $\mathcal{F}'(\pi)$ is defined at $\Psi(T)(f)$, and we have $\mathcal{F}'(\pi)(\Psi(T)(f)) = \Psi(S)(\mathcal{F}(\pi)(f))$. The morphism is called injective/surjective if each $\Psi(S)$ is injective/surjective, and an isomorphism if each $\Psi(S)$ is bijective and moreover $\mathcal{F}'(\pi)$ is defined *precisely* at all $f' \in \mathcal{F}'(T)$ such that $\mathcal{F}(\pi)$ is defined at $\Psi(T)^{-1}(f')$. ♢

Note that our morphisms $\mathcal{F} \to \mathcal{F}'$ are not precisely natural transformations, since we do not require that the diagram above commutes as a diagram of partially defined maps: we allow the partially defined map $\mathcal{F}'(\pi) \circ \psi(T)$ to have a larger domain than $\psi(S) \circ \mathcal{F}(\pi)$.

4.2. The component functor of an $\mathbf{FI}^{\mathbf{op}}$-scheme.

Definition 4.2.1. Let B be a finitely generated **FI**-algebra over a Noetherian ring K, so that $X = \mathrm{Spec}(B)$ is an affine $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over K. Then we define the contravariant functor $\mathcal{C}_X : \mathbf{FI} \to \mathbf{PF}$ on objects by

$$\mathcal{C}_X(S) = \{\text{the irreducible components of } X(S)\}$$

and on morphisms $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ as follows: $\mathcal{C}_X(\pi)$ is defined at some component $C \in \mathcal{C}_X(T)$ if (and only if) $X(\pi) : X(T) \to X(S)$ maps C dominantly into a component of $X(S)$. The functor $\mathcal{C}_X$ is called the *component functor* of X. ♢

Note that the condition that K is Noetherian and B is finitely generated implies that, indeed, $\mathcal{C}_X(S)$ is a finite set for each S.

Example 4.2.2. In Example 1.2.1, $\mathcal{C}_X$ is isomorphic to the functor $\mathbf{FI} \to \mathbf{PF}$ that assigns to the set S the set $\binom{S}{2}$ of two-element subsets and to $\pi : S \to T$ the partially defined map $\binom{T}{2} \to \binom{S}{2}$ that sends $\{i,j\}$ to $\{\pi^{-1}(i), \pi^{-1}(j)\}$ whenever this is defined. ♣

In the definition of the component functor we have not assumed that B is generated in width ≤ 1, and indeed larger **FI**-algebras also yield interesting examples.

Example 4.2.3. Let K be a field and let R be the **FI**-algebra that assigns to S the ring $R(S) = K[x_{i,j} \mid i,j \in S]$ and to a morphism $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the K-algebra homomorphism determined by $x_{i,j} \mapsto x_{\pi(i),\pi(j)}$. This **FI**-algebra is generated in width 2 by the two elements $x_{1,1}, x_{1,2} \in R([2])$.

It is well known that this **FI**-algebra is *not* Noetherian; the following example is closely related to [HS12, Example 3.8]. Let $I_d(S)$ be the ideal generated by all *cycle monomials* of the form $x_{i_1,i_2}x_{i_2,i_3}\cdots x_{i_k,i_1}$ where $3 \leq k \leq d$ and $i_1,\ldots,i_k$ are distinct. Then $I_3 \subsetneq I_4 \subsetneq \ldots$ is an infinite strictly increasing chain of ideals in R []. Let I_∞ be their union, and let $X = \mathrm{Spec}(R/I_\infty)$. A prime ideal P in $R(S)$ containing $I_\infty(S)$ contains at least one variable from every cycle of length at least 3. It follows that the minimal prime ideals of $R(S)/I_\infty(S)$ are in bijection to the *spanning trees* in the complete graph on the vertex set S. Recall that, by Cayley's formula, this number of spanning trees is n^{n-2} when $n := |S| \geq 2$. In particular, the number of $\mathrm{Sym}([n])$-orbits is at least $(n^{n-2})/n!$ and hence superpolynomial in n; this shows that in the Main Theorem the width-one condition cannot be dropped.

Furthermore, given a $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$, $\mathcal{C}_X(\pi)$ is defined on trees Δ with vertex set T as follows. If the induced subgraph of Δ on $\pi(S)$ is connected (and hence a tree), then $\mathcal{C}_X(\pi)(\Delta)$ is that tree but with the label $j \in \pi(S)$ replaced by $\pi^{-1}(j)$. Otherwise, $\mathcal{C}_X(\pi)$ is not defined at Δ. ♣

4.3. **The underlying species.** In our proof of the Main Theorem, we will give a fairly complete picture of the component functor of width-one $\mathbf{FI}^{\mathbf{op}}$-schemes, at least on sets $S \in \mathbf{FI}$ with $|S| \gg 0$. The first observation is that for *any* contravariant functor $\mathcal{F} : \mathbf{FI} \to \mathbf{PF}$ and any $\pi \in \mathrm{End}_{\mathbf{FI}}(S) = \mathrm{Sym}(S)$, $\mathcal{F}(\pi)$ is defined everywhere on $\mathcal{F}(S)$, and a bijection there. After all, by the properties of a contravariant functor $\mathrm{id}_{\mathcal{F}(S)} = \mathcal{F}(\pi \circ \pi^{-1}) = \mathcal{F}(\pi^{-1}) \circ \mathcal{F}(\pi)$. It follows that the functor from the category of finite sets with bijections to itself that sends S to $\mathcal{F}(S)$ and π to $\mathcal{F}(\pi)^{-1}$ is a covariant functor and hence a *species* in the sense of [Joy81]; we call this the *underlying species* of the $\mathcal{F}$. For the Main Theorem it would suffice to know the underlying species of the component functor $\mathcal{C}_X$ of X. However, to understand this species, we will also need to have some information on the partially defined maps $\mathcal{C}_X(\pi)$ where $\pi : S \to T$ is *not* a bijection.

4.4. **A property in width one.** The second observation on component functors concerns width-one $\mathbf{FI^{op}}$-schemes.

Lemma 4.4.1. *Suppose that X is a width-one affine $\mathbf{FI^{op}}$-scheme of finite type over a Noetherian ring K. Then there exists an n_0 such that for all $S, T \in \mathbf{FI}$ with $n_0 \leq |S| \leq |T|$ and all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$, the partially defined map $\mathcal{C}_X(\pi)$ is* surjective*.*

Proof. Take the n_0 from Proposition 2.8.1, so that $X(\pi) : X(T) \to X(S)$ is dominant for all $S, T \in \mathbf{FI}$ with $n_0 \leq |S| \leq T$. Then for each component of $X(S)$ there must be some component of $X(T)$ mapping dominantly into it. □

5. Proof of the Main Theorem

In this section, which takes up the remainder of the paper, we establish the Main Theorem. To do so, on the one hand we develop purely combinatorial tools (see §§5.2,5.3,5.5,5.6,5.7,5.8,5.9,) and on the other hand we establish algebraic results relating the component functors of width-one $\mathbf{FI^{op}}$-schemes to those combinatorial tools (see §§5.1,5.4,5.10). Finally, all is combined in §5.11 to establish the Main Theorem. We would like to highlight §5.7, where from a so-called model functor we extract certain groupoids acting on unions of rational cones, after which we use an orbit-counting lemma for groupoids from §5.5 to establish quasipolynomiality in that crucial case.

5.1. **The component functor of a wide-matrix space.** Let L be a finitely generated K-algebra, where K is a Noetherian ring, and $c \in \mathbb{Z}_{\geq 0}$. Then for each $S \in \mathbf{FI}$ we have a natural morphism $\mathrm{Mat}_{c,L}(S) \to \mathrm{Spec}(L)$ (corresponding to the natural embedding $L \to L[\mathrm{Mat}_{c,L}(S)]$), and the preimages of the irreducible components of L are the irreducible components of $\mathrm{Mat}_{c,L}(S)$. This establishes the following.

Lemma 5.1.1. *Let k be the number of minimal prime ideals of L. The component functor of $\mathrm{Mat}_{c,L}$ is isomorphic to the functor that assigns the set $[k]$ to each $S \in \mathbf{FI}$ and the identity on $[k]$ to each $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$. In particular, the number of $\mathrm{Sym}([n])$-orbits on $\mathcal{C}_{\mathrm{Mat}_{c,L}}([n])$ is equal to k.* □

5.2. **Elementary model functors.** We construct a class of contravariant functors $\mathbf{FI} \to \mathbf{PF}$ from which, as we will see, the component functor of a width-one $\mathbf{FI^{op}}$-scheme of finite type over a Noetherian ring is built up in a suitable sense.

Definition 5.2.1. Let $k \in \mathbb{Z}_{\geq 0}$, let G be a subgroup of $\mathrm{Sym}([k])$, and, for $S \in \mathbf{FI}$, let $\mathcal{E}(S)$ be a subset of $[k]^S$ that is preserved under the diagonal action of G on $[k]^S$. Assume, furthermore, that for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ the map $[k]^T \to [k]^S$, $\alpha \mapsto \alpha \circ \pi$ maps $\mathcal{E}(T)$ into $\mathcal{E}(S)$. Then the contravariant functor $\mathbf{FI} \to \mathbf{PF}$ that sends S to $\mathcal{E}(S)/G$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ to the (everywhere defined) map

$$\mathcal{E}(T)/G \to \mathcal{E}(S)/G, \quad G \cdot \alpha \mapsto G \cdot (\alpha \circ \pi)$$

is called an *elementary model functor* **FI** $\to$ **PF**. Note that the latter map is well-defined as G acts diagonally. $\diamondsuit$

5.3. A first quasipolynomial count.

Proposition 5.3.1. *Let $k \in \mathbb{Z}_{\geq 0}$, G a subgroup of* $\mathrm{Sym}([k])$ *and M a G-stable downward closed subset of $\mathbb{Z}_{\geq 0}^k$; that is, for all $\beta \in M$ and $g \in G$ we have $g\beta \in M$, and for all $\beta \in M$ and $j \in [k]$ with $\beta_j > 0$ we have $\beta - e_j \in M$, where e_j is the j-th standard basis vector in $\mathbb{Z}^k$. Then there exists a quasipolynomial f such that for $n \gg 0$ the number of G-orbits on the set M_n of elements $\beta \in M$ of* total degree $|\beta| := \sum_j \beta_j$ *equal to n equals $f(n)$.*

Proof. By the orbit-counting lemma, that number of orbits equals

$$\frac{1}{|G|}\sum_{g \in G} |M_n^g|$$

where $M_n^g = \{\alpha \in M_n \mid g\alpha = \alpha\}$. So it suffices to prove that each of the summands is a quasipolynomial for $n \gg 0$.

The set M has a so-called *Stanley decomposition* [Sta82]

$$M = \bigsqcup_{i=1}^{d} (\alpha_i + \mathbb{Z}_{\geq 0}^{I_i})$$

for suitable subsets $I_i \subseteq [k]$. Call the i-th term $N(i)$. Then, for each $g \in G$, $N(i)^g$ is the set of nonnegative integers points in a certain rational polyhedron, and its elements of degree $n \gg 0$ are counted by a quasipolynomial by [Sta97, Theorem 4.5.11 and Proposition 4.4.1]. $\square$

An immediate consequence is the following.

Corollary 5.3.2. *Let $S \mapsto \mathcal{E}(S)/G \subseteq [k]^S/G$ be an elementary model functor. Then $|(\mathcal{E}([n])/G)/\mathrm{Sym}([n])|$ equals some quasipolynomial in n, for all $n \gg 0$.*

Proof. Define a map $\mathcal{E}([n]) \to \mathbb{Z}_{\geq 0}^k$ by sending the vector α to its *count vector* β, i.e., the vector in which β_j is the number of $l \in [n]$ with $\alpha_l = j$. Note that this map is G-equivariant, so the image M_n is G-stable; and that the fibres are precisely the $\mathrm{Sym}([n])$-orbits. Furthermore, the fact that $\mathcal{E}$ is a model functor implies that the union $M = \bigcup_n M_n$ is downward closed. Now apply Proposition 5.3.1. $\square$

5.4. $\mathbf{FI^{op}}$-schemes of product type.

Elementary model functors are combinatorial models for the component functor of $\mathbf{FI^{op}}$-schemes of product type in the sense of Definition3.3.2, as follows.

Proposition 5.4.1. *Let L a Noetherian domain and let X be a width-one* $\mathbf{FI^{op}}$*-scheme of product type over L. Then the component functor $\mathcal{C}_X$ is isomorphic to an elementary model functor.*

Proof. By assumption, $X(S) = Z^S$, where Z is a fixed affine scheme over L, and each irreducible component of $X(S)$ maps dominantly into $\mathrm{Spec}(L)$. Let M be the fraction field of L, and let X_M be the base change of X to M. Since each irreducible component of $X(S)$ maps dominantly into $\mathrm{Spec}(L)$, basic properties of localisation imply that the morphism $X_M(S) \to X(S)$ is a bijection at the level of irreducible components. Furthermore, taking these bijections for all S, we obtain an isomorphism $\mathcal{C}_{X_M} \to \mathcal{C}_X$ of contravariant functors **FI** $\to$ **PF**.

Next let $\overline{M}$ be a separable closure of M and let $X_{\overline{M}}$ be the base change of X_M to $\overline{M}$. Then, for each $S \in \mathbf{FI}$, the morphism $X_{\overline{M}}(S) \to X_M(S)$ induces a surjection $\mathcal{C}_{X_{\overline{M}}}(S) \to \mathcal{C}_{X_M}(S)$, and the fibres are precisely the orbits of the Galois group $\mathrm{Gal}(\overline{M} : M)$ on $\mathcal{C}_{X_{\overline{M}}}(S)$ [Sta20, Tag 0364]. In other words, $\mathcal{C}_{X_M}(S)$ has a canonical bijection to $\mathcal{C}_{X_{\overline{M}}}(S)/\mathrm{Gal}(\overline{M} : M)$. To complete the proof, we need to analyse the component functor of $X_{\overline{M}}$.

To this end, let $Z_1, \ldots, Z_k$ be the irreducible components of the base change $Z_{\overline{M}}$. Then

$$X_{\overline{M}}(S) = Z_{\overline{M}}^S = \bigcup_{\alpha \in [k]^S} \prod_{i \in S} Z_i^{\alpha_i}$$

where each product over $i \in S$ is a product of irreducible varieties over the separably closed field $\overline{M}$, and hence irreducible. To construct our component functor, we just set $\mathcal{E}(S) := [k]^S$.

Finally, let G be the image of $\mathrm{Gal}(\overline{M} : M)$ in $\mathrm{Sym}([k])$ through its action on the irreducible components $Z_1, \ldots, Z_k$ of Z. Then the (image of the) action of $\mathrm{Gal}(\overline{M} : M)$ on irreducible components of $X_{\overline{M}}(S)$ corresponds precisely to the (image of the) diagonal action of G on $\mathcal{E}(S)$, and hence the orbit space $\mathcal{E}(S)/G$ is in bijection with the irreducible components of $X_M(S)$. This bijection, taken for all S, is an isomorphism from the elementary model functor given by $\mathcal{E}(S) = [k]^S$ and the group $G \subseteq \mathrm{Sym}([k])$. □

Remark 5.4.2. Note that the elementary model functors coming from $\mathbf{FI^{op}}$-schemes of product type all have $\mathcal{E}(S) = [k]^S$ rather than just $\mathcal{E}(S) \subseteq [k]^S$. The set of count vectors is therefore all of $\mathbb{Z}_{\geq 0}^k$. However, in our proof of the Main Theorem we will need to do induction over the poset of downward closed subsets of $\mathbb{Z}_{\geq 0}^k$; this requires the greater generality in the definition of elementary model functors.

Corollary 5.4.3. *The Main Theorem holds for affine $\mathbf{FI^{op}}$-schemes of product type over some Noetherian domain.*

Proof. This is an immediate corollary of Proposition 5.4.1 and Corollary 5.3.2. □

We are now in a position to prove the following result, most of which also follows from combining results from [VNNR20] (linearity of codimension) and [NR17] (the form of the Hilbert function).

Theorem 5.4.4. *Let X be an affine width-one $\mathbf{FI^{op}}$-scheme X of finite type over a Noetherian ring K. Assume that $X([n])$ is not the empty scheme for any n. Then for $n \gg 0$ the Krull dimension of $X([n])$ is eventually equal to an affine-linear polynomial in n, and the number of irreducible components $|\mathcal{C}_X([n])|$ is bounded from above by c^n for some constant $c \geq 1$.*

Proof. By Lemma 2.9.2 we may assume that $X = \mathrm{Spec}(B)$ is reduced, and by Proposition 2.8.1 we may assume that X is nice. Since $X([n])$ is not the empty scheme for any n, we have $1 \neq 0$ in B_0. By the Shift Theorem 3.1.1 and Proposition 3.3.1 there exists an $S_0 \in \mathbf{FI}$ and a nonzero $h \in B(S_0)$ such that $X' := (\mathrm{Sh}_{S_0} X)[1/h]$ is of product type; in particular, it sends $S \to Z^S$ for some reduced scheme Z of finite type over $L := B(S_0)[1/h]$.

Let Y be the closed $\mathbf{FI^{op}}$-subscheme of X defined by the vanishing of h. For any $S \in \mathbf{FI}$ we then have

$$\dim X(S_0 \sqcup S) = \max\{\dim(Y(S_0 \sqcup S)), \dim(X'(S))\}.$$

By Noetherianity (Theorem 2.7.1) we may assume that the theorem holds for Y. On the other hand, $\dim(X'(S)) = \dim(L) + |S| \cdot \dim(Z)$. We conclude that, for $n \gg 0$, $\dim(X([n]))$ is a maximum of two affine-linear functions of n, hence itself an affine-linear function of n. Similarly, to bound $|\mathcal{C}_X(S_0 \sqcup S)|$ we claim that

$$|\mathcal{C}_X(S_0 \sqcup S)| \leq |\mathcal{C}_Y(S_0 \sqcup S)| + (|S| + |S_0|)(|S| + |S_0| - 1) \cdots (|S| + 1)|\mathcal{C}_{X'}(S)|.$$

Indeed, if C is an irreducible component of $X(S_0 \sqcup S)$, then either C is contained in $Y(S_0 \sqcup S)$ (and then a component there) or else there exists an injection $\pi : S_0 \to S_0 \sqcup S$ such that $B(\pi)h$ is not identically zero on C. In the latter case, let $\sigma \in \mathrm{Sym}(S_0 \sqcup S)$ be any element with $\sigma \circ \pi = \mathrm{id}_{S_0}$. Then $B(\sigma)B(\pi)h = h$, and hence $C = X(\sigma)C'$ for a component C' of $X(S_0 \sqcup S)$ on which h is nonzero. These components correspond bijectively to components of $X'(S)$. This explains the second term, where the first $|S_0|$ factors count the number of possibilities for π.

Now the first term is bounded by an exponential function of $|S_0| + |S|$ by the induction hypothesis, and the second term is bounded by an exponential function by the proof of Proposition 5.4.1. Hence so is the sum. □

5.5. **The orbit-counting lemma for groupoids.** It turns out that in the general case of the Main Theorem, the (Galois) group that featured in the proof of Corollary 5.4.3, is replaced by a suitable *groupoid*. We briefly recall the relevant set-up.

Let G be a finite groupoid, that is, a category whose class of objects is a finite set Q and in which for any $p, q \in Q$ the set $G(p, q) := \mathrm{Hom}(p, q)$ is a finite set all of whose elements are isomorphisms. Rather than homomorphisms or isomorphisms, we will call these elements *arrows*.

For a groupoid to act on a finite set X, one first specifies an *anchor map* $a : X \to Q$. For $p \in Q$, set $X_p := a^{-1}(p)$. Then, an action of G on X consists of the data of a map $\varphi(g) : X_p \to X_q$ for each homomorphism $g : p \to q$, subject to the conditions that $\varphi(\mathrm{id}_p) = \mathrm{id}_{X_p}$ and $\varphi(h \circ g) = \varphi(h) \circ \varphi(g)$ for any two arrows $g : p \to q$ and $h : q \to r$. We often write gx instead of $\varphi(g)(x)$.

Write $G(p) := \bigcup_{q \in Q} G(p, q)$ for the set of arrows from p. For $x \in X_p$ we have a map $G(p) \to X, g \mapsto gx$. The image of this map is called the *orbit* of x and denoted $G(p)x$. On the other hand, we write $G(p,p)_x := \{g \in G(p,p) \mid gx = x\}$, the stabiliser of x in $G(p,p)$, which is a subgroup of the group $G(p,p)$. The map $G(p) \to G(p)x$ yields a bijection $G(p)/G(p,p)_x \to G(p)x$; here $G(p,p)_x$ acts freely on $G(p)$ by precomposition, so that $|G(p)x| = |G(p)|/|G(p,p)_x|$. Furthermore, for every element $y \in G(p)x$ we have $|G(a(y))| = |G(p)|$ and $|G(a(y), a(y))_y| = |G(p,p)_x|$.

Finally, for $g \in G(p,p)$ we write X_p^g for the set of elements $x \in X_p$ with $gx = x$. The following is a generalisation of the orbit-counting lemma for groups.

Lemma 5.5.1. *The number of orbits of G on X equals*

$$\sum_{p \in Q} \frac{1}{|G(p)|} \sum_{g \in G(p,p)} |X_p^g|.$$

Proof. We count the triples (p, g, x) with $p \in Q$ and $x \in X_p$ and $g \in G(p,p)$ with $gx = x$ and $|G(p)| = N$ in two different ways. If we first fix x, then we are forced

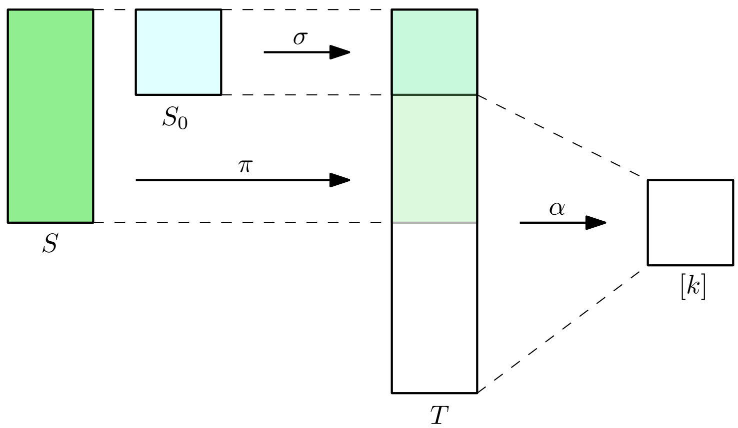

FIGURE 1. The construction of $\mathcal{F}(\pi)$ in Definition 5.6.1.

to take $p := a(x)$, and we obtain

$$\sum_{x\in X:|G(a(x))|=N} |G(a(x),a(x))_x| = \sum_{x\in X:|G(a(x))|=N} |G(a(x))|/|G(a(x))x|$$
$$= \sum_{x\in X:|G(a(x))|=N} N/|G(a(x))x|.$$

This is N times the number of orbits of G on the set of x with $|G(a(x))| = N$.

On the other hand, if we first fix p with $|G(p)| = N$ and $g \in G(p,p)$, then we find

$$\sum_{p\in Q:|G(p)|=N} \sum_{g\in G(p,p)} |X_p^g|.$$

Hence the number of orbits of G on the set of x with $|G(a(x))| = N$ equals

$$\frac{1}{N} \sum_{p\in Q:|G(p)|=N} \sum_{g\in G(p,p)} |X_p^g|.$$

Now sum over all possible values of N to obtain the formula in the lemma. □

5.6. **Model functors.** Elementary model functors are special cases of a more general class of functors $\mathbf{FI} \to \mathbf{PF}$, which we call *model functors*. Their construction is motivated by Theorem 3.1.1 and Proposition 5.4.1, as we will see below.

Fix $S_0 \in \mathbf{FI}$, a $k \in \mathbb{Z}_{\geq 0}$, and a subfunctor $\mathcal{E} : \mathbf{FI} \to \mathbf{PF}$ of the functor $S \mapsto [k]^S$, where we require that for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$, $\mathcal{E}(\pi)$ is defined everywhere. Hence, by passing to count vectors as in §5.3, $\mathcal{E}$ is uniquely determined by a downward closed subset of $\mathbb{Z}_{\geq 0}^k$.

Then we define a new functor $\mathcal{F} : \mathbf{FI} \to \mathbf{PF}$ on objects by

$$\mathcal{F}(S) = \{(\sigma,\alpha) \mid \sigma \in \mathrm{Hom}_{\mathbf{FI}}(S_0,S),\ \alpha \in \mathcal{E}([S \setminus \mathrm{im}(\sigma)])\}$$

and on a morphism $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ as follows: for $(\sigma,\alpha) \in \mathcal{F}(T)$ we set

$$\mathcal{F}(\pi)((\sigma,\alpha)) := \begin{cases} \text{undefined if } \mathrm{im}(\sigma) \not\subseteq \mathrm{im}(\pi)\text{; and} \\ (\sigma', \alpha \circ \pi|_{S\setminus \mathrm{im}(\sigma')}) \text{ where } \sigma' := \pi^{-1}\circ\sigma \text{ otherwise.} \end{cases}$$

Figure 1 depicts all relevant maps. At the level of species (so remembering the maps $\mathcal{F}(\pi)$ only when π is a bijection), this is an instance of a well-known construction: $\mathcal{F}$ is the product of the species that maps S to its set of bijections $S_0 \to S$ and the species that maps S to $\mathcal{E}(S)$.

Next let $\sim_S$ be an equivalence relation on $\mathcal{F}(S)$ for each S, and assume that these relations have the following three properties:

(1) if $(\sigma, \alpha) \sim_T (\sigma', \alpha')$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ has $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma) \cup \mathrm{im}(\sigma')$, then $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$;
(2) conversely, if $(\sigma, \alpha) \in \mathcal{F}(T)$, $(\sigma'', \alpha'') \in \mathcal{F}(S)$, and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ satisfy $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma)$ and $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'')$, then there exists a pair $(\sigma', \alpha') \in \mathcal{F}(T)$ with $\mathrm{im}(\sigma') \subseteq \mathrm{im}(\pi)$ such that $(\sigma', \alpha') \sim_T (\alpha, \sigma)$ and $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$; and
(3) if $(\sigma, \alpha) \sim_T (\sigma', \alpha')$ and $i, j \in T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$, then $\alpha(i) = \alpha(j) \Leftrightarrow \alpha'(i) = \alpha'(j)$.

The first property ensures that $\mathcal{F}/\sim: S \mapsto \mathcal{F}(S)/\sim_S$ is a functor $\mathbf{FI} \to \mathbf{PF}$ that comes with a canonical surjective morphism $\mathcal{F} \to \mathcal{F}/\sim$ in the sense of Definition 4.1.2; in particular, this implies that $\sim_S$ is preserved under the symmetric group $\mathrm{Sym}(S)$ acting on $\mathcal{F}(S)$. The second and third property will be crucial in §5.7.

Definition 5.6.1. A functor of the form $S \mapsto \mathcal{F}(S)/\sim_S$ as constructed above is called a *model functor*. ♢

Remark 5.6.2. Each elementary model functor $S \mapsto \mathcal{E}(S)/G$ is isomorphic to a model functor with $S_0 = \emptyset$ (so that we may leave out the σs from the pairs) and $\alpha \sim_S \alpha'$ if and only if $\alpha' \in G\alpha$. We will see that, conversely, a model functor gives rise to to certain groupoids that play the role of G. ♠

Remark 5.6.3. Informally, we wish to view (σ, α) as a word in $[k]^S$ in which the letters corresponding to $\mathrm{im}(\sigma) \subseteq S$ are concealed. If $(\sigma', \alpha') \sim_S (\sigma, \alpha)$, it corresponds to a different word in which the letters corresponding to $\mathrm{im}(\sigma')$ are concealed. Outside of $\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma')$, the two words should be equal up to some permutation of $[k]$ by axiom (3). Axioms (1) and (2) simply ask that these equivalent words behave well with respect to **FI**-morphisms. In the next section, we will roughly speaking attempt to 'discover' the information concealed in $\mathrm{im}(\sigma)$ by looking at equivalent σ' whose image does not contain $\mathrm{im}(\sigma)$.

5.7. **A second quasi-polynomial count.** We use the notation from §5.6. So $S \mapsto \mathcal{F}(S)/\sim_S$ is a model functor, where $\mathcal{F}(S)$ is the set of pairs (σ, α) with $\sigma \in \mathrm{Hom}_{\mathbf{FI}}(S_0, S)$ and $\alpha \in \mathcal{E}(S \setminus S_0) \subseteq [k]^{S \setminus S_0}$, and where $\sim_S$ is an equivalence relation on $\mathcal{F}(S)$ that satisfies the axioms (1),(2),(3) for model functors. The following theorem and its proof are elementary, but quite subtle—indeed, this is probably the most intricate part of the paper.

Theorem 5.7.1. *Let $S \mapsto \mathcal{F}(S)/\sim_S$ be a model functor* **FI** $\to$ **PF**. *Then there exists a quasipolynomial f such that the number of* $\mathrm{Sym}([n])$*-orbits on* $\mathcal{F}([n])/\sim_{[n]}$ *equals $f(n)$ for all $n \gg 0$.*

To prove the theorem, we introduce the notion of sub-model functor.

Definition 5.7.2. Suppose that we have, for each $S \in \mathbf{FI}$, a subset $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$ such that, first, for all $\pi \in \mathrm{Hom}_{\mathbf{FI}(S,T)}$ the pull-back map $[k]^T \to [k]^S$ that maps

$\mathcal{E}(T)$ into $\mathcal{E}(S)$ also maps $\mathcal{E}'(T)$ into $\mathcal{E}'(S)$; and second, for all $(\sigma, \alpha) \sim_S (\sigma', \alpha') \in \mathcal{F}(S)$ with $\alpha \in \mathcal{E}'(S \setminus \mathrm{im}(\sigma))$, we also have $\alpha' \in \mathcal{E}'(S \setminus \mathrm{im}(\sigma'))$. Then $S \mapsto \mathcal{F}'(S)/{\sim_S}$, where

$$\mathcal{F}'(S) := \{(\sigma, \alpha) \in \mathcal{F}(S) \mid \alpha \in \mathcal{E}'(S)\}$$

and where $\sim_S$ stands for the restriction of $\sim_S$ to $\mathcal{F}'(S)$ is a model functor called a *sub-model functor* of $\mathcal{F}$. ♢

Proof of Theorem 5.7.1. The proof of this theorem will take up the remainder of this subsection. This will involve an induction hypothesis for a sub-model functor $\mathcal{F}'$ of $\mathcal{F}$ and the construction of a certain groupoid for the complement $\mathcal{F} \setminus \mathcal{F}'$.

Let $M \subseteq \mathbb{Z}^k_{\geq 0}$ be the downward-closed set consisting of all the count vectors of elements in $\mathcal{E}(S)$ for S running over **FI**. Since, by Dickson's lemma, the set of downward-closed sets in $\mathbb{Z}^k_{\geq 0}$ satisfies the descending chain property, we may assume that the theorem holds for all model functors whose corresponding downward set is strictly contained in M.

Our goal is now to find $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$ that defines a sub-model functor $\mathcal{F}'$ of $\mathcal{F}$ such that the downward closed set M' of $\mathcal{E}'$ is strictly contained in M. Then we have

$$|\mathcal{F}(S)/{\sim_S}| = |\mathcal{F}'(S)/{\sim_S}| \; + \; |(\mathcal{F}(S) \setminus \mathcal{F}'(S))/{\sim_S}|$$

and this equality continues to hold if we mod out the action of $\mathrm{Sym}(S)$ on the three sets in question. Hence by the induction hypothesis we are done if we can show that the number of $\mathrm{Sym}(S)$-orbits on $(\mathcal{F}(S) \setminus \mathcal{F}'(S))/{\sim_S}$ grows quasipolynomially in $|S|$ for $|S| \gg 0$. In this induction argument, we may of course assume that M is not empty—otherwise, the quasipolynomial 0 will do.

To construct $\mathcal{E}'$ we proceed as follows. Let

$$A := \{I \subseteq [k] \mid \exists v \in M : v + \mathbb{Z}^I_{\geq 0} \subseteq M\};$$

here, and in the rest of the paper, we identify $\mathbb{Z}^I_{\geq 0}$ with $\mathbb{Z}^I_{\geq 0} \times \{0\}^{[k] \setminus I} \subseteq \mathbb{Z}^k_{\geq 0}$. Note that A is nonempty because M is. Let I be an inclusion-wise maximal element of A and set

$$d := \max \left\{ \sum_{l \in [k] \setminus I} v(l) \mid v + \mathbb{Z}^I_{\geq 0} \subseteq M \right\}.$$

This is well-defined, since if the sum of the entries in such v at positions outside I were unbounded, I would be contained in a strictly larger element of A. Choose $v \in M$ such that $v + \mathbb{Z}^I_{\geq 0} \subseteq M$ and $\sum_{l \in [k] \setminus I} v(l) = d$. Furthermore, we may and will assume that $v(l) \gg 0$ for all $l \in I$—however, to avoid technicalities, we make no attempt to specify exactly how large the $v(l)$ need to be.

Lemma 5.7.3. *The vectors in M that are componentwise $\geq v$ are precisely the vectors in $v + \mathbb{Z}^I_{\geq 0}$.*

Proof. This is clear if $I = \emptyset$ by maximality of d.

Suppose $I \neq \emptyset$, and let $v_j = v + \sum_{i \in I} j e_i$. Suppose that for every $j \in \mathbb{Z}_{\geq 0}$, there is a $w_j \geq v_j$ in M that is not in $v_j + \mathbb{Z}^I_{\geq 0}$. Then by Dickson's lemma the sequence $w_1|_{[k] \setminus I}, w_2|_{[k] \setminus I}, w_3|_{[k] \setminus I}, \ldots$ would contain an infinite subsequence, labelled by $i_1 < i_2 < \ldots$, that weakly increases componentwise. It follows that $w_{i_1} + \mathbb{Z}^I_{\geq 0} \subseteq M$ because M is downward closed and the entries of w_{i_j} in I diverge to infinity (and hence so do the entries of $w_{i_1} + (w_{i_j} - w_{i_1})|_I$). This would contradict the choice of v. Using $v_1 \leq v_2 \leq \ldots$, it follows that there is a maximal j for which a $w_j \geq v_j$ as above exists. Then v_{j+1} has the desired property. □

The set I is uniquely determined by v as the set of all positions $l \in [k]$ where $v(l)$ is very large. We call it the *frequent set* of v.

Write $v = v_0 + v_1$ with $v_0 \in \mathbb{Z}_{\geq 0}^{[k]\setminus I}$ and $v_1 \in \mathbb{Z}_{\geq 0}^{I}$. We now define $\mathcal{F}' : \mathbf{FI} \to \mathbf{PF}$ by setting $\mathcal{F}'(S)$ to be the set of pairs $(\sigma, \alpha) \in \mathcal{F}(S)$ for which there exists *no* pair $(\sigma', \alpha') \sim_S (\sigma, \alpha)$ such that the count vector of α' is in $\tilde{v} + \mathbb{Z}_{\geq 0}^{I}$, where $\tilde{v} := v + v_1 = v_0 + 2v_1$. Notice the factor 2; the relevance of this will become clear towards the end of the proof.

Lemma 5.7.4. *The association $S \mapsto \mathcal{F}'(S)/{\sim_S}$ is a sub-model functor of $\mathcal{F}$.*

Proof. By definition, $\mathcal{F}'(S)$ is a union of of $\sim_S$-equivalence classes, and uniquely determined by a subset $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$ of allowed second components α. So we need only prove that $\mathcal{F}'$ is preserved under morphisms.

Hence let $(\sigma, \alpha) \in \mathcal{F}'(T)$, let $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ satisfy $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma)$, and consider $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$. If $(\tilde{\sigma}, \tilde{\alpha}) \sim_S (\sigma'', \alpha'')$ where α'' has a count vector in $\tilde{v} + \mathbb{Z}_{\geq 0}^{I}$, then by axiom (2) for model functors there exists a pair $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$. This means that the count vector of α' is an element of M that is componentwise greater than or equal to the count vector of α'' and hence, by the choice of $\tilde{v}$, the count vector of α' lies in $\tilde{v} + \mathbb{Z}_{\geq 0}^{I}$. This contradicts the fact that $(\sigma, \alpha) \in \mathcal{F}'(T)$. Therefore, $\mathcal{F}(\pi)((\sigma, \alpha)) \in \mathcal{F}'(S)$. □

Furthermore, we observe that the downward closed subset M' corresponding to $\mathcal{E}'$ is strictly contained in M, since it does not contain $\tilde{v} \in M$. Hence the induction hypothesis applies to $\mathcal{F}'$.

Our task is therefore reduced to counting the $\mathrm{Sym}(S)$-orbits on the set of $\sim_S$-equivalence classes on $\tilde{\mathcal{F}}(S) := \mathcal{F}(S) \setminus \mathcal{F}'(S)$. Note that $\tilde{\mathcal{F}}$ is not a sub-model functor of $\mathcal{F}$; rather, $\tilde{\mathcal{F}}(S)$ consists of all pairs (σ, α) that are equivalent to some pair (σ', α') where α' has a count vector in $\tilde{v} + \mathbb{Z}_{\geq 0}^{I}$. Before counting these, we will work for a while with the larger set $\mathcal{F}''(S) \supseteq \tilde{\mathcal{F}}(S)$ consisting of all pairs $(\sigma, \alpha) \in \mathcal{F}(S)$ that are equivalent to some pair (σ', α') where the count vector of α' is in $v + \mathbb{Z}_{\geq 0}^{I} \supseteq \tilde{v} + \mathbb{Z}_{\geq 0}^{I}$.

For the time being, fix a pair $(\sigma, \alpha) \in \mathcal{F}''(T)$ where α has count vector in $v + \mathbb{Z}_{\geq 0}^{I}$. This implies that all elements in I occur very frequently among the entries of α, while all elements in $[k]\setminus I$ occur very infrequently; we call I the *frequent set* of α and of the pair (σ, α). Now let $(\sigma', \alpha') \sim_T (\sigma, \alpha)$. Since, by axiom (3) for model functors, the equality patterns of α' and α agree on $T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$, there exists a set $I' \subseteq [k]$ of the same cardinality as I, and a bijection $g = g((\sigma', \alpha'), (\sigma, \alpha)) : I \to I'$ such that, for $i \in T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$, we have $\alpha(i) = l \in I$ if and only if $\alpha'(i) = g(l)$. In particular, α', too, has a distinguished set $I' \subseteq [k]$ of elements that occur very frequently among its entries, while the complement occurs very infrequently. We call I' the frequent set of α'.

Furthermore, if also $(\sigma'', \alpha'') \sim_T (\sigma, \alpha)$, then we have

$$g((\sigma'', \alpha''), (\sigma', \alpha')) \circ g((\sigma', \alpha'), (\sigma, \alpha)) = g((\sigma'', \alpha''), (\sigma, \alpha)) \tag{1}$$

as a map from I to the frequent set I'' of α'', and we have

$$g((\sigma, \alpha), (\sigma, \alpha)) = \mathrm{id}_I \,. \tag{2}$$

Still using elements from the $\sim_T$-equivalence class of (σ, α), we define a relation $\equiv$ on T as follows: first, $\equiv$ is reflexive, and second, for $i \neq j$ we have $i \equiv j$ if and only if there exists $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $i, j \not\in \operatorname{im}(\sigma')$ and $\alpha'(i) = \alpha'(j) \in I' = g((\alpha', \sigma'), (\alpha, \sigma))I$.

Lemma 5.7.5. *Assume that* $i \equiv j$*. Then* $(\sigma, \alpha) \sim_T \mathcal{F}((i\ j))(\sigma, \alpha)$*, where* $(i\ j)$ *is the transposition of* i *and* j*.*

Proof. If $i = j$, then the statement is obvious. Otherwise, there exists a pair $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ such that α' is defined at i and j and takes the same value l in the frequent set of α'. Then $\mathcal{F}((i\ j))(\sigma', \alpha') = (\sigma', \alpha')$ and by axiom (1), $\mathcal{F}((i\ j))(\sigma, \alpha) \sim_T \mathcal{F}((i\ j))(\sigma', \alpha') = (\sigma', \alpha') \sim_T (\sigma, \alpha)$, as desired. □

Lemma 5.7.6. *The relation* $\equiv$ *is an equivalence relation on* T*.*

Proof. First note that—using axiom (3) for model functors—if $i \neq j$ satisfy $i \equiv j$, then in fact for *all* $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $i, j \in T \setminus \operatorname{im}(\sigma')$ we have $\alpha'(i) = \alpha'(j) \in I'$. Since $\equiv$ is reflexive and symmetric by definition, we only need to show transitivity. For this, assume that $i \equiv j$ and $j \equiv h$, where we may assume that i, j, h are all distinct. Let $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ be such that $\alpha'(j) = \alpha'(h) \in I'$. Now if α' is defined at i, then $i \equiv j$ implies that also $\alpha'(i) = \alpha'(j)$ so that $i \equiv h$. Assume that α' is not defined at i. Then let $i' \in T \setminus \{i, j, h\}$ be a position where α' *is* defined and such that $i' \equiv i$—this exists, because there exists a pair $(\sigma'', \alpha'') \sim_T (\sigma, \alpha)$ for which α'' is defined at i (and defined and equal at j); now set $l := \alpha''(i)$, an element in the frequent set of α'', and take for i' any element from $((\alpha'')^{-1}(l)) \setminus \{i, j, h\}$. Then by Lemma 5.7.5 we have $(\sigma', \alpha') \sim_T \mathcal{F}((i\ i'))(\sigma', \alpha')$ and the latter element *is* defined at i, j, h. This proves transitivity. □

So for all elements (σ', α') in the $\sim_T$-equivalence class of (σ, α) we have the same, well-defined equivalence relation $\equiv$ on T. Let $T_1 \subseteq T$ be the set of elements that form a singleton class; we call T_1 the *core* of (the $\sim_T$-equivalence class of) (σ, α). Note that $T_1 = T_{10} \sqcup T_{11}$ where T_{10} is the set of those positions in T that are in $\operatorname{im}(\sigma')$ for *all* $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ and T_{11} is the set of elements that are in $(\alpha')^{-1}([k] \setminus I')$ for some $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ and $I' = g((\sigma', \alpha'), (\sigma, \alpha))I$. We have $T_{10} \subseteq \operatorname{im}(\sigma)$ and also $T_{11} \subseteq \operatorname{im}(\sigma) \cup \alpha^{-1}([k] \setminus I)$; in particular, $|T_1|$ is bounded from above by $|S_0| + |\alpha^{-1}([k] \setminus I)| = |S_0| + d$, where d was the number used in the construction of $v \in M$.

The same reasoning applies to *any* element $(\sigma, \alpha) \in \mathcal{F}''(T)$: it unambiguously determines a subset $J \subseteq [k]$ (the frequent set of α) of cardinality $|J| = |I|$ and a subset $T_1 \subseteq T$ of some bounded size (the core of the pair), as well as a surjection $\tau : T \setminus T_1 \to J$ defined by $\tau(i) = l$ if and only if there exists a $(\alpha', \sigma') \sim_T (\alpha, \sigma)$ with α' defined at i and $\alpha'(i) = g((\alpha', \sigma'), (\alpha, \sigma))(l)$; and this surjection only has large fibres. Furthermore, passing to another element of the $\sim_T$-equivalence class, the core T_1 remains the same, J is acted upon by a bijection g to yield a J', and τ is composed with that same bijection. We now determine how certain morphisms transform the data J, T_1, τ.

Lemma 5.7.7. *Let* $\pi \in \operatorname{Hom}_{\mathbf{FI}}(S, T)$ *be such that* $\operatorname{im}(\pi)$ *contains the core* $T_1 \subseteq T$ *of (the* $\sim_T$*-equivalence class of)* (σ, α)*, and assume that* $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$ *lies in* $\mathcal{F}''(S)$*. Then the frequent set of* $\tilde{\alpha}$ *equals the frequent set* J *of* α*, the core* $S_1 \subseteq S$ *of* $(\tilde{\sigma}, \tilde{\alpha})$ *equals* $\pi^{-1}(T_1)$*, and the surjection* $\tilde{\tau} : S \setminus S_1 \to J$ *determined by* $(\tilde{\sigma}, \tilde{\alpha})$ *is the map* $\tau \circ (\pi|_{S \setminus S_1})$ *where* $\tau : T \setminus T_1 \to J$ *is the surjection determined by* (σ, α)*.*

Proof. That the frequent set J remains unchanged is immediate: the elements that appear frequently in $\tilde{\alpha}$ also appear frequently in α and vice versa.

For the statement about the core, it suffices to show that distinct $i, j \in S$ satisfy $i \equiv j$ in the equivalence relation on S defined by $(\tilde{\sigma}, \tilde{\alpha})$ if and only if $\pi(i) \equiv \pi(j)$ in the equivalence relation on T defined by (σ, α).

Let $i, j \in S$ be distinct and assume $i \equiv j$, so that there exists a pair $(\tilde{\sigma}', \tilde{\alpha}') \sim_S (\tilde{\sigma}, \tilde{\alpha})$ with $\tilde{\alpha}'(i) = \tilde{\alpha}'(j) \in J$. By axiom (2) there exists a pair $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $\mathcal{F}(\pi)((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$, and we find that $\alpha'(\pi(i)) = \alpha'(\pi(j)) \in J$, so $\pi(i) \equiv \pi(j)$.

Conversely, let $i, j \in S$ be distinct and assume that $\pi(i) \equiv \pi(j)$, so there exists a pair $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $\alpha'(\pi(i)) = \alpha'(\pi(j)) \in J$. Since $\mathrm{im}(\pi)$ contains T_1, it contains all elements of $\mathrm{im}(\sigma') \cap T_1$. Using Lemma 5.7.5 we may apply transpositions $\mathcal{F}((h\ h'))$ to (σ', α') for all $h \in \mathrm{im}(\sigma) \setminus \mathrm{im}(\pi) \subseteq T \setminus T_1$, where the $h' \equiv h$ are all chosen distinct, disjoint from $\mathrm{im}(\sigma)$ and from $\{i, j\}$, and inside $\mathrm{im}(\pi) \setminus T_1$, to arrive at a $(\sigma'', \alpha'') \sim_T (\alpha', \alpha')$ still satisfying $\alpha''(\pi(i)) = \alpha''(\pi(j))$ and now also satisfying $\mathrm{im}(\sigma'') \subseteq \mathrm{im}(\pi)$. So we may apply $\mathcal{F}(\pi)$ to (σ'', α'') and find that $i \equiv j$.

For the statement about $\tilde{\tau}$ let $i \in S \setminus S_1 = S \setminus \pi^{-1}(T_1)$. Then there exists a $(\tilde{\sigma}', \tilde{\alpha}') \sim_S (\tilde{\sigma}, \tilde{\alpha})$ such that $\tilde{\alpha}'$ is defined at i. By axiom (2) there exists a $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ with $\mathcal{F}(\pi)((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$. In particular, α' is defined at $\pi(i)$ and we have $\alpha'(\pi(i)) = \tilde{\alpha}'(i)$, so that $\tilde{\tau}(i) = \tau(\pi(i))$, as desired. □

A special case of the lemma is that where $S = T$, and we find that $\mathrm{Sym}(S)$ acts in the expected manner on the data consisting of the frequent set (namely, trivially) and on the core S_1 and the map τ. The cardinality of the core is an invariant under this action, and also preserved under the more general morphisms of Lemma 5.7.7.

The core is a finite subset of cardinality at most $|S_0|+d$. For each $e \in \{0, \ldots, |S_0|+d\}$ let $\mathcal{F}''_e(S)$ be the set of elements in $\mathcal{F}''(S)$ with a core of cardinality e, and set $\tilde{\mathcal{F}}_e(S) := \tilde{\mathcal{F}}(S) \cap \mathcal{F}''_e(S)$. We are done once we establish that for each e the set of $\mathrm{Sym}(S)$-orbits on $\tilde{\mathcal{F}}_e(S)/{\sim_S}$ is quasipolynomial in $|S|$ for $|S| \gg 0$.

We will decouple the core from the rest of S. A first justification for this is the following lemma.

Lemma 5.7.8. *The number of* $\mathrm{Sym}([e] \sqcup S)$*-orbits on* $A := \tilde{\mathcal{F}}_e([e] \sqcup S)/{\sim_{[e]\sqcup S}}$ *equals the number of* $\mathrm{Sym}([e]) \times \mathrm{Sym}(S)$*-orbits on the set* B *of elements in* $\tilde{\mathcal{F}}_e([e] \sqcup S)/{\sim_{[e]\sqcup S}}$ *with core equal to* $[e]$.

Proof. The inclusion map $B \to A$ induces a map $B/(\mathrm{Sym}([e]) \times \mathrm{Sym}(S)) \to A/\mathrm{Sym}([e] \sqcup S)$. This map is surjective since the core of any element in A can be moved into $[e]$ by an element of $\mathrm{Sym}([e] \sqcup S)$. To see that it is also injective, note that if $\pi \in \mathrm{Sym}([e] \sqcup S)$ and $(\sigma, \alpha), (\sigma', \alpha') \in B$ satisfy $\mathcal{F}(\pi)((\sigma, \alpha)) = (\sigma', \alpha')$, then π must preserve the common core $[e]$ of both tuples, hence $\pi \in \mathrm{Sym}([e]) \times \mathrm{Sym}(S)$. □

Consider a pair $(\sigma, \alpha) \in \mathcal{F}''_e([e] \sqcup T)$ with core equal to $[e]$. To such a pair we associate the quintuple $(J, \tau, \sigma_0, \sigma_1, \overline{\alpha})$ determined by:

1. the frequent set J of α;
2. the surjection $\tau : T \to J$;
3. the partially defined map $\sigma_0 : S_0 \to [e]$, which is the restriction of σ to $\sigma^{-1}([e])$;
4. the map $\sigma_1 : S_0 \setminus \mathrm{dom}(\sigma_0) \to J$ defined by $\sigma_1(i) = \tau(\sigma(i))$.
5. the restriction $\overline{\alpha}$ of α to $[e] \setminus \mathrm{im}(\sigma)$, which takes values in $[k] \setminus J$.

The quintuple remembers everything about the pair (σ, α) *except* the exact values (in T) of σ on $S_0 \setminus \mathrm{dom}(\sigma_0)$: of these, only their $\equiv$-classes are remembered; these can be read off from σ_1 (and τ). If another pair $(\sigma', \alpha') \in \mathcal{F}''_e([e] \sqcup T)$ with core equal to $[e]$ yields the same quintuple, then (σ', α') differs from (σ, α) by a permutation of T that permutes elements within their $\equiv$-equivalence classes. Hence, by repeatedly applying Lemma5.7.5, we find that $(\sigma', \alpha') \sim_{[e]\sqcup T} (\sigma, \alpha)$.

We will use the notation $\sim_{[e]\sqcup T}$ for the induced equivalence relation on such quintuples. Note that $\mathrm{Sym}([e])$ fixes (J and) τ, whereas $\mathrm{Sym}(T)$ fixes (J and) $\sigma_0, \sigma_1, \overline{\alpha}$. Also note that all components of the quintuple except τ can take only finitely many different values as T varies. We call $(J, \sigma_0, \sigma_1, \overline{\alpha})$ the quadruple determined by the quintuple (and *a fortiori* also determined by the pair (σ, α)).

We now come to the central tool for proving quasipolynomiality; note that now we work with $\tilde{\mathcal{F}}$ instead of $\mathcal{F}''$—recall that $\tilde{\mathcal{F}}$ and its complement $\mathcal{F}'$ were defined using $\tilde{v} = v + v_1 = v_0 + 2v_1$, while $\mathcal{F}'' \supseteq \tilde{\mathcal{F}}$ was defined using v.

Definition 5.7.9. Let G be the finite directed graphs whose vertices are all quadruples of pairs in $\tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$, where T runs through **FI**, and whose arrows from one quadruple $(J, \sigma_0, \sigma_1, \overline{\alpha})$ to $(J', \sigma'_0, \sigma'_1, \overline{\alpha'})$ are all bijections $g((\sigma', \alpha'), (\sigma, \alpha)) : J \to J'$ coming from pairs $(\sigma', \alpha') \sim_{[e]\sqcup T} (\sigma, \alpha)$ in $\tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$ and with the prescribed quadruples. ♢

Typically, the same bijection g will arise from more than one pair of pairs with the prescribed quadruples; it then only appears once as an arrow g between those quadruples. The following proposition will be used below to establish that G is, in fact, a groupoid.

Proposition 5.7.10. *Let $(J, \tau, \sigma_0, \sigma_1, \overline{\alpha})$ be the quintuple of a pair $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$ and let g be an arrow in G from $(J, \sigma_0, \sigma_1, \overline{\alpha})$ to $(J', \sigma'_0, \sigma'_1, \overline{\alpha'})$. Then $(J', g \circ \tau, \sigma'_0, \sigma'_1, \overline{\alpha'})$ is also the quintuple of some pair in $\tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$, and that quintuple is $\sim_{[e]\sqcup T}$-equivalent to the original quintuple. Furthermore, all quintuples equivalent to the original quintuple arise in this manner.*

Proof. The last statement is immediate: such an equivalent quintuple comes from an equivalent pair $(\sigma', \alpha') \sim_{[e]\sqcup T} (\sigma, \alpha)$, and for the arrow we can take $g = g((\sigma', \alpha'), (\sigma, \alpha))$.

For the first statement, let $(\tilde{\sigma}, \tilde{\alpha}) \sim_{[e]\sqcup S} (\tilde{\sigma}', \tilde{\alpha}')$ be pairs with the given quadruples such that $g = g((\tilde{\sigma}', \tilde{\alpha}'), (\tilde{\sigma}, \tilde{\alpha}))$.

We will replace these equivalent pairs by smaller pairs, the first of which we can relate to (σ, α) so as to apply axiom (2). The details are as follows.

For each $l \in J$ let m_l be the minimum of $|\alpha^{-1}(l)|$ and $|\tilde{\alpha}^{-1}(l)|$ and set $n_l := |\sigma_1^{-1}(l)|$. Define an injection π from $[e] \sqcup \bigsqcup_{l \in J}([m_l] \sqcup [n_l])$ to $[e] \sqcup S$ that is the identity on $[e]$, sends each $[m_l]$ injectively to m_l elements in $\alpha^{-1}(l) \subseteq S$ and sends each $[n_l]$ bijectively to the elements in $\mathrm{im}(\tilde{\sigma}) \cap S$ where τ takes the value l. This construction ensures that $\mathcal{F}(\pi)$ is defined at $(\tilde{\sigma}, \tilde{\alpha})$. It might *a priori* not be defined at $(\tilde{\sigma}', \tilde{\alpha}')$, because $\mathrm{im}(\tilde{\sigma}') \cap S$ might not be contained in $\mathrm{im}(\pi)$. But if it is not, then using Lemma 5.7.5 we can replace $(\tilde{\sigma}', \tilde{\alpha}')$ by a $\sim_S$-equivalent pair, with the same quadruple and with the same bijection $g : J \to J'$, such that $\mathcal{F}(\pi)$ *is* defined at $(\tilde{\sigma}', \tilde{\alpha}')$. Now replace $(\tilde{\sigma}, \tilde{\alpha})$ and $(\tilde{\sigma}', \tilde{\alpha}')$ by their images under $\mathcal{F}(\pi)$.

The point of distinguishing $\tilde{\mathcal{F}}$ and $\mathcal{F}''$ is that these images may not be in $\tilde{\mathcal{F}}$. The reason is that while $\tilde{v} + \mathbb{Z}^I_{\geq 0}$ is closed under componentwise minimum, the count vectors of pairs in $\tilde{\mathcal{F}}$ with a fixed frequent set J may not quite be closed under componentwise minimum: the number of entries that α has in the frequent set is not quite invariant under the equivalence relations $\sim$, although it is up to a bounded difference. This is remedied by allowing the componentwise minimum to have a count vector in the bigger set $v + \mathbb{Z}^I_{\geq 0}$.

Hence the new $(\tilde{\sigma}, \tilde{\alpha})$ and $(\tilde{\sigma}', \tilde{\alpha}')$ are in $\tilde{\mathcal{F}}''([e] \sqcup \bigsqcup_{l \in J}([m_l] \sqcup [n_l]))$ and are related by the same bijection g. Now construct similarly an injection $\pi' : [e] \sqcup \bigsqcup_{l \in J}([m_l] \sqcup [n_l]) \to [e] \sqcup T$ which is the identity on $[e]$, sends $[m_l]$ injectively into the set $\alpha^{-1}(l)$, and sends $[n_l]$ bijectively to the set of elements in $\mathrm{im}(\sigma) \cap T$ where τ takes the value l. Then $(\tilde{\sigma}, \tilde{\alpha}) = \mathcal{F}(\pi')((\sigma, \alpha))$. Now apply axiom (2) to find that there exists a pair $(\sigma', \alpha') \sim_{[e]\sqcup T} (\sigma, \alpha)$ such that $\mathcal{F}(\pi')((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$. The pair (σ', α') has the required quintuple. Moreover, since the pair is $\sim_{[e]\sqcup T}$-equivalent to (σ, α), so are their quintuples. □

Proposition 5.7.11. *The finite graph G is a groupoid with objects its vertices (quadruples), arrows as in Definition 5.7.9, and composition maps given by composing the bijections g.*

Proof. That G has identity arrows follows from (2), and that all arrows are invertible follows from (1) combined with (2). It remains to check that composition is well-defined. So let g_1 be an arrow in G from a vertex $q := (J, \sigma_0, \sigma_1, \overline{\alpha})$ to a vertex $q' := (J', \sigma'_0, \sigma'_1, \overline{\alpha'})$, and let g_2 be an arrow in G from $q'' := (J'', \sigma''_0, \sigma''_1, \overline{\alpha''})$ to q. We need to show that $g_1 \circ g_2$ is an arrow in G from q'' to q'. Now the existence of the arrow g_2 means that there are a finite set T and pairs $(\sigma, \alpha), (\sigma'', \alpha'') \in \tilde{F}_e([e] \sqcup T)$ with core $[e]$ and quadruples q, q'', respectively, such that $g_2 = g((\sigma, \alpha), (\sigma'', \alpha''))$.

Let $(J, \tau, \sigma_0, \sigma_1, \overline{\alpha})$ be the quintuple of (σ, α). By Proposition 5.7.10, $(J', g_1 \circ \tau, \sigma'_0, \sigma'_1, \overline{\alpha'})$ is the quintuple of another pair $(\sigma', \alpha') \in \tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$ and $(\sigma', \alpha') \simeq_{[e]\sqcup T} (\sigma, \alpha)$. Then $g_1 = g((\sigma', \alpha'), (\sigma, \alpha))$, and $g_1 \circ g_2 = g((\sigma', \alpha'), (\sigma'', \alpha''))$ is an arrow from q'' to q', as desired. □

Consider the class of quintuples arising from pairs $(\sigma', \alpha') \in \tilde{\mathcal{F}}([e] \sqcup T), T \in \mathbf{FI}$ with core $[e]$. This class of quintuples comes with a natural anchor map to the objects of G, namely, the map that forgets τ. The following says that equivalence classes of quintuples are precisely orbits under G.

Corollary 5.7.12. *The groupoid G acts on the set of quintuples of pairs in $\tilde{\mathcal{F}}([e] \sqcup T)$ with core $[e]$, for T varying through* $\mathbf{FI}$*, via the anchor map that sends a quintuple to its corresponding quadruple. The orbits of this action are precisely the $\sim_{[e]\sqcup T}$-equivalence classes. The action of G commutes with the action of* $\mathrm{Sym}(T)$*.*

Proof. The action of an arrow $g : (J, \sigma_0, \sigma_1, \overline{\alpha}) \to (J', \sigma'_0, \sigma'_1, \overline{\alpha'})$ on a quintuple $(J, \tau, \sigma_0, \sigma_1, \overline{\alpha})$ yields the quintuple $(J', g \circ \tau, \sigma'_0, \sigma'_1, \overline{\alpha'})$, and the axioms for a groupoid action are readily verified. The action commutes with $\pi \in \mathrm{Sym}(T)$ because $(g \circ \tau) \circ \pi = g \circ (\tau \circ \pi)$. That the G-orbits on quintuples coming from pairs $(\sigma, \alpha) \in \tilde{\mathcal{F}}([e] \sqcup T)$ with core $[e]$ are precisely the $\sim_{[e]\sqcup T}$-equivalence classes follows from the last statement of Proposition 5.7.10. □

Recall that, by Lemma 5.7.8, we need to count the elements of $\tilde{\mathcal{F}}([e] \sqcup T)$ with core $[e]$ up to $\sim_{[e]\sqcup T}$ as well as up to the action of $\mathrm{Sym}([e]) \times \mathrm{Sym}(T)$. Modding out the action of $\mathrm{Sym}(T)$ is now straightforward: it just consists of replacing $\tau : T \to J$ by its count vector u in $\mathbb{Z}^J_{\geq 0}$. Thus now the groupoid G acts on quintuples

$(J, u, \sigma_0, \sigma_1, \overline{\alpha})$ coming from pairs in $\tilde{\mathcal{F}}_e([e] \sqcup T)$ with core $[e]$, where u is a vector in $\mathbb{Z}_{\geq 0}^J$. Also the group $\mathrm{Sym}([e])$ acts on such quintuples, indeed, it acts on the corresponding quadruples and fixes u. We need to count the quintuples up to the action of G and $\mathrm{Sym}([e])$. To do so, we first observe that $\mathrm{Sym}([e])$ acts by automorphisms of G on the objects of G: if there is an arrow $g : q \to q'$, then there is also an arrow $\pi(q) \to \pi(q')$ with the same label g, for each $\pi \in \mathrm{Sym}([e])$. To stress that this arrow has a different source and target, we write $\pi(g : q \to q')\pi^{-1}$ for that arrow $\pi(q) \to \pi(q')$.

We now combine these actions into that of a larger groupoid $\tilde{G}$ with the same ground set as G and with arrows $q \to q''$ all pairs $(\pi, g : q \to q')$ where g is an arrow from the quadruple q to some quadruple q' and $\pi \in \mathrm{Sym}([e])$ maps q' to q''. The composition $(\pi', g') \circ (\pi, g)$, where $g' : q'' \to q'''$ and $\pi'(q''') = q''''$, is defined as $(\pi' \circ \pi, \pi^{-1}(g' : q'' \to q''')\pi \circ g)$. It is straightforward to see that $\tilde{G}$ is, indeed, a finite groupoid acting on quintuples. Now we are left to count orbits of $\tilde{G}$ on quintuples.

Proposition 5.7.13. *There exists a quasipolynomial f such that, for $n \gg 0$, the number of orbits of $\tilde{G}$ on quintuples $(J, u \in \mathbb{Z}_{\geq 0}^J, \sigma_0, \sigma_1, \overline{\alpha})$ arising from pairs $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup [n])$ with core $[e]$ equals $f(n)$.*

Proof. Applying the orbit-counting lemma for groupoids (Lemma 5.5.1), it suffices to show that for each object of $\tilde{G}$, which is a quadruple $q = (J, \sigma_0, \sigma_1, \overline{\alpha})$, and for each arrow $(\pi, g) : q \to q$ in $\tilde{G}$, the number of fixed points of (π, g) on quintuples with the given quadruple q is quasipolynomial for $n \gg 0$. Such a quintuple is fixed if and only if the count vector $u \in \mathbb{Z}_{\geq 0}^J$ is fixed by g—indeed, π acts trivially on the count vector.

Now we figure out the structure of the set of count vectors arising from such quintuples. First, if u is the count vector of a quintuple (with the fixed quadruple q) arising from $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup [n])$, and if $l \in J$ and $i \in [n]$ such that $\alpha(i) = l$, then $u - e_l$ is the count vector of the quintuple arising from the pair $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$, where $\pi : [e] \sqcup [n-1] \to [e] \sqcup [n]$ is the identity on $[e]$ and increasing from $[n-1] \to [n]$ and does not hit i. This suggest that the set of count vectors that we are considering is downward closed. However, it may be that $(\tilde{\sigma}, \tilde{\alpha})$ is in $\mathcal{F}''([e] \sqcup [n-1]) \setminus \tilde{\mathcal{F}}([e] \sqcup [n-1])$.

On the other hand, if we had started with $(\sigma, \alpha) \in \mathcal{F}''([e] \sqcup [n]) \setminus \tilde{\mathcal{F}}([e] \sqcup [n])$ and applied such an element $\mathcal{F}(\pi)$ to it, the result would not have been an element of $\tilde{\mathcal{F}}([e] \sqcup [n-1])$.

This shows that the set of relevant count vectors is the difference $N \setminus N'$, where N and N' are downward closed sets in $\mathbb{Z}_{\geq 0}^J$. Hence, using the Stanley decomposition as in the proof of Proposition 5.3.1, the fixed points u that we are counting are the lattice points in a finite disjoint union of rational cones, each given as the intersection of a linear space (the eigenspace of g in $\mathbb{R}^J$ with eigenvalue 1) and a finite union of sets of the form $u + \mathbb{Z}_{\geq 0}^{J'}$ with $J' \subseteq J$. The number of such u is a quasipolynomial in $n = \sum_l u(l)$ for $n \gg 0$ for each of the finitely many choices of (π, g), hence so is their sum. □

This completes the proof of Theorem 5.7.1.

□

5.8. **Pre-component functors.** We will see that, if X is a width-one **FI**-scheme of finite type over a Noetherian ring K, then we can cover $X(S)$ by means of closed, irreducible subsets parameterised by a finite number of model functors evaluated at S. Then, of course, the irreducible components of $X(S)$ are among these. But to ensure that we are neither double-counting components of $X(S)$ nor counting closed subsets that are strictly contained in components, we need to keep track of the inclusions among these closed subsets. At the combinatorial level, this is done using compatible quasi-orders.

Definition 5.8.1. Let $a \in \mathbb{Z}_{\geq 0}$, for each $b \in [a]$ let $k_b \in \mathbb{Z}_{\geq 0}$ and let $S \mapsto \mathcal{F}_b(S)/\sim_{b,S}$ be a model functor, where $\mathcal{F}_b(S)$ is the set of pairs (σ, α) with $\sigma : S_{0,b} \to S$ and $\alpha \in \mathcal{E}_b(S) \subseteq [k_b]^S$.

Suppose that we are given, for each $S \in \mathbf{FI}$, a quasi-order $\preceq_S$ (i.e., a reflexive and transitive relation) on the disjoint union

$$\mathcal{F}(S) := \bigsqcup_{b=1}^{a} \mathcal{F}_b(S).$$

This collection of quasi-orders is called *compatible* if

1. whenever $\mathcal{F}_b(S) \ni (\sigma, \alpha) \preceq_S (\sigma', \alpha') \in \mathcal{F}_{b'}(S)$, we have $b \leq b'$;
2. the restriction of $\preceq_S$ to each $\mathcal{F}_b(S)$ equals the equivalence relation $\sim_{b,S}$; and
3. for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ and $(\sigma, \alpha) \in \mathcal{F}_b(T)$ and $(\sigma', \alpha') \in \mathcal{F}_{b'}(T)$ with $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma) \cup \mathrm{im}(\sigma')$ we have
$$(\sigma, \alpha) \preceq_T (\sigma', \alpha') \Rightarrow \mathcal{F}_b(\pi)((\sigma, \alpha)) \preceq_S \mathcal{F}_{b'}(\pi)((\sigma', \alpha')).$$

◊

Note that the condition that $\mathrm{im}(\pi)$ contains both $\mathrm{im}(\sigma)$ and $\mathrm{im}(\sigma')$ implies that $\mathcal{F}_b(\pi)((\sigma, \alpha))$ and $\mathcal{F}_{b'}(\pi)((\sigma', \alpha'))$ are both defined.

Definition 5.8.2. In the setting of the previous definition, we introduce an equivalence relation $\sim_S$ on $\mathcal{F}(S)$ by $(\sigma, \alpha) \sim_S (\sigma', \alpha')$ if both $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$ and $(\sigma', \alpha') \preceq_S (\sigma, \alpha)$. Note that, by the first and second axioms for pre-component functors, $(\sigma, \alpha) \in \mathcal{F}_b(S)$ is $\sim_S$-equivalent to $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$ if and only if $b = b'$ and $(\sigma, \alpha) \sim_{b,S} (\sigma', \alpha')$.

The quasi-order $\preceq_S$ induces a partial order on the set

$$\mathcal{F}(S)/\sim_S = \bigsqcup_b (\mathcal{F}_b(S)/\sim_{b,S})$$

of equivalence classes. We define the functor $\mathcal{C} : \mathbf{FI} \to \mathbf{PF}$ on objects by

$$\mathcal{C}(S) := \{\text{the maximal elements of } \mathcal{F}(S)/\sim_S\}$$

and on a morphism $\pi : S \to T$ as follows. Let $c \in \mathcal{C}(T)$ be a maximal equivalence class. If c contains some element $(\sigma, \alpha) \in \mathcal{F}_b(T)$ at which $\mathcal{F}_b(\pi)$ is defined (i.e., with $\mathrm{im}(\sigma) \subseteq \mathrm{im}(\pi)$), and if, moreover, $c' := [\mathcal{F}_b(\pi)((\sigma, \alpha))]_{\sim_S}$ is also maximal in $\mathcal{F}(S)/\sim_S$, then we set $\mathcal{C}(\pi)(c) := c' \in \mathcal{C}(S)$. By the axioms for compatible quasi-orders, this is independent of the choice of the representative (σ, α) of c—subject to the requirement that $\mathcal{F}_i(\pi)$ be defined at that representative.

A functor $\mathcal{C} : \mathbf{FI} \to \mathbf{PF}$ obtained in this manner is called a *pre-component functor.* ◊

5.9. **The final quasipolynomial count.** We retain the notation from §5.8: for each $b \in [a]$ we have a model functor $S \mapsto \mathcal{F}_b(S)/{\sim_{b,S}}$, and on the disjoint unions $\mathcal{F}(S) := \bigsqcup_b \mathcal{F}_b(S)$ (for $S \in \mathbf{FI}$) we have compatible quasi-orders $\preceq_S$.

Theorem 5.9.1. *Let $\mathcal{C}$ be the pre-component functor corresponding to the data above. Then the number of* $\mathrm{Sym}(S)$*-orbits on $\mathcal{C}(S)$ is a quasipolynomial in $|S|$ for all S with $|S| \gg 0$.*

The proof requires the proof technique used in §5.7, and takes up the rest of this subsection.

Proof. By Theorem 5.7.1, we know that the number of $\mathrm{Sym}(S)$-orbits on the disjoint union of model functors $\bigsqcup_b(\mathcal{F}_b(S)/{\sim_{b,S}})$ is eventually quasipolynomial in $|S|$. From this disjoint union we will remove, for each $b \in [a]$, the $\sim_{b,S}$-classes of pairs (σ, α) for which there exists a $b' > b$ and a $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$ with $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$. We will see that, for a fixed $b \in [a]$, the $\mathrm{Sym}(S)$-orbits on these deletions are also counted by a quasipolynomial.

To this end, fix $b \in [a]$ and let $M \subseteq \mathbb{Z}_{\geq 0}^{k_b}$ be the downward closed subset corresponding to $\mathcal{E}_b$. From §5.7 we recall that, to prove that the number of $\mathrm{Sym}(S)$-orbits on $\mathcal{F}_b(S)/\sim_{b,S}$ is quasipolynomial in $|S|$ for $|S| \gg 0$, we performed induction on M using Dickson's lemma. We do the same here.

Lemma 5.9.2. *Fix $b \in [a]$. For $|S| \gg 0$, the number of* $\mathrm{Sym}(S)$*-orbits on pairs $(\sigma, \alpha) \in \mathcal{F}_b(S)$ that are $\preceq$ some pair $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$ for some $b' > b$ is quasipolynomial in $|S|$.*

Proof. We construct the sub-functor $\mathcal{E}'_b \subseteq \mathcal{E}_b$ and the corresponding sub-functor $\mathcal{F}'_b \subseteq \mathcal{F}_b$ as in §5.7, as well as the difference $\tilde{\mathcal{F}}_b(S) := \mathcal{F}_b(S) \setminus \mathcal{F}'_b(S)$. By induction on M using Dickson's lemma, we may assume that the lemma holds for the sub-model functor $S \mapsto \mathcal{F}'_b(S)/\sim_{b,S}$. On the other hand, in §5.7 we counted the $\mathrm{Sym}(S)$-orbits on $\tilde{\mathcal{F}}_b(S)/{\sim_{b,S}}$, for all S, as a finite sum of orbit counts of certain finite groupoids. More precisely, for finitely many values of a nonnegative integer e, we there considered the pairs $(\alpha, \sigma) \in \tilde{\mathcal{F}}_b([e] \sqcup S)$ with core equal to $[e]$. Each of these pairs gives rise to a quintuple $(J, u \in \mathbb{Z}_{\geq 0}^J, \sigma_0, \sigma_1, \overline{\alpha})$ where $J \subseteq [k_b]$ is the frequent set of α, and we showed that the $\mathrm{Sym}(S)$-orbits on $\sim_{b,[e]\sqcup S}$-equivalence classes of such pairs (α, σ) are in bijection with the orbits of a certain groupoid on the sum-$|S|$ level set of a difference $N \setminus N'$ where $N, N' \subseteq \mathbb{Z}_{\geq 0}^{k_b}$ are downward closed; this difference is where the count vector $u \in \mathbb{Z}_{\geq 0}^J \subseteq \mathbb{Z}_{\geq 0}^{k_b}$ lives.

Now suppose that $(\sigma, \alpha) \preceq_{[e]\sqcup S} (\sigma', \alpha')$ for some $(\sigma', \alpha') \in \tilde{\mathcal{F}}_{b'}([e] \sqcup S)$ with $b' > b$, and let $i \in S$ be such that $l := \alpha(i) \in J$. If i happens to be in $\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma')$, then choose $j \in S \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$ such that $\alpha(i) = \alpha(j) = l$—this can be done since l appears frequently in α (although, to be precise, when choosing the vector $\tilde{v}$ in the construction of $\tilde{F}_b$, we now have to make sure that its large entries values are large even compared to the finitely many numbers $k_{b'}, b' \in [a]$).

By Lemma 5.7.5, $(\sigma, \alpha) \sim_{b,[e]\sqcup S} (\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}_b((i\ j))((\sigma, \alpha))$, and the axioms for compatible quasi-orders imply that $(\tilde{\sigma}, \tilde{\alpha}) \preceq_{[e]\sqcup S} (\tilde{\sigma}', \tilde{\alpha}') := \mathcal{F}_{b'}((i\ j))((\sigma', \alpha'))$. Moreover, we have achieved that $i \in S \setminus (\mathrm{im}(\tilde{\sigma}) \cup \mathrm{im}(\tilde{\sigma}'))$.

Now let $\pi : [e] \sqcup S \setminus \{i\} \to [e] \sqcup S$ be the inclusion map. Then $\mathcal{F}_b(\pi)$ and $\mathcal{F}_{b'}(\pi)$ are defined at $(\tilde{\sigma}, \tilde{\alpha})$ and $(\tilde{\sigma}', \tilde{\alpha}')$, respectively, and we have

$$\mathcal{F}_b(\pi)((\tilde{\sigma}, \tilde{\alpha})) \preceq_{[e]\sqcup S\setminus\{i\}} \mathcal{F}_{b'}(\pi)((\tilde{\sigma}', \tilde{\alpha}')).$$

The count vector in $\mathbb{Z}_{\geq 0}^J$ of the left-hand side is just $u - e_l$.

We conclude that the set of count vectors of quintuples corresponding to pairs in $\tilde{\mathcal{F}}_{b,e}([e] \sqcup S)$ with core $[e]$ whose $\sim_{[e]\sqcup S}$-equivalence class is not maximal in $\tilde{F}([e] \sqcup S)/\sim_{[e]\sqcup S}$ is downward-closed, or more precisely the intersection of a downward-closed set with the earlier difference $N \setminus N'$ of downward-closed sets, hence again a difference of downward-closed subsets of $\mathbb{Z}_{\geq 0}^{k_b}$. The same orbit-counting argument with groupoids as in §5.7 applies, and shows that the number of $\mathrm{Sym}(S)$-orbits on pairs in $\tilde{\mathcal{F}}(S)$ that are not maximal in $\preceq$ is a quasipolynomial. □

This concludes the proof of Theorem 5.9.1.

□

5.10. **Component functors "are" pre-component functors.** We are now ready to establish our main result on component functors of width-one $\mathbf{FI}^{\mathbf{op}}$-schemes.

Theorem 5.10.1. *Let X be a width-one affine $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over a Noetherian ring K. Then there exists a pre-component functor $\mathcal{C}$ and a morphism $\varphi : \mathcal{C} \to \mathcal{C}_X$ such that φ is an isomorphism at the level of species.*

In other words, for each $S \in \mathbf{FI}$, $\varphi(S)$ is a bijection $\mathcal{C}(S) \to \mathcal{C}_X(S)$ such that for all $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the diagram

$$\begin{array}{ccc} \mathcal{C}(T) & \xrightarrow{\varphi(T)} & \mathcal{C}_X(T) \\ {\scriptstyle \mathcal{C}(\pi)} \downarrow & & \downarrow {\scriptstyle \mathcal{C}_X(\pi)} \\ \mathcal{C}(S) & \xrightarrow[\varphi(S)]{} & \mathcal{C}_X(S) \end{array}$$

commutes in the following sense: if $\mathcal{C}(\pi)$ is defined at c, $\mathcal{C}_X(\pi)$ is defined at $\varphi(T)(c)$, and the identity $\varphi(S)(\mathcal{C}(\pi)(c)) = \mathcal{C}_X(\pi)(\varphi(T)(c))$ holds.

The proof of this theorem takes up the rest of this subsection.

Proof. Let $X = \mathrm{Spec}(B)$ be a width-one $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over a Noetherian ring K. Without loss of generality, we may and will assume that X is reduced and nice. By the Shift Theorem and Proposition 3.3.1 there exist $S_0 \in \mathbf{FI}$ and $h \in B(S_0)$ such that $X' := \mathrm{Sh}_{S_0} X[1/h] = \mathrm{Spec}(B')$ is of product type in the sense of Definition 3.3.2. In particular, $B'_0 = B(S_0)[1/h]$ is a domain, X' is isomorphic to $S \mapsto Z^S$ where $Z := Z'([1])$, and each irreducible component of Z^S maps dominantly into $\mathrm{Spec}(B'_0)$.

For an $S \in \mathbf{FI}$, let $Z(S)$ be the open subset of $X(S)$ defined by

$$Z(S) := \{p \in X(S) \mid \exists \sigma : S_0 \to S : (\sigma h)(p) \neq 0\}.$$

Let $Y(S) := X(S) \setminus Z(S)$. Note that $Y \subsetneq X$ is a proper closed $\mathbf{FI}^{\mathbf{op}}$-subscheme of $X(S)$, but $Z(S)$ is not (quite) functorial in S: for $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ and $p \in Z(T)$ it might happen that $X(\pi)(p)$ lies in $Y(S)$ rather than in $Z(S)$.

However, if $\sigma \in \mathrm{Hom}_{\mathbf{FI}}(S_0,T)$ and $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ satisfy $\mathrm{im}(\sigma) \subseteq \mathrm{im}(\pi)$, then $X(\pi)$ maps the points of $Z(T)$ where $\sigma(h)$ is nonzero to points of $Z(S)$ where $(\pi^{-1}\sigma)(h)$ is nonzero. Consequently, in spite of the fact that Z is not an $\mathbf{FI}^{\mathbf{op}}$-scheme, as in Definition 4.2.1 we associate to Z the component functor $\mathcal{C}_Z : \mathbf{FI} \to \mathbf{PF}$ that assigns to S the set of irreducible components of $Z(S)$ and to $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S,T)$ the partially defined map $\mathcal{C}_Z(\pi) : \mathcal{C}_Z(T) \to \mathcal{C}_Z(S)$ that at a component $c \in \mathcal{C}_Z(T)$ is defined and takes the value $c' \in \mathcal{C}_Z(S)$ if and only if $X(\pi)$ maps c dominantly into c'.

Now $\mathcal{C}_X(S)$ is the union of $\mathcal{C}_Z(S)$ and the components in $\mathcal{C}_Y(S)$ that are not contained in the closure in $X(S)$ of any component of $Z(S)$.

By Noetherianity (Theorem 2.7.1), we may assume that there exists a morphism from a pre-component functor $\mathcal{C}_1$ to $\mathcal{C}_Y$ that is an isomorphism at the level of species. We now construct a model functor $\mathcal{C}_2$ and a morphism $\mathcal{C}_2 \to \mathcal{C}_Z$ that is also an isomorphism at the level of species.

Let L be the fraction field of B'_0 and let X'_L be the base change of X' to L. Since every irreducible component of $X'(S)$ maps dominantly into $\operatorname{Spec}(B'_0)$, the morphism $\mathcal{C}_{X'_L} \to \mathcal{C}_{X'}$ is an isomorphism. So for each $S \in \mathbf{FI}$, the irreducible components of $X'_L(S)$ are in bijection with the irreducible components of $X(S_0 \sqcup S)[1/h] \subseteq Z(S_0 \sqcup S)$. Consequently, for each injection $\sigma : S_0 \to S$ we now have an injective map

$$\{\text{irreducible components of } X'_L(S \setminus \operatorname{im}(\sigma))\} \to \mathcal{C}_Z(S), \quad c \mapsto X(\tilde{\sigma})(c) \tag{3}$$

where $\tilde{\sigma} : S \to S_0 \sqcup (S \setminus \operatorname{im}(\sigma))$ is the bijection that equals σ^{-1} on $\operatorname{im}(\sigma)$ and the identity on $S \setminus \operatorname{im}(\sigma)$. The image of this injective map consists precisely of the irreducible components of $Z(S)$ on which $\sigma(h)$ is not identically zero. As we vary σ, we thus obtain all irreducible components of $Z(S)$, indeed typically multiple times.

Let $\overline{L}$ be a separable closure of L, and let $X'_{\overline{L}}$ be the base change of X' to $\overline{L}$. Now we have a surjective morphism $\mathcal{C}_{X'_{\overline{L}}} \to \mathcal{C}_{X'_L}$ whose fibres are Galois orbits. More precisely, let $C_1, \ldots, C_k$ be the irreducible components of $X'_{\overline{L}}([1]) = Z_{\overline{L}}$, and let G be the image of the Galois group $\operatorname{Gal}(\overline{L}/L)$ in $\operatorname{Sym}([k])$ through its action on the components $C_1, \ldots, C_k$. Then, as we have seen in §5.4, $\mathcal{C}_{X'_{\overline{L}}}$ is isomorphic to the functor $S \mapsto \mathcal{E}(S) := [k]^S$. Furthermore, $\mathcal{C}_{X'} \cong \mathcal{C}_{X'_L}$ is isomorphic to the elementary model functor $S \mapsto \mathcal{E}(S)/G$.

Now we construct the functor $\mathcal{F} : \mathbf{FI} \to \mathbf{PF}$ by

$$\mathcal{F}(S) := \{(\sigma, \alpha) \mid \sigma : S_0 \to S, \alpha \in \mathcal{E}(S \setminus \operatorname{im}(\sigma))\}.$$

Then, by the above, we have a surjective morphism

$$\Psi : \mathcal{F} \to \mathcal{C}_Z;$$

concretely, $\Psi(S)$ takes $(\sigma, \alpha) \in \mathcal{F}(S)$, computes the component of $X'_{\overline{L}}(S \setminus \operatorname{im}(\sigma))$ corresponding to α, its image in $C_{X'_L}(S \setminus \operatorname{im}(\sigma))$ (modding out the Galois group), and then applies the map (3). To simplify notation, we will write $\Psi((\sigma, \alpha))$ instead of $\Psi(S)((\sigma, \alpha))$.

This surjection is by no means a bijection: even ignoring the Galois groups for a moment, on the left we have pairs of a component in $Z(S)$ and a specified $\sigma : S_0 \to S$ such that $\sigma(h)$ is not identically zero on that component; and on the right we just have components of $Z(S)$. A single component of $Z(S)$ may admit many different such maps σ. We therefore introduce an equivalence relation $\sim_S$ on $\mathcal{F}(S)$ by $(\sigma, \alpha) \sim_S (\sigma', \alpha') :\Leftrightarrow \Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$. In text we will sometimes opress the S and say that (σ, α) is *equivalent* to (σ', α').

We will prove that the equivalence relation $\sim_S$ satisfies the axioms in the definition of a model functor (§5.6).

Lemma 5.10.2. *Suppose that* $\mathcal{F}(T) \ni (\sigma, \alpha) \sim_T (\sigma', \alpha') \in \mathcal{F}(T)$ *and let* $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ *with* $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma) \cup \mathrm{im}(\sigma')$. *Then*

$$\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$$

holds.

Proof. The assumptions assert that $\Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$ and that $\mathcal{C}_Z(\pi)$ is defined at this component of $Z(T)$. By construction, we have

$$\Psi(\mathcal{F}(\pi)((\sigma, \alpha))) = \mathcal{C}_Z(\pi)(\Psi((\sigma, \alpha))) = \mathcal{C}_Z(\pi)(\Psi((\sigma', \alpha'))) = \Psi(\mathcal{F}(\pi)((\sigma', \alpha'))),$$

so that $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$, as desired. □

This lemma establishes axiom (1) for the equivalence relations $\sim_S$. We continue with axiom (2).

Lemma 5.10.3. *Let* $(\sigma, \alpha) \in \mathcal{F}(T)$ *and let* $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ *satisfy* $\mathrm{im}(\pi) \supseteq \mathrm{im}(\sigma)$. *Assume that* $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'') \in \mathcal{F}(S)$. *Set* $\sigma' := \pi \circ \sigma''$. *Then there exists an* $\alpha' \in \mathcal{E}(T)$ *such that* (σ', α') *lies in* $\mathcal{F}(T)$, *is* $\sim_T$*-equivalent to* (σ, α), *and satisfies* $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$.

Proof. Let $C := \Psi((\sigma, \alpha))$ be the corresponding component of $Z(T)$ and D its image in $Z(S)$ under $X(\pi)$. The fact that $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'') \in \mathcal{F}(S)$ implies that $\sigma''(h)$ is not identically zero on D. Then $\sigma'(h) = \pi(\sigma''(h))$ is not identically zero on C, and hence $C = \Psi((\sigma', \tilde{\alpha}'))$ for a suitable $\tilde{\alpha}' \in \mathcal{E}(T)$. Then $\mathcal{F}(\pi)((\sigma', \tilde{\alpha}')) = (\sigma'', \tilde{\alpha}'')$ for $\tilde{\alpha}'' := \tilde{\alpha}' \circ \pi|_{S \setminus \mathrm{im}(\sigma'')}$ and we have $(\sigma'', \tilde{\alpha}'') \sim_S (\sigma'', \alpha'')$. Since the first component σ'' of these pairs is the same, it follows that $\alpha'' = g\tilde{\alpha}''$ for some $g \in G$. Now set $\alpha' := g\tilde{\alpha}'$. As the action of G commutes with $\mathcal{F}(\pi)$, we have $(\sigma'', \alpha'') = \mathcal{F}(\pi)((\sigma', \alpha'))$, and since $\Psi((\sigma', \alpha')) = C$, we have $(\sigma', \alpha') \sim_T (\sigma, \alpha)$. □

Next, we extablish axiom (3) for model functors.

Lemma 5.10.4. *Assume that* $(\sigma, \alpha) \in \mathcal{F}(T)$ *is* $\sim_T$*-equivalent to* $(\sigma', \alpha') \in \mathcal{F}(T)$. *Then for all all* $i, j \in T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$ *we have*

$$\alpha(i) = \alpha(j) \Leftrightarrow \alpha'(i) = \alpha'(j).$$

Proof. Set $C := \Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$, an irreducible component of $Z(T)$. Set $D := X(\mathrm{im}(\sigma))[1/(\sigma h)]$, and note that the inclusion $\mathrm{im}(\sigma) \to T$ yields a dominant morphism $C \to D$. Similarly, the inclusion $\mathrm{im}(\sigma') \to T$ yields a dominant morphism $C \to D' := X(\mathrm{im}(\sigma'))[1/(\sigma' h)]$.

Furthermore, let $\tilde{D}$ the image of C in $X(\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))[1/(\sigma h), 1/(\sigma' h)]$ under the morphism coming from the inclusion $\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma') \to T$. Note that $X(\sigma)$ maps D isomorphically onto $\mathrm{Spec}(B_0')$ and $X(\sigma')$ maps D' isomorphically onto $\mathrm{Spec}(B_0')$, and that $\tilde{D}$ maps dominantly into D and into D'.

Now let M be the field of rational functions on D, and M' the field of rational functions on D'. Note that σ and σ' give rise to isomorphisms from L to M and M', respectively. We extend these isomorphisms to isomorphisms from $\overline{L}$ to separable closures $\overline{M}$ and $\overline{M'}$ of M, M'. This yields isomorphisms from $\mathrm{Gal}(\overline{L}/L)$ to the Galois groups $\mathrm{Gal}(\overline{M}/M)$ and $\mathrm{Gal}(\overline{M'}/M)$.

The base change $C_{\overline{M}}$ equals

$$\bigcup_{\beta \in G\alpha} \prod_{i \in T \setminus \mathrm{im}(\sigma)} C_{\beta_i};$$

here the product is over $\overline{M}$ and C_{β_i} is regarded as a variety over $\overline{M}$ via the isomorphism $\overline{L} \to \overline{M}$; and similarly for (σ', α'). The base change $\tilde{D}_{\overline{M}}$ splits as a similar union of products, but now over $\beta \in G\alpha|_{\mathrm{im}(\sigma')\setminus\mathrm{im}(\sigma)}$.

Let $\tilde{M} \supseteq M \cup M'$ be the field of rational functions of $\tilde{D}$, and let $\overline{\tilde{M}} \supseteq \overline{M} \cup \overline{M'}$ be a separable closure of $\tilde{M}$. The components of the base change $C_{\overline{\tilde{M}}}$ can then be computed in two different ways: by first doing a base change to $\overline{M}$ or by first doing a base change to $\overline{M'}$. For the first route, we have to analyse what happens to the product over $\overline{M}$

$$\prod_{i \in T\setminus\mathrm{im}(\sigma)} C_{\beta_i}$$

when doing a base change to $\overline{\tilde{M}}$. Write this product as $V \times_{\overline{M}} W$, where V is the product over all $i \in \mathrm{im}(\sigma') \setminus \mathrm{im}(\sigma)$, and hence an irreducible component of $\tilde{D}_{\overline{M}}$; and W is the product over all $i \in T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$. The function field $\tilde{M}$ of the irreducible M-scheme $\tilde{D}$ embeds into the function field of any component of $\tilde{D}_{\overline{M}}$, hence in particular into $\overline{M}(V)$, and this field extension is algebraic, so that $\overline{\tilde{M}} \cong \overline{\overline{M}(V)}$. So the base change to $\overline{\tilde{M}}$ of the product $V \times_{\overline{M}} W$ is just the base change of W over the separably closed field $\overline{M}$ with the field extension $\overline{\overline{M}(V)}$ and hence still irreducible (e.g. by [Sta20, Tag 020J]).

Summarising, we obtain a bijection from the irreducible components of $C_{\overline{M}}$, which are labelled by elements of $G\alpha$, to the irreducible components of $C_{\overline{\tilde{M}}}$, and that bijection is evidently $\mathrm{Sym}(T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$-equivariant. As a consequence, for $i, j \in T\setminus(\mathrm{im}(\sigma)\cup\mathrm{im}(\sigma'))$ we have $\alpha(i) = \alpha(j)$ if and only if the transposition $(i\ j)$ preserves some (and then each) component in $C_{\overline{M}}$, if and only if that transposition preserves some (and then each) component in $C_{\overline{\tilde{M}}}$.

Similarly, we obtain a bijection from the irreducible components of $C_{\overline{M'}}$, which are labelled by elements of $G\alpha'$, to the irreducible components of $C_{\overline{\tilde{M}}}$, with the same remark about compatibility with the transposition $(i\ j)$. Combining these results, we find that for $i, j \in T \setminus (\mathrm{im}(\sigma) \cup \mathrm{im}(\sigma'))$ we have $\alpha(i) = \alpha(j)$ if and only if $\alpha'(i) = \alpha'(j)$. □

We have now concluded the proof that $\mathcal{C}_2 : S \mapsto \mathcal{F}(S)/\sim_S$ is a model functor; and by construction, the map $\mathcal{C}_2 \to \mathcal{C}_Z$ induced by Ψ is a morphism that is an isomorphism at the level of species.

Finally, we combine the pre-component functor $\mathcal{C}_1$ (mapping onto $\mathcal{C}_Y$) and the model functor $\mathcal{C}_2$ (mapping onto $\mathcal{C}_Z$) as follows: we extend the compatible quasi-order $\preceq_S$ in the definition $\mathcal{C}_1$ as follows: on $\mathcal{F}(S)$, $\preceq_S$ is just the same as $\sim_S$, and furthermore, if the class in $\mathcal{C}_1$ of a pair (σ, α) represents a component C of $Y(S)$ that is contained in the closure of the component represented by the class in $\mathcal{C}_2$ of a pair $(\sigma', \alpha') \in \mathcal{F}(S)$, then we set $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$. This yields a quasi-order $\preceq_S$ satisfying axioms (1)–(2) in the definition of pre-component functors, and the proof that this quasi-order also satisfies axiom (3) is similar to the proof of Lemma 5.10.2.

With this quasi-order, the pre-component functor $\mathcal{C}_1$ and the model functor $\mathcal{C}_2$ are combined into a pre-component functor $\mathcal{C}_3$ with an obvious morphism to $\mathcal{C}_X$ that is an isomorphism at the level of species. Here we use that, while components of $Y(S)$ can be strictly contained in the closure of components of $Z(S)$, the opposite cannot happen: on each component of $Y(S)$ all functions $\sigma(h)$ with $\sigma \in \mathrm{Hom}_{S_0,S}$

are identically zero, while on each component of $Z(S)$ at least one of these functions is nonzero. □

5.11. **Proof of the Main Theorem.**

Proof of Theorem 1.1.1. Before the Main Theorem we introduced $\mathbf{FI}^{\mathbf{op}}$-schemes slightly differently than we do in §2, but the two definitions are equivalent via Remark 2.2.3. So we may assume that X is a width-one $\mathbf{FI}^{\mathbf{op}}$-scheme of finite type over a Noetherian ring K in the sense of §2.

By Theorem 5.10.1, the component functor $\mathcal{C}_X$ of X is, at the level of species, isomorphic to a pre-component functor $\mathcal{C}$. In particular, for all S, the number of $\mathrm{Sym}(S)$-orbits on $\mathcal{C}_X(S)$ equals the number of $\mathrm{Sym}(S)$-orbits on $\mathcal{C}(S)$. By Theorem 5.9.1, this number is a quasipolynomial in $|S|$ for all sufficiently large S. This proves the Main Theorem and concludes our paper. □

References

[AH07] Matthias Aschenbrenner and Christopher J. Hillar. Finite generation of symmetric ideals. *Trans. Am. Math. Soc.*, 359(11):5171–5192, 2007.

[BD11] Andries E. Brouwer and Jan Draisma. Equivariant Gröbner bases and the two-factor model. *Math. Comput.*, 80:1123–1133, 2011.

[BDES21] Arthur Bik, Jan Draisma, Rob H. Eggermont, and Andrew Snowden. The geometry of polynomial functors. 2021. manuscript in preparation.

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules: a new approach to stability for S_n-representations. *Duke Math. J.*, 164(9):1833–1910, 2015.

[CEFN14] Thomas Church, Jordan S. Ellenberg, Benson Farb, and Rohit Nagpal. FI-modules over Noetherian rings. *Geom. Topol.*, 18(5):2951–2984, 2014.

[Coh67] Daniel E. Cohen. On the laws of a metabelian variety. *J. Algebra*, 5:267–273, 1967.

[Coh87] Daniel E. Cohen. Closure relations, Buchberger's algorithm, and polynomials in infinitely many variables. In *Computation theory and logic*, volume 270 of *Lect. Notes Comput. Sci.*, pages 78–87, 1987.

[DEKL16] Jan Draisma, Rob H. Eggermont, Robert Krone, and Anton Leykin. Noetherianity for infinite-dimensional toric varieties. *Algebra & Number Theory*, 9(8):1857–1880, 2016.

[Dra10] Jan Draisma. Finiteness for the k-factor model and chirality varieties. *Adv. Math.*, 223:243–256, 2010.

[Dra19] Jan Draisma. Topological Noetherianity of polynomial functors. *J. Am. Math. Soc.*, 32:691–707, 2019.

[GN18] Sema Güntürkün and Uwe Nagel. Equivariant Hilbert series of monomial orbits. *Proc. Am. Math. Soc.*, 146(6):2381–2393, 2018.

[HKL13] Christopher J. Hillar, Robert Krone, and Anton Leykin. EquivariantGB Macaulay2 package. Webpage: `people.math.gatech.edu/~rkrone3/EquivariantGB.html`, 2013.

[HS12] Christopher J. Hillar and Seth Sullivant. Finite Gröbner bases in infinite dimensional polynomial rings and applications. *Adv. Math.*, 221:1–25, 2012.

[Joy81] Andre Joyal. Une théorie combinatoire des séries formelles. *Adv. Math.*, 42:1–82, 1981.

[KLS17] Robert Krone, Anton Leykin, and Andrew Snowden. Hilbert series of symmetric ideals in infinite polynomial rings via formal languages. *J. Algebra*, 485:353–362, 2017.

[NR17] Uwe Nagel and Tim Römer. Equivariant Hilbert series in non-Noetherian polynomial rings. *J. Algebra*, 486:204–245, 2017.

[NR19] Uwe Nagel and Tim Römer. FI- and OI-modules with varying coefficients. *J. Algebra*, 535:286–322, 2019.

[NS20] Rohit Nagpal and Andrew Snowden. Symmetric subvarieties of infinite affine space. 2020. Preprint, `arXiv:2011.09009`.

[SS17] Steven V. Sam and Andrew Snowden. Gröbner methods for representations of combinatorial categories. *J. Am. Math. Soc.*, 30(1):159–203, 2017.

[Sta82] Richard P. Stanley. Linear Diophantine equations and local cohomology. *Invent. Math.*, 68:175–193, 1982.

[Sta97] Richard P. Stanley. *Enumerative combinatorics. Vol. 1.*, volume 49. Cambridge: Cambridge University Press, 1997.

[Sta20] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2020.

[VNNR18] Dinh Van Le, Uwe Nagel, Hop D. Nguyen, and Tim Römer. Castelnuovo-mumford regularity up to symmetry. *Int. Math. Res. Not.*, 2018. to appear, `arXiv:1806.00457`.

[VNNR20] Dinh Van Le, Uwe Nagel, Hop D. Nguyen, and Tim Römer. Codimension and projective dimension up to symmetry. *Math. Nachr.*, 293(2):346–362, 2020.