EQUATIONS AND MULTIDEGREES FOR INVERSE SYMMETRIC MATRIX PAIRS

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We compute the equations and multidegrees of the biprojective variety that parametrizes pairs of symmetric matrices that are inverse to each other. As a consequence of our work, we provide an alternative proof for a result of Manivel, Michałek, Monin, Seynnaeve and Vodička that settles a previous conjecture of Sturmfels and Uhler regarding the polynomiality of maximum likelihood degree.

1. Introduction

The purpose of this paper is to study the biprojective variety that parametrizes pairs of symmetric matrices that are inverse to each other. Let $\mathbb{S}^n$ be the space of symmetric $n \times n$ matrices over the complex numbers $\mathbb{C}$. Let $\mathbb{P}^{m-1}$ be the projectivization $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$ of $\mathbb{S}^n$, where $m = \binom{n+1}{2}$. We are interested in the biprojective variety $\Gamma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ given as follows

$$\Gamma := \overline{\left\{(M, M^{-1}) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1};$$

i.e., the closure of all possible pairs of an invertible symmetric matrix and its inverse.

Our main results are determining the equations and multidegrees of the biprojective variety Γ. Before presenting them, we establish some notation.

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Let $X = (X_{i,j})_{1 \le i,j \le n}$ and $Y = (Y_{i,j})_{1 \le i,j \le n}$ be generic symmetric matrices; i.e., $X_{i,j}$ and $Y_{i,j}$ are new variables over $\mathbb{C}$. Let R be the standard graded polynomial ring $R = \mathbb{C}[X_{i,j}]$, and S be the standard bigraded polynomial ring $S = \mathbb{C}[X_{i,j}, Y_{i,j}]$ where $\mathrm{bideg}(X_{i,j}) = (1,0)$ and $\mathrm{bideg}(Y_{i,j}) = (0,1)$.

Let $\mathfrak{J} \subset S$ be ideal of the defining equations of Γ.

As $\dim(\Gamma) = m-1$, for each $i, j \in \mathbb{N}$ with $i + j = m-1$, one considers the *multidegree* $\deg^{i,j}(\Gamma)$ *of* Γ *of type* (i,j). Geometrically, $\deg^{i,j}(\Gamma)$ equals the number of points in the intersection of Γ with the product $L \times M \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, where $L \subset \mathbb{P}^{m-1}$ and $M \subset \mathbb{P}^{m-1}$ are general linear subspaces of dimension $m-1-i$ and $m-1-j$, respectively. Following the notation of [17, §8.5], we say that the *multidegree polynomial* of Γ is given by

$$\mathcal{C}(\Gamma; t_1, t_2) \; := \sum_{i+j=m-1} \deg^{i,j}(\Gamma)\, t_1^{m-1-i} t_2^{m-1-j} \; \in \; \mathbb{N}[t_1, t_2]$$

(also, see [2, Theorem A], [1, Remark 2.9]).

A fundamental idea in our approach is to reduce the study of Γ to instead considering the biprojective variety of pairs of symmetric matrices with product zero. Let $\Sigma \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ be the biprojective variety parametrized by pairs of symmetric matrices with product zero; i.e., by pairs of symmetric matrices $(M,N) \in \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ such that $MN = 0$. The ideal of defining equations of Σ is clearly given by

$$I_1(XY),$$

where $I_1(XY)$ denotes the ideal generated by the 1×1-minors (i.e, the entries) of the matrix XY. Similarly, since $\dim(\Sigma) = m-2$, we define the multidegree polynomial

$$\mathcal{C}(\Sigma; t_1, t_2) \; := \sum_{i+j=m-2} \deg^{i,j}(\Sigma)\, t_1^{m-1-i} t_2^{m-1-j} \; \in \; \mathbb{N}[t_1, t_2]$$

of Σ.

The theorem below provides the defining equations of Γ. It also shows that the study of $\mathcal{C}(\Gamma; t_1, t_2)$ can be substituted to considering $\mathcal{C}(\Sigma; t_1, t_2)$ instead. Our proof depends on translating our questions in terms of Rees algebras and on using the results of Kotsev [11].

Theorem A. Under the above notations, the following statements hold:

(i) $\mathfrak{J}$ is a prime ideal given by

$$\mathfrak{J} = I_1\left(XY - b\mathrm{Id}_n\right) = \begin{pmatrix} \sum_{k=1}^n X_{i,k} Y_{k,j}, & 1 \le i \ne j \le n \\ \sum_{k=1}^n X_{i,k} Y_{k,i} - \sum_{k=1}^n X_{j,k} Y_{k,j}, & 1 \le i, j \le n \end{pmatrix}$$

where $b = (XY)_{1,1} = \sum_{l=1}^n X_{1,k} Y_{k,1} \in S$ and Id_n denotes the $n \times n$ identity matrix.

(ii) We have the following equality relating multidegree polynomials

$$t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2) \;=\; (t_1 + t_2) \cdot \mathcal{C}(\Gamma; t_1, t_2).$$

Our second main result is obtaining general formulas for the multidegrees of Γ and Σ. Here our approach depends on previous computations that were made by Nie, Ranestad and Sturmfels [18], and by von Bothmer and Ranestad [6]. The formula we obtained is expressed in terms of a function on subsequences of $\{1, \dots, n\}$. Let

$$\psi_i = 2^{i-1}, \quad \psi_{i,j} = \sum_{k=i}^{j-1} \binom{i+j-2}{k} \quad \text{when } i < j,$$

and for any $\alpha = (\alpha_1, \dots, \alpha_r) \subset \{1, \dots, n\}$ let

$$\psi_\alpha = \begin{cases} \mathrm{Pf}\,(\psi_{\alpha_k, \alpha_l})_{1 \le k < l \le n} & \text{if } r \text{ is even}, \\ \mathrm{Pf}\,(\psi_{\alpha_k, \alpha_l})_{0 \le k < l \le n} & \text{if } r \text{ is odd}, \end{cases}$$

where $\psi_{\alpha_0, \alpha_k} = \psi_{\alpha_k}$ and Pf denotes the Pfaffian. For any $\alpha \subset \{1, \dots, n\}$, the complement $\{1, \dots, n\} \setminus \alpha$ is denoted by α^c. By an abuse of notation we set $\psi_\emptyset = 1$.

Theorem B. Under the above notations, the following statements hold:

(i) The multidegree polynomial of Σ is determined by the equation

$$t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2) \;=\; \sum_{d=0}^{m} \beta(n,d)\, t_1^{m-d} t_2^d,$$

where

$$\beta(n,d) := \sum_{\substack{\alpha \subset \{1,\dots,n\} \\ ||\alpha|| = d}} \psi_\alpha \psi_{\alpha^c};$$

in the last sum α runs over all strictly increasing subsequences of $\{1, \dots, n\}$, including the case $\alpha = \emptyset$, and $||\alpha||$ denotes the sum of the entries of α.

(ii) For each $0 \le d \le m-1$, we have the equality

$$\deg^{m-1-d,d}(\Gamma) \;=\; \sum_{j=0}^{d} (-1)^j \beta(n, d-j).$$

Our last interest is on the maximum likelihood degree (ML-degree) of the general linear concentration model (see [19], [16] for more details). Let $\mathcal{L}$ be a general linear subspace of dimension d in $\mathbb{S}^n$, and denote by $\mathcal{L}^{-1}$ the $(d-1)$-dimensional projective subvariety of $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$ obtained by inverting the matrices in $\mathcal{L}$. From [19, Theorem 1], the ML-degree of the general linear concentration model, denoted as $\phi(n,d)$, is equal to the degree of the projective variety $\mathcal{L}^{-1}$. From the way Γ is defined, it then follows that

$$\phi(n,d) = \deg^{m-d,d-1}(\Gamma). \tag{1}$$

So, the computation of the invariants $\phi(n,d)$ can be reduced to determining the multidegrees of Γ (which we did in Theorem B).

Finally, by using Theorem B and a result of Manivel, Michałek, Monin, Seynnaeve and Vodička regarding the polynomiality in n of the function $\psi_{\{1,\ldots,n\}\setminus\alpha}$ (see Theorem 4.1), we obtain an alternative proof to a previous conjecture of Sturmfels and Uhler (see [19, p. 611]).

Corollary C (Manivel-Michałek-Monin-Seynnaeve-Vodička; [13, Theorem 1.3])**.** For each $d \geq 1$, the function $\phi(n,d)$ coincides with a polynomial of degree $d-1$ in n.

The basic outline of this paper is as follows. In Section 2, we compute the defining equations of Γ. In Section 3, we determine the multidegrees of Γ and Σ. In Section 4, we show the polynomiality of $\phi(n,d)$.

2. The defining equations of Γ

During this section, we compute the defining equations of the variety Γ. The following setup is used throughout the rest of this paper.

Setup 2.1. Let $X = (X_{i,j})_{1\leq i,j\leq n}$ and $Y = (Y_{i,j})_{1\leq i,j\leq n}$ be generic symmetric matrices over $\mathbb{C}$. Let R be the standard graded polynomial ring $R = \mathbb{C}[X_{i,j}]$, and S be the standard bigraded polynomial ring $S = \mathbb{C}[X_{i,j}, Y_{i,j}]$ where $\operatorname{bideg}(X_{i,j}) = (1,0)$ and $\operatorname{bideg}(Y_{i,j}) = (0,1)$. Let $I = I_{n-1}(X)$ be the ideal of $(n-1)\times(n-1)$-minors of X. Let t be a new indeterminate. The Rees algebra $\mathcal{R}(I) := \bigoplus_{n=0}^{\infty} I^n t^n \subset R[t]$ of I can be presented as a quotient of S by using the map

$$\begin{array}{rccc} \Psi : S & \longrightarrow & \mathcal{R}(I) \subset R[t] \\ Y_{i,j} & \mapsto & Z_{i,j}t, \end{array}$$

where $Z_{i,j} \in R$ is the signed minor obtained by deleting i-th row and the j-th column. We set $\operatorname{bideg}(t) = (-n+1,1)$, which implies that Ψ is bihomogeneous of degree zero, and so $\mathcal{R}(I)$ has a natural structure of bigraded S-algebra.

Our point of departure comes from the following simple remarks.

Remark 2.2. For any matrix $M \in \mathbb{S}^n$, we denote its adjoint matrix as M^+. For any $M \in \mathbb{S}^n$ with $\det(M) \neq 0$, since $M^{-1} = \frac{1}{\det(M)} M^+$, it follows that M^{-1} and M^+ represent the same point in $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$. Thus, we have that Γ can be equivalently described as

$$\Gamma = \overline{\left\{(M, M^+) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

Denote by $\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}$ the rational map determined by signed minors $Z_{i,j}$, that is,

$$\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}, \quad (X_{1,1} : X_{1,2} : \cdots : X_{n,n}) \mapsto (Z_{1,1} : Z_{1,2} : \cdots : Z_{n,n}).$$

Therefore, we obtain that Γ coincides with

$$\Gamma = \overline{\operatorname{graph}(\mathcal{F})} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1},$$

the closure of the graph of the rational map $\mathcal{F}$.

Remark 2.3. Notice that $I = I_{n-1}(X)$ by construction is the base ideal of the rational map $\mathcal{F}$ – the ideal generated by a linear system defining the rational map. So, it is a basic result that the Rees algebra $\mathcal{R}(I)$ coincides with the bihomogeneous coordinate ring of the closure of the graph of $\mathcal{F}$. By Remark 2.2, the bihomogeneous coordinate ring of Γ is given by the Rees algebra $\mathcal{R}(I)$. Hence, in geometrical terms, we have the identification

$$\Gamma = \operatorname{BiProj}(\mathcal{R}(I)) \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

In more algebraic terms: the ideal $\mathfrak{J} \subset S$ considered in the Introduction coincides with the defining equations of the Rees algebra, that is, $\mathfrak{J} = \operatorname{Ker}(\Psi)$. For the relations between rational maps and Rees algebras, see, e.g., [3, Section 3].

In general the Rees algebra is a very difficult object to study, but, under the present conditions we shall see that it coincides with the symmetric algebra $\operatorname{Sym}(I)$ of I (i.e., the ideal I is of linear type). So, the main idea is to bypass the Rees algebra and consider the symmetric algebra instead.

From a graded presentation of I

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{(Z_{1,1}, Z_{1,2}, \ldots, Z_{n,n})} I \to 0,$$

the symmetric algebra $\operatorname{Sym}(I)$ automatically gets the presentation

$$\operatorname{Sym}(I) \cong S/I_1\left([Y_{i,j}] \cdot \varphi\right) \tag{2}$$

and obtains a natural structure of bigraded S-algebra (for more details on the symmetric algebra, see, e.g., [4, §A2.3]). In general, we have a canonical exact sequence of bigraded S-modules relating both algebras

$$0 \to \mathcal{K} \to \mathrm{Sym}(I) \to \mathcal{R}(I) \to 0,$$

where $\mathcal{K}$ equals the R-torsion of $\mathrm{Sym}(I)$ (see [15]). However, in the present case, we shall see that $\mathrm{Sym}(I) = \mathcal{R}(I)$.

We are now ready to compute the defining equations of Γ.

Proof of Theorem A (i). Due to Remark 2.3, it suffices to compute the defining equations of the Rees algebra $\mathcal{R}(I)$. From [11, Theorem A] we have that I is of linear type, i.e., the canonical map

$$\mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I)$$

is an isomorphism. So, $\mathfrak{J}$ coincides with the ideal of defining equations of $\mathrm{Sym}(I)$. By using [10] or [5] we obtain an explicit R-free resolution for the ideal I which is of the form $0 \to J_3 \to J_2 \xrightarrow{\varphi} J_1 \to R \to R/I \to 0$. From the presentation φ of I, we obtain the ideal

$$\mathfrak{J} = I_1\left([Y_{i,j}] \cdot \varphi\right)$$

of defining equations of the symmetric algebra (see (2) above). Therefore, $\mathcal{R}(I) = \mathrm{Sym}(I)$ is a bigraded S-algebra presented by the quotient

$$\mathcal{R}(I) = \mathrm{Sym}(I) \cong S/\mathfrak{J},$$

and from the description of φ (the syzygies of I) given in [10] or [5] we obtain

$$\mathfrak{J} = \begin{pmatrix} \sum_{k=1}^{n} X_{i,k}Y_{k,j}, & 1 \le i \neq j \le n \\ \sum_{k=1}^{n} X_{i,k}Y_{k,i} - \sum_{k=1}^{n} X_{j,k}Y_{k,j}, & 1 \le i,j \le n \end{pmatrix}.$$

Finally, it is clear that $\mathfrak{J} = I_1\left(XY - b\mathrm{Id}_n\right)$. □

3. Computation of the multidegrees of Γ

In this section, we concentrate on computing the multidegrees of Γ. The idea is to reduce this computation to instead compute the multidegrees of Σ and then to use previous results obtained in [18] and [6].

For each $r_1, r_2 \in \mathbb{N}$, we define the following ideal

$$J(r_1,r_2) := I_1(XY) + I_{r_1+1}(X) + I_{r_2+1}(Y) \subset S.$$

The following proposition yields a primary decomposition of the ideal $I_1(XY)$ in terms of the ideals $J(r_1,r_2)$. Its proof is easily obtained by using results from [11]. Similarly, the properties of $I_1(XY)$ described below are known in a more geometric language (see [9, Proposition 16] and the references given therein).

Proposition 3.1. The following statements hold:

(i) If $r_1+r_2 \leq n$, then $J(r_1,r_2)$ is a prime ideal.

(ii) The ideal $I_1(XY)$ is equidimensional of dimension $m = \binom{n+1}{2}$ and radical with primary decomposition

$$I_1(XY) = \bigcap_{r=0}^{n} J(r,n-r).$$

Proof. (i) From [11, Proposition 4.5], we have that $B(r_1,r_2) = S/J(r_1,r_2)$ is a domain, so the result is clear.

(ii) By [11, Lemma 4.6], we know that the canonical map

$$\frac{S}{I_1(XY)} \to \prod_{r=0}^{n} \frac{S}{J(r,n-r)}$$

is injective. So, it is clear that $I_1(XY) = \bigcap_{r=0}^{n} J(r,n-r)$. The dimension of the Rees algebra is equal to $\dim(\mathcal{R}(I)) = \dim(R)+1 = m+1$ (see, e.g., [7, Theorem 5.1.4]). By (5), $S/I_1(XY) \cong \mathcal{R}(I)/b\mathcal{R}(I)$, and so Krull's Principal Ideal Theorem (see, e.g., [14, Theorem 13.5]) yields that

$$\dim(S/J(r,n-r)) = \dim(\mathcal{R}(I)) - 1 = m$$

for each $0 \leq r \leq n$. Therefore, the result follows. □

We now recall how to define the multidegree polynomial $\mathcal{C}(\mathcal{R}(I);t_1,t_2)$ of $\mathcal{R}(I)$ by using the Hilbert series of $\mathcal{R}(I)$ (see [17, §8.5]). We can write the Hilbert series

$$\mathrm{Hilb}_{\mathcal{R}(I)}(t_1,t_2) := \sum_{v_1,v_2 \in \mathbb{N}} \dim_{\mathbb{C}}\left([\mathcal{R}(I)]_{v_1,v_2}\right) t_1^{v_1} t_2^{v_2} \in \mathbb{N}[[t_1,t_2]]$$

in the following way

$$\mathrm{Hilb}_{\mathcal{R}(I)}(t_1,t_2) = \frac{K(\mathcal{R}(I);t_1,t_2)}{(1-t_1)^m(1-t_2)^m},$$

where $K(\mathcal{R}(I);t_1,t_2)$ is called the K-polynomial of $\mathcal{R}(I)$ (for instance, by just computing a bigraded free S-resolution of $\mathcal{R}(I)$). Then, we define

$$\mathcal{C}(\mathcal{R}(I);t_1,t_2) := \text{sum of the terms of } K(\mathcal{R}(I);1-t_1,1-t_2) \text{ of degree } = m-1.$$

Additionally, we remark that $m-1$ is the minimal degree of the terms of

$$K\left(\mathcal{R}(I);1-t_1,1-t_2\right).$$

In a similar way, we define the multidegree polynomials

$$\mathcal{C}\big(S/I_1(XY);t_1,t_2\big) \quad \text{and} \quad \mathcal{C}\big(S/J(r,n-r);t_1,t_2\big)$$

for each $0 \le r \le n$.

The multidegrees of the particular cases $S/J(0,n)$ and $S/J(n,0)$ are easily handled by the following remark.

Remark 3.2. Since $J(0,n) = I_1(X) = (X_{i,j})$ and $J(n,0) = I_1(Y) = (Y_{i,j})$, it follows from the definition of multidegrees that

$$\mathcal{C}\big(S/J(0,n);t_1,t_2\big) = t_1^m \quad \text{and} \quad \mathcal{C}\big(S/J(n,0);t_1,t_2\big) = t_2^m.$$

For notational purposes, we denote by $\mathfrak{N} := (X_{i,j}) \cap (Y_{i,j}) \subset S$ the irrelevant ideal in the current biprojective setting. We have the following equivalent descriptions of Γ and Σ in terms of the BiProj construction

$$\Gamma = \operatorname{BiProj}(\mathcal{R}(I)) = \big\{P \in \operatorname{Spec}(\mathcal{R}(I)) \mid P \text{ is bihomogeneous and } P \not\supseteq \mathfrak{N}\mathcal{R}(I)\big\} \tag{3}$$

and

$$\Sigma = \operatorname{BiProj}\left(T\right) = \big\{P \in \operatorname{Spec}\left(T\right) \mid P \text{ is bihomogeneous and } P \not\supseteq \mathfrak{N}T\big\}, \tag{4}$$

where $T = S/I_1(XY)$. For more details on the BiProj construction, the reader is referred to [8, §1].

Next, we have a remark showing that the multidegree polynomial of Γ as introduced before coincides with the multidegree polynomial of the Rees algebra $\mathcal{R}(I)$.

Remark 3.3. Due to (3), the fact that $\left(0 :_{\mathcal{R}(I)} \mathfrak{N}^\infty\right) = 0$ and [1, Remark 2.9], it follows that

$$\mathcal{C}\left(\Gamma;t_1,t_2\right) = \mathcal{C}\left(\mathcal{R}(I);t_1,t_2\right).$$

On the other hand, the following remark shows that the multidegree polynomials of Σ and $S/I_1(XY)$ do not agree. Indeed, the minimal primes $J(0,n)$ and $J(n,0)$ of $I_1(XY)$ are irrelevant from a geometric point of view, and so they are taken into account in the multidegree polynomial of $I_1(XY)$ but not in the one of Σ.

Remark 3.4. For ease of notation, set $T = S/I_1(XY)$. Directly from (4), we get that

$$\Sigma = \mathrm{BiProj}(T) = \mathrm{BiProj}\left(\frac{T}{(0:_T \mathfrak{N}^\infty)}\right) = \mathrm{BiProj}\left(\frac{S}{\bigcap_{r=1}^{n-1} J(r,n-r)}\right).$$

Let $T' = S/\bigcap_{r=1}^{n-1} J(r,n-r)$. Thus, since $(0:_{T'} \mathfrak{N}^\infty) = 0$, [1, Remark 2.9] yields the equality $\mathcal{C}(\Sigma;t_1,t_2) = \mathcal{C}(T';t_1,t_2)$. Therefore, from Proposition 3.1, Remark 3.2 and the additivity of multidegrees (see [17, Theorem 8.53]) we obtain the equality

$$\mathcal{C}(S/I_1(XY);t_1,t_2) = t_1^m + t_2^m + \mathcal{C}(\Sigma;t_1,t_2).$$

The next result provides an important relation between the multidegrees of Γ and Σ.

Proof of Theorem A (ii). First, we note the following trivial equality

$$\mathfrak{J} + bS = I_1(XY - b\mathrm{Id}_n) + bS = I_1(XY).$$

As $\mathcal{R}(I) \cong S/\mathfrak{J}$ is clearly a domain and $\mathrm{bideg}(b) = (1,1)$, we obtain the short exact sequence

$$0 \to \mathcal{R}(I)(-1,-1) \xrightarrow{\cdot b} \mathcal{R}(I) \to S/I_1(XY) \to 0 \tag{5}$$

and that $\dim(S/I_1(XY)) = \dim(\mathcal{R}(I)) - 1$. Consequently, we get the following equality relating Hilbert series

$$\mathrm{Hilb}_{S/I_1(XY)}(t_1,t_2) = (1-t_1t_2)\cdot \mathrm{Hilb}_{\mathcal{R}(I)}(t_1,t_2).$$

It then follows that $K(S/I_1(XY);t_1,t_2) = (1-t_1t_2)\cdot K(\mathcal{R}(I);t_1,t_2)$, and the substitutions $t_1 \mapsto 1-t_1, t_2 \mapsto 1-t_2$ yield the equation

$$K(S/I_1(XY);1-t_1,1-t_2) = (t_1+t_2-t_1t_2)\cdot K(\mathcal{R}(I);1-t_1,1-t_2).$$

By choosing the terms of minimal degree in both sides of the last equation, we obtain

$$\mathcal{C}(S/I_1(XY);t_1,t_2) = (t_1+t_2)\cdot \mathcal{C}(\mathcal{R}(I);t_1,t_2),$$

and so the result follows Remark 3.3 and Remark 3.4. □

In [18] it was introduced the notion of algebraic degree of semidefinite programming. By using [18, Theorem 10], these invariants can be seen as the multidegrees of $S/J(r,n-r)$ for $0 < r < n$.

Theorem 3.5 (Nie - Ranestad - Sturmfels; [18, Theorem 10])**.** For $0 < r < n$, we have that

$$\mathcal{C}\big(S/J(n-r,r);t_1,t_2\big) \;=\; \sum_{d=0}^{m} \delta(d,n,r)\, t_1^{m-d} t_2^{d},$$

where $\delta(d,n,r)$ denotes the algebraic degree of semidefinite programming.

We now present the following explicit formula for the algebraic degree of semidefinite programming that was obtained in [6].

Theorem 3.6 (von Bothmer - Ranestad; [6, Theorem 1.1])**.** The algebraic degree of semidefinite programming is equal to

$$\delta(d,n,r) \;=\; \sum_{\alpha} \psi_\alpha \psi_{\alpha^c},$$

where the sum runs over all strictly increasing subsequences $\alpha = \{\alpha_1,\ldots,\alpha_{n-r}\}$ of $\{1,\ldots,n\}$ of length $n-r$ and sum $\alpha_1+\cdots+\alpha_r = d$, and α^c is the complement $\{1,\ldots,n\}\setminus\alpha$.

After the previous discussions, we can now compute the multidegrees of Γ.

Proof of Theorem B. (i) First, we concentrate on computing the multidegrees of $S/I_1(XY)$. By using the additivity of multidegrees (see [17, Theorem 8.53]) together with Proposition 3.1, we obtain the following equality

$$\mathcal{C}\big(S/I_1(XY)\big) \;=\; \sum_{r=0}^{n} \mathcal{C}\big(S/J(r,n-r);t_1,t_2\big).$$

Hence, by combining Theorem 3.5, Remark 3.2 and Theorem 3.6 it follows that

$$\begin{aligned}\mathcal{C}\big(S/I_1(XY);t_1,t_2\big) &= t_1^m + t_2^m + \sum_{r=1}^{n-1}\sum_{d=0}^{m} \delta(d,n,r)\, t_1^{m-d}t_2^{d}\\ &= \sum_{d=0}^{m}\left(\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\ ||\alpha||=d}} \psi_\alpha\psi_{\alpha^c}\right) t_1^{m-d}t_2^{d},\end{aligned}$$

where in the last equation α runs over all strictly increasing subsequences of $\{1,\ldots,n\}$, including the case $\alpha=\emptyset$, and $||\alpha||$ denotes the sum of the entries of α. Notice that $\psi_{\{1,\ldots,n\}} = 1$ (see [12, Proposition A.15]) and that by an abuse of notation we are setting $\psi_\emptyset = 1$. Finally, by setting

$$\beta(n,d) \;=\; \sum_{\substack{\alpha\subset\{1,\ldots,n\}\\ ||\alpha||=d}} \psi_\alpha\psi_{\alpha^c},$$

the result of this part follows from Remark 3.4.

(ii) Notice that Theorem A (ii) yields the equation

$$\beta(n,d) = \deg^{d,m-1-d}(\Gamma) + \deg^{d-1,m-d}(\Gamma).$$

Since the ideal $\mathfrak{J}$ of defining equations of $\mathcal{R}(I)$ is symmetric under swapping the variables $X_{i,j}$ and $Y_{i,j}$, it follows that $\deg^{d,m-1-d}(\Gamma) = \deg^{m-1-d,d}(\Gamma)$ for all $0 \le d \le m-1$. Accordingly, we have the equality

$$\beta(n,d) = \deg^{m-1-d,d}(\Gamma) + \deg^{m-d,d-1}(\Gamma).$$

Therefore, the equation $\deg^{m-1-d,d}(\Gamma) = \sum_{j=0}^{d} (-1)^j \beta(n,d-j)$ is obtained iteratively. $\square$

4. Polynomiality of ML-degree

During this short section, we show Corollary C. Our proof is an easy consequence of Theorem B and the following result.

Theorem 4.1 (Manivel-Michałek-Monin-Seynnaeve-Vodička; [13, Theorem 4.3])**.** Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a strictly increasing subsequence of $\{1,\ldots,n\}$. For $n \ge 0$ the function

$$P_\alpha(n) := \begin{cases} \psi_{\{1,\ldots,n\}\setminus\alpha} & \text{if } \alpha \subset \{1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$$

is a polynomial in n of degree $||\alpha|| = \alpha_1 + \cdots + \alpha_r$.

Finally, we provide our proof for the polynomiality of $\phi(n,d)$.

Proof of Corollary C. By using Theorem B (ii) and (1) we obtain the equation

$$\phi(n,d) = \deg^{m-d,d-1}(\Gamma) = \sum_{j=0}^{d-1} (-1)^j \beta(n,d-1-j).$$

Therefore, it suffices to show that

$$\beta(n,d) = \sum_{\substack{\alpha\subset\{1,\ldots,n\} \\ ||\alpha||=d}} \psi_\alpha \psi_{\alpha^c}$$

in n of degree d. Since ψ_α does not depend on n, the result follows directly from Theorem 4.1. $\square$

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