Extended abstract
Schur apolarity

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Abstract

Structured tensors are multilinear objects with prescribed symmetries. The *structured rank* of such objects is of extreme interest in application. The aim of this talk is to investigate specific classes of structured tensors whose rank 1 elements are parametrized by points of $SL(n)$-rational homogeneous varieties. This family also includes Veronese varieties (symmetric tensors), Grassmann varieties (skew-symmetric tensors) and Flag varieties. Inspired by the apolarity theory developed for symmetric tensors, together with its recent analogue for skew-symmetric tensors, we present an apolarity action compatible with the structure of the tensors in the respective irreducible representations of $SL(n)$. By the time of the MEGA conference an algorithm will be developed for the structured rank in the case of small rank. In this extended abstract we present the rigorous theory and we focus on certain examples regarding the rank of tensors with respect to certain Flag varieties.

Introduction

The aim of this extended abstract is to set the foundation of a new concept which we call *Schur apolarity theory* that extends the classical one for homogeneous polynomials. This theory is the basis of a wide spread of topics like the dimensions of secant varieties of rational homogeneous varieties and the computation of the rank of tensors with a specific structure induced by the action of $SL(n)$.

In Section 1 we recall some basic facts borrowed from Representation theory which we need to build our theory. The main reference for this is [1]. In Section 2 we describe the structured tensors we are going to study together with their structured rank. The known facts can be recovered in [1] and [2]. In Section 3 we describe the Schur apolarity action and we prove the Schur apolarity lemma. Section 4 is devoted to describe some expected algorithm to discriminate tensors of small rank.

1 Some facts from Representation theory

Let V be a n-dimensional vector space over $\mathbb{C}$ and consider the tensor product $V^{\otimes d}$, with $d \geq 1$ an integer. The symmetric group on d elements $\mathfrak{S}_d$ and the group $SL(n)$ act on this tensor product

$$\begin{aligned} \sigma \cdot (v_1 \otimes \cdots \otimes v_d) &:= v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \quad \text{and} \\ g \cdot (v_1 \otimes \cdots \otimes v_d) &:= (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_d), \end{aligned}$$

for all $\sigma \in \mathfrak{S}_d$, $g \in SL(n)$ and $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$. Using these two actions it is a classical construction that all the irreducible representations of $SL(n)$ can be built. We follow here the construction of [1] that we briefly recall for the reader convenience. Consider a *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k < n$, i.e. a non decreasing sequence of positive integers which sum up to $|\lambda|$. From now on every partition will have length strictly less then n. We may represent it by its *Young diagram* which is a picture with λ_1 boxes in a row, λ_2 boxes below it and so on, all left justified. A *standard Young tableau T of shape* λ, or simply standard tableau of shape λ, is a Young diagram of shape λ with a filling given by the numbers $1, \ldots, d$ in such a way that when reading from top to bottom and from left to right all the sequences are strictly increasing. For example if $\lambda = (2, 2)$, then a possible standard tableau of shape $(2, 2)$ is

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,.$$

Given a tableau T of shape λ one can construct an endomorphism $c_\lambda \in \operatorname{End}(V^{\otimes d})$ called *Young symmetrizer*. It is defined as the map which sends and element $v_1 \otimes \cdots \otimes v_d$ to

$$\sum_{\tau \in C_\lambda} \sum_{\sigma \in R_\lambda} \operatorname{sgn}(\tau) v_{\tau(\sigma(1))} \otimes \cdots \otimes v_{\tau(\sigma(d))}$$

where R_λ and C_λ are the subgroups of $\mathfrak{S}_d$ which keep fixed the rows and the columns of the chosen tableau T respectively. Naively at first one symmetrize “along rows” and then skew-symmetrize “along columns”. The image $\mathbb{S}_\lambda V := c_\lambda(V^{\otimes d})$ is called *Schur module* and it is an irreducible representation of $SL(n)$. From general facts of the theory, given two different tableaux of same shape, the images of the symmetrizers are isomorphic as representations. Moreover one can prove that all the irreducible representations of $SL(n)$ arise in this way. By the definition we have the inclusion

$$\mathbb{S}_\lambda V \subset \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_h} V$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_h)$ is a non decreasing sequence of integer whose diagram is *conjugated* to the one of λ, i.e. its diagram is obtained from the one of λ transposing it as if it would be a matrix. The product of exterior powers will be denoted compactly with $\bigwedge^{\lambda'} V$. Note that if $\lambda = (d)$ or $\lambda = (1^k)$, where 1^k stands for 1 repeated k times, then $\mathbb{S}_{(d)}V = \mathrm{Sym}^d V$ and $\mathbb{S}_{(1^k)}V = \bigwedge^k V$ respectively.

2 Rank and algebraic varieties

In this section we recall some notions linked to the definition of *rank* and algebraic varieties. Let X be a non degenerate irreducible algebraic variety $X \subset \mathbb{P}^N$ and let p be a point of $\mathbb{P}^N$. We define its X*-rank* as the least integer r such that there exists r points $p_1, \dots, p_r \in X$ such that $p \in \langle p_1, \dots, p_r \rangle$. The Zariski closure of the points of $\mathbb{P}^N$ of X-rank r is an irreducible variety $\sigma_r(X)$ called the *r-th secant variety of X*.

Consider a class of varieties which are homogeneous by the action of the group $SL(n)$. Recall that an algebraic variety is homogeneous by the action of a group G if there is an action

$$G \times X \longrightarrow X$$

which maps (g, x) to $g \cdot x \in X$, where $id_G \cdot x = x$ and such an action is transitive. From more general facts of the theory [2], every irreducible representation W of $SL(n)$ has a unique element v up to scalars whose span is fixed by the action of a Borel subgroup contained in $SL(n)$. Such an element is called *highest weight vector*. After some work it can be proved that the orbit X of the action of $SL(n)$ on v is closed in $\mathbb{P}(W)$ and hence it is a homogeneous projective variety. Since the irreducible representations of $SL(n)$ are known, we get that the rational homogeneous varieties obtained by the action of $SL(n)$ are completely described. If W is the Schur module $\mathbb{S}_\lambda V$ with $\lambda = (\lambda_1^{h_1}, \dots, \lambda_k^{h_k})$, where $h_1 + \cdots + h_k < n$, then the respective minimal orbit in $\mathbb{P}(\mathbb{S}_\lambda V)$ is

$$\mathbb{F}(d_k, \dots, d_1; n) = \{(V_k, \dots, V_1) : V_k \subset \cdots \subset V_1 \subset V,\ \dim V_i = d_i\} \subset \prod_{i=0}^{k-1} \mathbb{G}(d_{k-i}, V)$$

embedded with $\mathcal{O}(a_k, \dots, a_1)$. We have the relations

$$d_{k-i+1} = \sum_{j=1}^{i} h_j \text{ and } a_i = \lambda_{k-(i-1)} - \lambda_{k-(i-2)},\ \text{setting } \lambda_{k+1} = 0.$$

Remark that given λ as above, with this notation the conjugate partition λ' can be written as $\lambda' = (d_1^{a_1}, \dots, d_k^{a_k})$. Note that the two extremes of this family are the Veronese varieties, for $s = 1$ and $d_1 = 1$ with any a_1, and the Grassmann varieties, for $s = 1$ and a_1 with any d_1. All the other elements of this family are Flag varieties.

Definition 1. The points of the minimal orbit $X_\lambda \subset \mathbb{P}(\mathbb{S}_\lambda V)$, i.e. points of X_λ-rank 1, are of the form

$$(v_1 \wedge \cdots \wedge v_{d_1})^{\otimes a_1} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{d_k})^{\otimes a_k}$$

and they represent the flag $\langle v_1, \ldots, v_{d_k} \rangle \subset \cdots \subset \langle v_1, \ldots, v_{d_1} \rangle$. We call these tensors *points of λ-rank* 1 to underline the connection with the partition λ. The points of the ambient space which can be written as a minimal linear combination of r points of X are called *points of λ-rank r*. The Zariski closure of the set of points of λ-rank at most r is the *r-th secant variety of X*. Remark that the notion of λ-rank is the same as the notion of X-rank of a point classically used in the theory of secant varieties, where X is the minimal orbit we are considering.

3 Schur apolarity

The interest of this study is to develop an apolarity theory suitable for these varieties as an instrument able to detect the λ-rank of a point. At first we need to recall some more classical definitions [1], [2].

Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_h)$, we say that $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i, possibly setting some μ_i equal to zero. Pictorially the diagram of μ will fit perfectly in the diagram of λ in the left upper corner. With this data, if we remove the diagram of μ from the one of λ one can obtain the so called *skew Young diagram*. A filling with the integers from 1 to $|\lambda| - |\mu|$ such that when reading from top to bottom and from left to right will turn a skew Young diagram into a *skew Young tableau*. For example if $\lambda = (3, 2, 1)$ and $\mu = (1, 1)$, then $\mu \subset \lambda$ and a possible skew Young tableau is

$$\begin{array}{|c|c|c|}\hline \blacksquare & 1 & 3 \\ \hline \blacksquare & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Given such a tableau, one can repeat the construction of the Young symmetrizer. However this time the image of the endomorphism can be possibly reducible as representation. We denote such image with $\mathbb{S}_{\lambda/\mu} V$.

A second tool we need is a ring in which the apolarity must take place. For this purpose we consider the symmetric algebra

$$\begin{aligned} A^\bullet(V) :&= \mathrm{Sym}^\bullet \left(\bigwedge^1 V \otimes \cdots \otimes \bigwedge^n V \right) \simeq \\ &\simeq \bigoplus_{(a_1, \ldots, a_n) \in \mathbb{N}^n} \mathrm{Sym}^{a_1}(\bigwedge^1 V) \otimes \cdots \otimes \mathrm{Sym}^{a_n}(\bigwedge^n V). \end{aligned}$$

The ring we will use is the quotient $\mathbb{S}^\bullet(V) := A^\bullet(V)/I^\bullet(V)$, where the ideal $I^\bullet(V)$ is the two-sided ideal generated by the *Plücker relations*, i.e. by the elements

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) \tag{3.1}$$
$$- \sum_{i=1}^{p} (v_1 \wedge \dots v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q) \tag{3.2}$$

for all $p \geq q \geq 1$. Note that only the symmetric product is involved in the definition of the ring. It can be proved that all the possible Schur modules constructed on V are contained in this ring.
The last ingredient we need to define our apolarity action is the definition of the *skew-symmetric apolarity action*. This definition has been gave a couple years ago in [3]. Given two integers $h \leq k$ we define it as the map

$$\lrcorner_\wedge : \bigwedge^h V^* \otimes \bigwedge^k V \longrightarrow \bigwedge^{k-h} V$$

$$\lrcorner_\wedge \left((\alpha_1 \wedge \cdots \wedge \alpha_h) \otimes (v_1 \wedge \cdots \wedge v_k) \right) \mapsto \sum_R \operatorname{sign}(R) \cdot \det(h_i(v_{r_j})) \cdot v_{\overline{R}}$$

where the sum runs over all the possible ordered sets $R = \{r_1, \dots, r_j\} \subset \{1, \dots, k\}$, while $\overline{R}$ is the set $\{r_{j+1}, \dots, r_n\} = \{1, \dots, n\} \setminus R$ and $v_{\overline{R}} = v_{r_{j+1}} \wedge \cdots \wedge v_{r_n}$. The symbol $\operatorname{sign}(R)$ stands for sign of the permutation which sends $(1, \dots, k)$ to $(r_1, \dots, r_k, \dots)$ keeping the order of the other elements.

Definition 2. The *Schur apolarity action* is a map

$$\varphi : \mathbb{S}^\bullet(V) \otimes \mathbb{S}^\bullet(V^*) \longrightarrow \mathbb{S}^\bullet(V)$$

such that when restricted to a product of two irreducible Schur modules $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$ with $\mu \not\subset \lambda$ it is the zero map, otherwise if $\mu \subset \lambda$ then it is the restriction of the map

$$\tilde{\varphi} : \bigwedge^{\lambda'} V \otimes \bigwedge^{\mu'} V^* \longrightarrow \bigwedge^{\lambda'/\mu'} V$$

given by the product of skew-symmetric apolarity actions

$$\lrcorner_\wedge : \bigwedge^{\lambda'_i} V \otimes \bigwedge^{\mu'_i} V^* \to \bigwedge^{\lambda'_i - \mu'_i} V.$$

The symbol λ'/μ' stands for the sequence $(\lambda'_1 - \mu'_1, \dots, \lambda'_h - \mu'_h)$ but it is not in general a partition.

For example let V be a vector space of dimension at least 4 and $\lambda = (3, 2, 1)$ and $\mu = (1, 1)$, so that $\mu \subset \lambda$. Consider the standard tableaux

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad \text{and} \quad S = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

to construct the modules $\mathbb{S}_{(3,2,1)}V$ and $\mathbb{S}_{(1,1)}V^*$. The image of two elements $t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_1 \in \mathbb{S}_{(3,2,1)}V$ and $s = \alpha_1 \wedge \alpha_2 \in \mathbb{S}_{(2,1)}V^*$ via the Schur apolarity is

$$\begin{aligned} \varphi(t \otimes s) &= \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) \\ \alpha_2(v_1) & \alpha_2(v_2) \end{pmatrix} v_1 \wedge v_2 \otimes v_1 \otimes v_3 \\ &- \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_3) \\ \alpha_2(v_1) & \alpha_2(v_3) \end{pmatrix} v_1 \wedge v_2 \otimes v_1 \otimes v_2 \\ &+ \det \begin{pmatrix} \alpha_1(v_2) & \alpha_1(v_3) \\ \alpha_2(v_2) & \alpha_2(v_3) \end{pmatrix} v_1 \wedge v_2 \otimes v_1 \otimes v_1. \end{aligned}$$

The order of the factors of the product are determined by the choice of T and S. Two facts can be proved.

Proposition 3.1. *Let λ and μ two partitions such that $\mu \subset \lambda$. The image of the Schur apolarity action restricted to the product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$ is contained in the module $\mathbb{S}_{\lambda/\mu}V$. Moreover given a different choice of tableaux of shape λ and μ, the respective Schur apolarity actions will have isomorphic kernels and images.*

We give here just an idea of the proof. At first choose two partitions λ and μ such that $\lambda \subset \mu$ and consider the product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$. Then consider an element of the basis of $\mathbb{S}_\lambda V$ and one of the basis of $\mathbb{S}_\mu V^*$. If is enough to prove the thesis for these two elements. There is an easy way to construct a "canonical basis" for this module in [4]. Fix a basis $e_1, \dots, e_n$ of V. Given a tableaux T of shape λ, consider a *semistandard tableau* S of shape λ, which is the Young diagram of λ filled with integers from 1 to n, strictly increasing along columns but allowing non decreasing sequences along rows. The couple (T, S) is usually regarded as *bitableau*. The integers i appearing in S represent the element e_i of V. At this point one can build an element $e_S = e_{i_1} \otimes \cdots \otimes e_{i_d}$, where e_{i_j} is e_k if there is a k in the box of S which corresponds to the same box in T in which there is a j. Then we apply c_λ to e_S to obtain an element of $\mathbb{S}_\lambda V$. The collection of the elements $c_\lambda(e_S)$ for all semistandard tableau S forms a basis for $\mathbb{S}_\lambda V$. For example if

$$(T, S) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right)$$

then $e_S = e_1 \otimes e_3 \otimes e_2$ and $c_\lambda(e_S) = e_1 \wedge e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_1$. Pictorially we may represent this last element using the tableau

$$c_\lambda(e_S) = \left(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau} - \begin{ytableau} 2 & 3 \\ 1 \end{ytableau} - \begin{ytableau} 2 & 1 \\ 3 \end{ytableau} + \begin{ytableau} 3 & 1 \\ 2 \end{ytableau} \right) \tag{3.3}$$

In each of these tableau the number i identifies the element e_i. Therefore take the tensor product of the respective vectors with the order prescribed by T in order to recover $c_\lambda(e_S)$. This construction obviously applies also to $\mathbb{S}_\mu V^*$. Then one has just to translate the Schur apolarity action in terms of the elements written as in (3.3). The result will be a sum of skew Young tableaux of shape λ/μ with proper coefficients determined by the apolarity. Note that the diagram λ/μ may contain disjoint subdiagrams, i.e. diagrams which do not share any column or row. If one collects the addends in groups in which they share the same coefficients and the same fillings in the disjoint subdiagrams, then one can note that all these elements satisfy symmetries conditions prescribed by a skew Young tableau. Hence the image belong to $\mathbb{S}_{\lambda/\mu}V$. With the same argument one can see that a different choice of standard tableaux of shape λ and μ will produce two maps with isomorphic images and kernels.

Definition 3. Let $f \in \mathbb{S}_\lambda V$ be a fixed element, for some suitable λ. For $\mu \subset \lambda$, the restricted Schur apolarity action $\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \to \mathbb{S}_{\lambda/\mu} V$ induce a map

$$\mathcal{C}_f^{\lambda,\mu} : \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

defined as $\mathcal{C}_f^{\lambda,\mu}(h) := \varphi(f \otimes h)$. This map is called *Schur catalecticant map.* The *orthogonal set to* f is

$$f^\perp := \{s \in \mathbb{S}^\bullet(V^*) \ : \ \varphi(f \otimes s) = 0\}.$$

It can be proved that it is an ideal.

One classic feature of the apolarity theory concerning symmetric and skew-symmetric tensors is the *apolarity lemma.* For the classic theory see [5] and [6]. For the skew-symmetric case see [3]. We are going to present our version in a moment. Such a lemma tells us that given an element $f \in \mathbb{S}_\lambda V$, we may find its λ-rank and also a decomposition if and only if a specific ideal is found inside the orthogonal set $f^\perp$. In the classic setting this ideal is the ideal of a finite set of points of $\mathbb{P}^n$, the ones which determine the decomposition of the symmetric tensor. Even in the skew-symmetric case we have an ideal of points but this time it is defined in the exterior algebra. We will give here a suitable definition which respect the common idea of all the apolarity theory, the evaluation.

Let $\lambda = (\lambda_1^{h_1}, \dots, \lambda_k^{h_k})$ with $h_1 + \cdots + h_k < n$. Let $\lambda' = (d_1^{a_1}, \dots, d_k^{a_k})$ be the conjugate partition, where

$$d_{k-i+1} = \sum_{j=1}^{i} h_j \quad \text{and} \quad a_i = \lambda_{k-(i-1)} - \lambda_{k-(i-2)}, \quad \text{setting} \quad \lambda_{k+1} = 0$$

so that the minimal orbit inside $\mathbb{S}_\lambda V$ is $\mathbb{F}(d_k,\dots,d_1;n)$ embedded with $\mathcal{O}(a_k,\dots,a_1)$. Let $v\in\mathbb{S}_\lambda V$ be a point of λ-rank 1, i.e. something like

$$(v_1\wedge\dots\wedge v_{d_1})^{\otimes a_1}\otimes\dots\otimes(v_1\wedge\dots\wedge v_{d_k})^{\otimes a_k}$$

and it represents the flag $\langle v_1,\dots,v_{d_k}\rangle\subset\dots\subset\langle v_1,\dots,v_{d_1}\rangle$. Denote them $V_{d_k},\dots,V_{d_1}$ respectively. Then we have that V_{d_1} is cut out by some $n-d_1$ linear equations $l_1,\dots,l_{n-d_1}$. The space V_{d_2} will be given by the same equations and also $l_{n-d_1+1},\dots,l_{n-d_2}$. In general V_{d_j} will be given by the equations of $V_{d_{j-1}}$ together with $l_{n-d_{j-1}+1},\dots,l_{n-d_j}$.

Definition 4. Let $v\in\mathbb{S}_\lambda V$ be a point of λ-rank 1 with the notation introduced above. The *ideal of* v, denoted with $I(v)$ is defined as the ideal generated in $\mathbb{S}^\bullet(V^*)$ as

$$I(v):=(l_1,\dots,l_{n-d_1},l_{n-d_1+1}^{\lambda_k+1},\dots,l_{n-d_2}^{\lambda_k+1},\dots,l_{n-d_{k-1}+1}^{\lambda_2+1},\dots,l_{n-d_k}^{\lambda_2+1}).$$

Given $v_1,\dots,v_r\in\mathbb{S}_\lambda V$ of $\lambda-$rank 1, the *ideal of* $v_1,\dots,v_r$ is defined as

$$I(v_1,\dots,v_r):=\bigcap_{i=1}^r I(v_i).$$

To state the *Schur apolarity lemma* we need a technical result.

Proposition 3.2. *Let $v_1,\dots,v_r\in\mathbb{S}_\lambda V$ of $\lambda-$rank 1 and let the $I(v_1,\dots,v_r)$ the ideal defined as above. Then we have*

$$I(v_1,\dots,v_r)^\perp=\left[\bigcap_{i=1}^r I(v_i)\right]^\perp=\sum_{i=1}^r I(v_i)^\perp,$$

where $I(v_i)^\perp=\{v\in\mathbb{S}^\bullet(V):\varphi(v\otimes h)=0,$ for all $h\in I(v_i)\}$.

Proof. Firstly we prove this fact for $r=2$ and after we prove the general case by induction. It is easy to see that given two sets $A,\ B\subset\mathbb{S}^\bullet(V^*)$ such that $A\subset B$, then via orthogonality we get $A^\perp\supset B^\perp$. Consider two ideals $I(v_1)$ and $I(v_2)$. Clearly one has that $I(v_1)\cap I(v_2)\subset I(v_i)$ for $i=1,2$, and this implies that $I(v_i)^\perp\subset(I(v_1)\cap I(v_2))^\perp$ for $i=1,2$. From this we get the inclusion $I(v_1)^\perp+I(v_2)^\perp\subset(I(v_1)\cap I(v_2))^\perp$. For the other inclusion, since $I(v_i)^\perp\subset I(v_1)^\perp+I(v_2)^\perp$ for $i=1,2$, then we get $I(v_i)\supset(I(v_1)^\perp+I(v_2)^\perp)^\perp$ for $i=1,2$ and hence $I(v_1)\cap I(v_2)\supset(I(v_1)^\perp+I(v_2)^\perp)^\perp$. Applying the orthogonality one gets $(I(v_1)\cap I(v_2))^\perp\subset I(v_1)^\perp+I(v_2)^\perp$ and the thesis.
Now assume that the proposition is true for $r-1$ and we prove it for r. We have that

$$\left(\bigcap_{i=1}^r I(v_i)\right)^\perp=\left(\bigcap_{i=1}^{r-1} I(v_i)\cap I(v_r)\right)^\perp=\left(\bigcap_{i=1}^{r-1} I(v_i)\right)^\perp+I(v_r)^\perp=\sum_{i=1}^r I(v_i)^\perp$$

where in the last two equalities we have applied the induction hypothesis. This concludes the proof. $\square$

We are now ready to state the main result of the document. Such a result is achieved in three steps using three classes of partitions. At first we prove it for *rectangular* partitions, i.e. sequences like (λ_1^k) with $k < n$. The related minimal orbits are Grassmann varieties $\mathbb{G}(k, V)$ embedded with $\mathcal{O}(\lambda_1)$.

Lemma 3.3 (of Schur apolarity - rectangular case)**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_1) = (\lambda_1^k)$ be a rectangular partition. Let $v_1, \ldots, v_r \in \mathbb{S}_\lambda V$ of $\lambda-$rank 1, i.e.*

$$v_i = v_1^i \wedge \cdots \wedge v_k^i \otimes \cdots \otimes v_1^i \wedge \cdots \wedge v_k^i$$

and let $f \in \mathbb{S}_\lambda V$. The following are equivalent:

(1) *there exists $c_1, \ldots, c_r \in \mathbb{K}$ such that $f = c_1 v_1 + \cdots + c_r v_r$,*

(2) *$I(v_1, \ldots, v_r) \subseteq f^\perp$.*

Proof. Assume that $f = c_1 v_1 + \cdots + c_r v_r$, and let us prove (2). Let $g \in I(v_1, \ldots, v_r) = \bigcap_{i=1}^r I(v_i)$, where

$$I(v_i) = (l_1, \ldots, l_{n-k}). \tag{3.4}$$

The intersection of these ideals give rise to elements of $\mathbb{S}^\bullet(V^*)$ such that when performing the Schur apolarity with f the result is 0, hence $I(v_1, \ldots, v_r) \subseteq f^\perp$. Suppose now that $I(v_1, \ldots, v_r) \subseteq f^\perp$ for some $v_1, \ldots, v_r \in \mathbb{S}_\lambda V$ of $\lambda-$rank 1, and some $f \in \mathbb{S}_\lambda V$. This is equivalent to say

$$(f) \subseteq I(v_1, \ldots, v_r)^\perp,$$

where $I(v_1, \ldots, v_r)^\perp = \{g \in \mathbb{S}^\bullet(V) : \varphi(g \otimes h) = 0, \text{ for all } h \in I(v_1, \ldots, v_r)\}$. In particular,

$$f \in (f) \cap \mathbb{S}_\lambda V \subseteq I(v_1, \ldots, v_r)^\perp \cap \mathbb{S}_\lambda V,$$

and by Proposition 3.2

$$I(v_1, \ldots, v_r)^\perp = \left[\bigcap_{i=1}^r I(v_i)\right]^\perp = \sum_{i=1}^r I(v_i)^\perp.$$

Now fix an i in $\{1, \ldots, r\}$. We claim that $I(v_i)^\perp \cap \mathbb{S}_\lambda V = \langle v_i \rangle$. Clearly we have $\langle v_i \rangle \subset I(v_i)^\perp \cap \mathbb{S}_\lambda V$. For the other inclusion, note that any element in $\mathbb{S}_\lambda V$ can be seen as an element of $\wedge^{\lambda'} V$ modulo the Plücker relations (3.1) related to $\mathbb{S}_\lambda V$, hence we may write

$$t = t_1^1 \wedge \cdots \wedge t_k^1 \otimes t_1^2 \wedge \cdots \wedge t_k^2 \otimes \cdots \otimes t_1^{\lambda_1} \wedge \cdots \wedge t_k^{\lambda_1} \in \mathbb{S}_\lambda V$$

Consider the subspaces V_j generated by $t_1^j, \ldots, t_k^j$. If we prove that every subspace V_j is equal to $W = \langle v_1^i, \ldots, v_k^i \rangle$ we have finished.

We will show this inductively on j. For $j=1$, the piece of degree 1 of $I(v_i)$ is given by linear combinations of $l_1,\dots,l_{n-k}$. Then if we perform the Schur apolarity action between this linear equations and any element t as above and we impose the result to be 0, we get that $V_1 = W$. Now assume that $V_1 = \cdots = V_{h-1} = W$, i.e. our element looks like

$$t = v_1^i \wedge \cdots \wedge v_k^i \otimes \cdots \otimes v_1^i \wedge \cdots \wedge v_k^i \otimes t_1^h \wedge \cdots \wedge t_k^h \otimes \cdots \otimes t_1^n \wedge \cdots \wedge t_k^n.$$

Observe that the products $g_p^{h-1} l_q$, with $q \in \{1,\dots,n-k\}$ and g_p is the dual element of some v_p^i, belong to the ideal $I(v_i)$. Since the elements we want to contract belong to tensor products of different algebras, write these last products as elements of $(\bigwedge^1 V)^{\otimes h}$, i.e.

$$g_p^{h-1} l_q = \frac{1}{h!}(l_q \otimes g_p \otimes \cdots \otimes g_p + g_p \otimes l_q \otimes g_p \otimes \cdots \otimes g_p + \cdots + g_p \otimes \cdots \otimes g_p \otimes l_q).$$

Note that l_q appears in every addend once. This means that when we contract, the first $h-1$ addends give 0 as result since we already know that the equation l_q is 0 on the subspaces $V_1,\dots,V_{h-1}$. When it comes for the last addend, the equation g_p cannot be zero on the first $h-1$ subspaces. The only way to get a zero is that V_h satisfies the equation l_q. Hence we get that V_h satisfies the equations $l_1,\dots,l_{n-k}$, i.e. $V_h \subseteq W$ as a subspace. For dimensional reasons we get $V_h = W$ and inductively $t = v_i$. This allows us to say that $f \in \sum_{i=1}^r I(v_i)^\perp \cap \mathbb{S}_\lambda V = \langle v_1,\dots,v_r\rangle$. This concludes the proof. $\square$

A this point we consider partitions whose Young diagram is union of two rectangles, i.e. $(\lambda_1^{h_1}, \lambda_2^{h_2})$. The related varieties are Flag varieties $\mathbb{F}(h_1, h_1+h_2)$ embedded with $\mathcal{O}(\lambda_2, \lambda_1-\lambda_2)$. For the moment set $h = h_1 + h_2$.

Lemma 3.4 (of Schur apolarity - union of two rectangles)**.** *Let $v_1,\dots,v_r \in \mathbb{S}_\lambda V$ of $\lambda-$rank 1 with λ union of two rectangles, i.e.*

$$v_i = v_1^i \wedge \cdots \wedge v_h^i \otimes \cdots \otimes v_1^i \wedge \cdots \wedge v_h^i \otimes v_1^i \wedge \cdots \wedge v_{h_1}^i \otimes \cdots \otimes v_1^i \wedge \cdots \wedge v_{h_1}^i,$$

and let $f \in \mathbb{S}_\lambda V$. The following are equivalent:

(1) *there exists $c_1,\dots,c_r \in \mathbb{K}$ such that $f = c_1 v_1 + \cdots + c_r v_r$,*

(2) *$I(v_1,\dots,v_r) = \bigcap_{i=1}^r I(v_i) \subseteq f^\perp$, where $I(v_i)$ is defined as above.*

Proof. Assume that $f = c_1 v_1 + \cdots + c_r v_r$. Then every $g \in I(v_1,\dots,v_r)$ kills by definition every v_i, and hence in particular g kills f.
Suppose that $I(v_1,\dots,v_r) \subset f^\perp$. This is equivalent to say that $(f) \subset I(v_1,\dots,v_r)^\perp$. In particular

$$f \in (f) \cap \mathbb{S}_\lambda V \subset I(v_1,\dots,v_r)^\perp \cap \mathbb{S}_\lambda V.$$

Moreover, by Proposition 3.2

$$I(v_1,\dots,v_r)^\perp = \left[\bigcap_{i=1}^r I(v_i)\right]^\perp = \sum_{i=1}^r I(v_i)^\perp.$$

If we prove that $I(v_i)^\perp \cap \mathbb{S}_\lambda V = \langle v_i\rangle$, where

$$I(v_i)^\perp = \{g \in \mathbb{S}^\bullet(V) : \varphi(g\otimes h) = 0, \text{ for all } h \in I(v_i)\}$$

for all i's, we have done. Fix an $i \in \{1,\dots,r\}$. Clearly it holds that $\langle v_i\rangle \subset I(v_i)\cap \mathbb{S}_\lambda V$. A general element of $\mathbb{S}_\lambda V$ can be seen as an element of $\wedge^{\lambda'} V$ modulo the Plücker relations, i.e. it has the form

$$t = t_1^1 \wedge\cdots\wedge t_h^1 \otimes\cdots\otimes t_1^{\lambda_2}\wedge\cdots\wedge t_h^{\lambda_2}\otimes t_1^{\lambda_2+1}\wedge\cdots\wedge t_{h_1}^{\lambda_2+1}\otimes\cdots\otimes t_1^{\lambda_1}\wedge\cdots\wedge t_{h_1}^{\lambda_1}.$$

Let us call the subspaces $U_j = \langle t_1^j,\dots,t_h^j\rangle$ for $j = 1,\dots,\lambda_2$, $W_j = \langle t_1^j,\dots,t_{h_1}^j\rangle$ for $j = \lambda_2+1\dots,\lambda_1$, $U = \langle v_1^i,\dots,v_h^i\rangle$ and $W = \langle v_1^i,\dots,v_{h_1}^i\rangle$. Hence we have to prove that $U_j = U$ for all $j = 1,\dots,\lambda_2$, and $W_j = W$ for all $j = \lambda_2+1,\dots,\lambda_1$. The keypoint of the proof is to show the above equalities at first for the U_j's, and then for the W_j's in order with respect the indices. The fact that $U_j = U$ follows exactly in the same way it has been proved in Lemma 3.3. Let us see the equality $W_j = W$, for $j = \lambda_2+1,\dots,\lambda_1$. Using the Lemma 3.3 again, the subspaces W_j must satisfy the equations that defines U and hence $W_j \subset U$ for all j. In order to show that the W_j's satisfy the added equations, we will proceed by induction on j.
Consider W_{λ_2+1} and the powers $l_q^{\lambda_2+1} \in I(v)$, with $q = n-h+1,\dots,n-h_1$. We want to contract these powers with t. In order to do so, write $l_q^{\lambda_2+1}$ as elements of $(\wedge^1 V)^{\otimes\lambda_1+1}$, i.e.

$$l_q^{\lambda_2+1} = l_q\otimes\cdots\otimes l_q.$$

When we perform the contraction, we are evaluating the linear form l_q in the subspaces $U_1,\dots,U_{\lambda_2},W_{\lambda_2+1}$. Since we already know that l_q is not zero in the first λ_2 spaces, we must have that W_{λ_2+1} satisfies $l_q = 0$ for all q. This proves that $W_j = W$. Now suppose that $W_{\lambda_2+1} = \cdots = W_{l-1} = W$ for some $\lambda_2+1 < l \le \lambda_1$ and let us prove that $W_l = W$. To this end, consider the elements $g_p^{l-\lambda_2-1} l_q^{\lambda_2+1} \in I(v)$ where g_p is the dual element of some $v_1,\dots,v_{h_1}$. In order to contract with t, we embed this element in $(\wedge^1 V)^{\otimes h}$, i.e.

$$g_p^{l-\lambda_2-1} l_q^{\lambda_2+1} = \frac{1}{l!}(g_p\otimes\cdots\otimes g_p\otimes l_q\otimes\cdots\otimes l_q+\cdots+l_q\otimes\cdots\otimes l_q\otimes g_p\otimes\cdots\otimes g_p).$$

We may distinguish some cases keeping in mind the hypothesis $W_{\lambda_1+1} = \cdots = W_{l-1} = W$. Consider the addend in which the g_p's appear in the $(\lambda_2+1),\dots,(l-1)-$th component. This means that we are evaluating g_p at $W_{\lambda_2+1},\dots,W_{l-1}$, and that we have to perform the evaluation of l_q on the remaining subspaces. From the hypothesis, g_p is different from 0 on $W_{\lambda_2+1},\dots,W_{l-1}$, and also l_q is different from 0 on $U_1,\dots,U_{\lambda_2}$. Hence, the only way to get a 0 from this contraction is that l_q must vanish on W_l. All the other addends are such that when contracting with t, we evaluate l_q on some W_j of the hypothesis and hence it is 0. This proves that $W_l = W$ and this concludes the proof. $\square$

The final case is the one given by any partition $\lambda = (\lambda_1^{h_1},\dots,\lambda_k^{h_k})$, with $h_1 + \cdots + h_k < n$.

Lemma 3.5 (of Schur apolarity)**.** *Let $\lambda = (\lambda_1^{h_1},\dots,\lambda_k^{h_k})$ be any partition, let $\lambda' = (d_1^{a_1},\dots,d_k^{a_k})$ be the conjugate partition to λ, where*

$$d_{k-i+1} = \sum_{j=1}^{i} h_j \quad \textit{and} \quad a_i = \lambda_{k-(i-1)} - \lambda_{k-(i-2)},\ \textit{for all } i = 1,\dots,k.$$

Let $v_1,\dots,v_r \in \mathbb{S}_\lambda V$ of $\lambda-$rank 1, i.e.

$$v_i = (v_1^i \wedge \cdots \wedge v_{d_1}^i)^{\otimes a_1} \otimes (v_1^i \wedge \cdots \wedge v_{a_2}^i)^{\otimes h_2} \otimes \cdots \otimes (v_1^i \wedge \cdots \wedge v_{d_k}^i)^{\otimes a_k},$$

and let $f \in \mathbb{S}_\lambda V$. The following are equivalent:

(1) *there exists $c_1,\dots,c_r \in \mathbb{K}$ such that $f = c_1v_1 + \cdots + c_rv_r$,*

(2) *$I(v_1,\dots,v_r) = \bigcap_{i=1}^r I(v_i) \subseteq f^\perp$, where $I(v_i)$ is defined as above.*

Proof. Suppose that $f = c_1v_1 + \cdots + c_rv_r$. Then for every $g \in I(v_1,\dots,v_r)$, since g kills every v_i, it kills also f.
On the other hand, assume that $I(v_1,\dots,v_r) \subset f^\perp$, or equivalently that $(f) \subset I(v_1,\dots,v_r)^\perp$. Clearly we have

$$f \in (f) \cap \mathbb{S}_\lambda V \subset I(v_1,\dots,v_r)^\perp \cap \mathbb{S}_\lambda V.$$

Moreover by Proposition (3.2)

$$I(v_1,\dots,v_r)^\perp = \left[\bigcap_{i=1}^r I(v_i)\right]^\perp = \sum_{i=1}^r I(v_i)^\perp.$$

If we prove that $I(v_i)^\perp \cap \mathbb{S}_\lambda V = \langle v_i \rangle$ for all i's we have finished. Fix an i in $\{1,\dots,r\}$. Clearly we have that $I(v_i)^\perp \cap \mathbb{S}_\lambda V \supset \langle v_i \rangle$. A general tensor in $\mathbb{S}_\lambda V$ can be seen as an element of $\bigwedge^\lambda V$ modulo the Plücker relations (3.1) related to $\mathbb{S}_\lambda V$, i.e. it has the form

$$\begin{aligned} t =& t_1^1 \wedge \cdots \wedge t_{d_1}^1 \otimes \cdots \otimes t_1^{\lambda_k} \wedge \cdots \wedge t_{d_1}^{\lambda_k} \otimes t_1^{\lambda_k+1} \wedge \cdots \wedge t_{d_2}^{\lambda_k+1} \otimes \dots \\ & \cdots \otimes t_1^{\lambda_{k-1}} \wedge \cdots \wedge t_{d_2}^{\lambda_{k-1}} \otimes \cdots \otimes t_1^{\lambda_2+1} \wedge \cdots \wedge t_{d_k}^{\lambda_2+1} \otimes \cdots \otimes t_1^{\lambda_1} \wedge \cdots \wedge t_{d_k}^{\lambda_1}. \end{aligned}$$

Denote with $V_{d_1}, \dots, V_{d_k}$ the chain of subspaces

$$\langle v_1^i, \dots, v_{d_1}^i \rangle \supset \langle v_1^i, \dots, v_{d_2}^i \rangle \supset \cdots \supset \langle v_1^i, \dots, v_{d_k}^i \rangle.$$

We have to prove that $t \in I(v_i)^\perp \cap \mathbb{S}_\lambda V$ is such that

$$\langle t_1^{m_j}, \dots, t_{d_j}^{m_j} \rangle = V_{d_j}$$

for $j = 1, \dots, k$ and $\lambda_{k-j+2} + 1 \leq m_j \leq \lambda_{k-j+1}$, where we have already set $\lambda_{k+1} = 0$. This fact is achieved working consecutively on every group identified by $j = 1, \dots, k$.

For $j = 1$ and $1 \leq m \leq \lambda_k$, we work as we have done in the proof of the Lemma 3.3, i.e. we start contracting the tensor t with the elements of $I(v_1, \dots, v_r)_1$ and we impose the result to be 0. This will give us $\langle t_1^1, \dots, t_{d_1}^1 \rangle = V_{d_1}$. Then, proceeding by "induction" until $m_1 = \lambda_k$ as done in the proof of the Lemma 3.3, we get $\langle t_1^{m_1}, \dots, t_{d_1}^{m_1} \rangle = V_{d_1}$ for all $1 \leq m_1 \leq \lambda_k$.

Now assume that we have proved our thesis for all the subspaces until a certain $j-1$ and let us prove it for j. Both the Lemma 3.3 and 3.4 come to help. Indeed, the first one allows us to say that the subspaces $\langle t_1^{m_j}, \dots, t_{d_j}^{m_j} \rangle$ are contained in $V_{d_{j-1}}$ for all m_j since, by contracting and imposing the result to be 0, these subspaces must satisfy the linear equations that defines $V_{d_{j-1}}$. We have to prove that $\langle t_1^{m_j}, \dots, t_{d_j}^{m_j} \rangle$ satisfies in addition the equations that cut out V_{d_j} from $V_{d_{j-1}}$ for all m_j. This fact follows using the products

$$g^{m_j - \lambda_{k-j+2} - 1} l_q^{\lambda_{k-j+2}+1} \in I(v_1, \dots, v_r),$$

where $q \in \{n - d_{j-1} + 1, \dots, n - d_j\}$ and g is the dual element of a generator of $V_{d_{j-1}}$. Emulating what we have done in the proof of the Lemma 3.4, contracting this elements with t and imposing it to be 0 will allow us to say that $\langle t_1^{m_j}, \dots, t_{d_j}^{m_j} \rangle$ is contained in V_{d_j} and for a dimensional count $\langle t_1^{m_j}, \dots, t_{d_j}^{m_j} \rangle = V_{d_j}$. This concludes the proof. $\square$

Let us see some examples. In the following $\{e_1, \dots, e_n\}$ denotes a basis of $\mathbb{C}^n$ while $\{x_1, \dots, x_n\}$ is the dual basis of $(\mathbb{C}^n)^*$.

Example 3.6. Let $\lambda = (2,1)$ and consider $t = e_2 \wedge e_3 \otimes e_1 - e_1 \wedge e_2 \otimes e_3 \in \mathbb{S}_{(2,1)}\mathbb{C}^3$. We show that its $(2,1)$-rank is 2 and we show a decomposition. In this case the minimal orbit in $\mathbb{P}(\mathbb{S}_{(2,1)}\mathbb{C}^3)$ is the Flag variety $\mathbb{F}(1,2;3)$ and the ideal of a point p of $(2,1)$-rank 1 has the form $I(p) = (l_1, l_2^2)$. We compute the catalecticant $\mathcal{C}_t^{(2,1),(1)} : \mathbb{S}_{(1)}(\mathbb{C}^3)^* \to \mathbb{S}_{(2,1)/(1)}\mathbb{C}^3$ to chase linear forms which kills t. One can get the 9×3 matrix

$$\mathcal{C}_t^{(2,1),(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has rank 3. Hence $\ker \mathcal{C}_t^{(2,1),(1)} = \langle 0 \rangle$ and since no linear forms have been found, the tensor t has not $(2,1)$-rank 1. Now suppose that t has $(2,1)$-rank 2. In this case $t = t_1 + t_2$ where t_i have $(2,1)$-rank 1 for $i = 1, 2$. Suppose that l_i is the linear form that defines the plane in the flag associated to t_i for $i = 1, 2$. Clearly l_1 and l_2 must be different, otherwise the tensor will have $(2,1)$-rank 1. Hence now we look for a product $l_1 l_2$ in $t^\perp$. Recall that

$$I(t_1, t_2) = I(t_1) \cap I(t_2) = (l_1, l_3^2) \cap (l_2, l_4^2)$$

and so we must have that $I(t_1, t_2)$ is generated in degree 2 and one of the generator is $l_1 l_2$. The other generators may have greater degree. Since $(3), (4) \not\subset (2,1)$, we cannot argue which are the generators in degree greater or equal then 3. Hence it is more convenient to look for the product of two distinct linear forms in $t^\perp$. For this purpose consider the map $\mathcal{C}^{(2,1),(2)} : \mathbb{S}_{(2)}(\mathbb{C}^3)^* \to \mathbb{S}_{(2,1)/(2)}\mathbb{C}^3$. We get that

$$\mathcal{C}_t^{(2,1),(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 3. In this case $\ker \mathcal{C}_t^{(2,1),(2)} = \langle x_1^2, x_2^2, x_3^2 \rangle$. Observe that we have the element $(x_1 + x_3)(x_1 - x_3) \in \ker \mathcal{C}_t^{(2,1),(2)}$. The vanishing of these two linear forms is given by the planes $\langle e_1 - e_3, e_2 \rangle$ and $\langle e_1 + e_3, e_2 \rangle$ in $\mathbb{C}^3$ respectively. Finally it is enough to solve the linear system

$$t = (a(e_1 - e_3) + be_2) \wedge e_2 \otimes (a(e_1 - e_3) + be_2) + (c(e_1 + e_3) + de_2) \wedge e_2 \otimes (c(e_1 + e_3) + de_2).$$

A solution is given by $a = \frac{1}{2}$, $c = -\frac{1}{2}$ and $b = d = 0$, hence

$$t = \frac{1}{2}(e_1 - e_3) \wedge e_2 \otimes (e_1 - e_3) - \frac{1}{2}(e_1 + e_3) \wedge e_3 \otimes (e_1 + e_3)$$

has $(2,1)$-rank 2.

Example 3.7. Let $\lambda = (3,2,1)$ and consider the element

$$t = e_1 \wedge e_2 \wedge e_3 \otimes e_1 \wedge e_2 \otimes e_3 - e_1 \wedge e_2 \wedge e_3 \otimes e_2 \wedge e_3 \otimes e_1 \in \mathbb{S}_{(3,2,1)}\mathbb{C}^4.$$

We show also in this case that the tensor has $(3,2,1)$-rank 2. In particular we will use the previous example to compute the rank of t. Collecting the first exterior product we can write t as

$$\begin{aligned} t &= e_1 \wedge e_2 \wedge e_3 \otimes (e_1 \wedge e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_1) \\ &= e_1 \wedge e_2 \wedge e_3 \otimes \left(\frac{1}{2}(e_1 + e_3) \wedge e_2 \otimes (e_1 + e_3) - \frac{1}{2}(e_1 - e_3) \wedge e_2 \otimes (e_1 - e_3) \right) \\ &= \frac{1}{2}(e_1 + e_3) \wedge e_2 \wedge e_3 \otimes (e_1 + e_3) \wedge e_2 \otimes (e_1 + e_3) + \\ &\qquad - \frac{1}{2}(e_1 - e_3) \wedge e_2 \wedge e_3 \otimes (e_1 - e_3) \wedge e_2 \otimes (e_1 - e_3) \end{aligned}$$

which is a decomposition of $(3,2,1)$-rank 2. If we show that t has not $(3,2,1)$-rank 1 we obtain the claim. recall that a point t_1 of $(2,1)$-rank 1 has the ideal $I(t_1) = (l_1, l_2^2, l_3^3)$. Hence if we do not find one of the three generators in $t^\perp$ we can conclude that t has not $(3,2,1)$-rank 1. Observe that the linear form x_4 kills t and hence $x_4 \in t^\perp$. Moreover it is the unique linear form in $t^\perp$. Indeed consider the catalecticant map $\mathcal{C}_t^{(3,2,1),(1)} : \mathbb{S}_{(1)}(\mathbb{C}^4)^* \to \mathbb{S}_{(3,2,1)/(1)}\mathbb{C}^4$. Since we have the decomposition

$$\mathbb{S}_{(3,2,1)/(1)}\mathbb{C}^4 \simeq \mathbb{S}_{(2,2,1)}\mathbb{C}^4 \oplus \mathbb{S}_{(3,1,1)}\mathbb{C}^4 \oplus \mathbb{S}_{(3,2)}\mathbb{C}^4$$

the map $\mathcal{C}_t^{(3,2,1),(1)}$ is described by a 116×4 matrix and it is not advisable to write it down. However the images of the basis $x_1, \ldots, x_4$ of the basis are

$$\begin{aligned} &\mathcal{C}_t^{(3,2,1),(1)}(x_1) = e_2 \wedge e_3 \otimes (e_1 \wedge e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_1) \\ &\mathcal{C}_t^{(3,2,1),(1)}(x_1) = -e_1 \wedge e_3 \otimes (e_1 \wedge e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_1) \\ &\mathcal{C}_t^{(3,2,1),(1)}(x_1) = e_1 \wedge e_2 \otimes (e_1 \wedge e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_1) \\ &\mathcal{C}_t^{(3,2,1),(1)}(x_4) = 0. \end{aligned}$$

The first three images are clearly linearly independent and hence the rank of the catalecticant map is 3. Moreover $\ker \mathcal{C}_t^{(3,2,1),(1)} = \langle x_4 \rangle$. We have found a linear form in $t^\perp$ and hence we still could not declare that the $(2,1)$-rank of t is 1 or 2. Hence we proceed the haunting looking for squares of linear forms in $t^\perp$. Obviously such squares must be different from x_4^2. For this purpose consider the catalecticant map $\mathcal{C}_t^{(3,2,1),(2)} : \mathbb{S}_{(2)}(\mathbb{C}^4)^* \to \mathbb{S}_{(3,2,1)/(2)}\mathbb{C}^4$. The module $\mathbb{S}_{(3,2,1)/(2)}\mathbb{C}^4$ is reducible and can be written as

$$\mathbb{S}_{(3,2,1)/(2)}\mathbb{C}^4 \simeq \mathbb{S}_{(2,1,1)}\mathbb{C}^4 \oplus \mathbb{S}_{(2,2)}\mathbb{C}^4 \oplus \mathbb{S}_{(3,1)}\mathbb{C}^4$$

and hence the $\mathcal{C}^{(3,2,1),(2)}$ is a 80×64 matrix. Since it is not handy to write such a matrix we argue the research of such square as we have done above. Consider the induced basis of $\mathbb{S}_{(2)}(\mathbb{C}^4)^* = \mathrm{Sym}^4(\mathbb{C}^4)^*$ given by $\{x_1^2, x_1x_2, \ldots, x_4^2\}$. One can see that the only elements of the basis which are not sent to zero are $x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2$. Their images are

$$\begin{aligned}
&\mathcal{C}_t^{(3,2,1),(2)}(x_1^2) = e_2 \wedge e_3 \otimes e_2 \otimes e_3,\\
&\mathcal{C}_t^{(3,2,1),(2)}(x_1x_2) = -e_2 \wedge e_3 \otimes e_1 \otimes e_3 - e_1 \wedge e_3 \otimes e_2 \otimes e_3 - e_2 \wedge e_3 \otimes e_3 \otimes e_1,\\
&\mathcal{C}_t^{(3,2,1),(2)}(x_1x_3) = e_1 \wedge e_2 \otimes e_2 \otimes e_3 + e_2 \wedge e_3 \otimes e_2 \otimes e_1,\\
&\mathcal{C}_t^{(3,2,1),(2)}(x_2^2) = e_1 \wedge e_3 \otimes e_1 \otimes e_3 + e_1 \wedge e_3 \otimes e_3 \otimes e_1,\\
&\mathcal{C}_t^{(3,2,1),(2)}(x_2x_3) = e_1 \wedge e_3 \otimes e_2 \otimes e_1 - e_1 \wedge e_2 \otimes e_1 \otimes e_3 - e_1 \wedge e_2 \otimes e_3 \otimes e_1,\\
&\mathcal{C}_t^{(3,2,1),(2)}(x_3^2) = e_1 \wedge e_2 \otimes e_2 \otimes e_1,
\end{aligned}$$

and one can see that such images are linearly independent. Hence the rank of $\mathcal{C}_t^{(3,2,1),(2)}$ is 6 and $\ker \mathcal{C}_t^{(3,2,1),(2)} = \langle x_1x_4, x_2x_4, x_3x_4, x_4^2\rangle$. Finally one can easily see that there are no squares of linear forms in this kernel but x_4^2. Since we have excluded this case, we conclude that t has $(3,2,1)$-rank 1.

4 Expected algorithms

This section is devoted to the computational part of the talk regarding algorithms able to detect tensors of small rank. We do expect that this part will be ready by the time of the MEGA conference. As suggested by the examples of the previous section, by the Lemma (3.4) it is enough to find an ideal of points inside the orthogonal of an element to detect a possible decomposition of it. However this is not always easy because the ideal of points are intersection of ideals and sometimes it can be a bit tricky to find their generators. In any case, as depicted by the example above we may be able to discriminate whether a tensor has small rank or not and an algorithm probably will follow.

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