SYMMETRIES IN AM/GM-BASED OPTIMIZATION

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Abstract. The arithmetic mean/geometric mean-inequality (AM/GM-inequality) facilitates classes of non-negativity certificates and of relaxation techniques for polynomials and, more general, for exponential sums. These certificates are known under the acronyms SONC (sums of non-negative circuit polynomials) and SAGE (sums of arithmetic-geometric exponentials). Here, we present a first systematic study of the AM/GM-based techniques in the presence of symmetries under the action of a permutation group. As a primary ingredient, we prove a symmetry-adapted representation theorem and develop techniques to reduce the size of the resulting relative entropy programs. We study non-convexity phenomena of the set of minimal points of SONC polynomials and SAGE exponentials and we show that the cones of symmetric SONC polynomials and symmetric SAGE exponentials differ from the corresponding non-negativity cones already in quite restricted situations.

1. Introduction

Deciding whether a real function only takes non-negative values is a fundamental question in real algebraic geometry. Non-negativity certificates and optimization approaches are tightly related to each other by observing that the infimum f^* of a function $f : \mathbb{R}^n \to \mathbb{R}$ can be expressed as the largest $\lambda \in \mathbb{R}$ for which $f - \lambda$ is non-negative on $\mathbb{R}^n$:

$$f^* = \inf\{f(x) \ : \ x \in \mathbb{R}^n\} \ = \ \sup\{\lambda \in \mathbb{R} \ : \ f - \lambda \text{ is non-negative on } \mathbb{R}^n\}.$$

Both in the context of polynomials and in the broader context of exponential sums, the last years have seen strong interest in non-negativity certificates and optimization techniques based on the arithmetic mean/geometric mean-inequality (AM/GM inequality). More precisely, an exponential sum (or *signomial*) supported on a finite subset $\mathcal{A} \subset \mathbb{R}^n$ is a linear combination $\sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T \cdot x)$ with real coefficients c_α. In particular cases, the non-negativity of the real function defined by an exponential sum can be decided via the arithmetic-geometric mean inequality. For example, for support points $\alpha_0, \ldots, \alpha_m \in \mathbb{R}^n$ and coefficients $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^n_+$ satisfying $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i \alpha_i = \alpha_0$, the exponential sum

$$\sum_{i=1}^m \lambda_i \exp(\alpha_i^T x) - \exp(\alpha_0^T x)$$

is non-negative on $\mathbb{R}^n$ as a consequence of the weighted arithmetic-geometric mean inequality, namely $\sum_{i=1}^m \lambda_i e^{\alpha_i^T x} \geqslant \prod_{i=1}^m (e^{\alpha_i^T x})^{\lambda_i}$. Clearly, sums of such exponential sums are non-negative as well. Note that exponential sums can be seen as a generalization of

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polynomials: when $\mathcal{A} \subset \mathbb{N}^n$, the transformation $x_i = \ln y_i$ gives polynomial functions $y \mapsto \sum_{\alpha \in \mathcal{A}} c_\alpha y^\alpha$ on $\mathbb{R}^n_{>0}$.

These AM/GM-based certificates appear to be particularly useful in sparse settings. In the specialized situation of polynomials, they can be seen as an alternative to nonnegativity certificates based on sums of squares. The ideas of these approaches go back to Reznick [27] and have been recently brought back into the focus of the developments by Pantea, Koeppl, and Craciun [25], Chandrasekaran and Shah [5] ("*SAGE*" cone: *sums of arithmetic-geometric exponentials*) and Iliman and de Wolff [17] ("*SONC*" cone: *sums of non-negative circuit polynomials*), see also [20] for a generalized, uniform framework. The AM/GM certificates can be effectively obtained by relative entropy programming (see [5, 6]), and in restricted settings these relative entropy programs become geometric programs [18]. These techniques have been extended to cover constrained situations, prominently by the work of Murray, Chandrasekaran and Wierman based on partial dualization [23]. This method can also be approached from sublinear circuits, see [24]. Furthermore, in the setting of polynomials, the AM/GM-based approaches can be combined with sums of squares [19]. Other recent approaches to sparse polynomials besides the ones based on the AM/GM inequality can be found in the sparse moment hierarchies [32, 33].

From an algebraic point of view, a problem is *symmetric* when it is invariant under some group action. Symmetries are ubiquitous in the context of polynomials and optimization, since they manifest both in the problem formulation and the solution set. This often allows to reduce the complexity of the corresponding algorithmic questions. Regarding the set of solutions, it was observed by Terquem as early as in 1840 that a symmetric polynomial does not always have a fully symmetric minimizer (see also Waterhouse's survey [34]). However, in many instances, the set of minimizers contains highly symmetric points (see [13, 21, 28, 30]). With respect to problem formulations, symmetry reduction has provided essential advances in many situations (see, for example, [3, 7, 8]), especially in the context of sums of squares (see [2, 4, 14, 15, 26, 29]).

The current paper departs from the question to which extent symmetries can be exploited in AM/GM-based optimization assuming that the problem affords symmetries. We provide a first systematic study of the AM/GM-based approaches in G-invariant situations under the action of a group G. Our focus is on symmetry-adapted representation theorems, on algorithmic symmetry reduction techniques, on the set of minimizers of the symmetric problems and on the cone of symmetric SAGE exponentials and SONC polynomials.

Our contributions. 1. We prove a symmetry-adapted decomposition theorem and develop a symmetry-adapted relative entropy formulation for SONC polynomials and SAGE exponentials in a general G-invariant setting. This adaption reduces the size of the resulting relative entropy programs or geometric programs, see Theorem 3.2, Theorem 3.3 and Corollary 3.5. As revealed by these statements, the gain depends on the orbit structure of the group action.

By a result of Jie Wang [31] in a non-symmetric setting, if there is only one point in the interior of the convex hull of the support of the "positive monomials", then there is equivalence between being a SONC polynomial (or similarly, a SAGE exponential) and being non-negative. Our symmetric decomposition raises the natural question whether a symmetric version of Wang's result applies if a symmetric polynomial has only one orbit of interior points. We can answer this positively in the particular case where the positive part of the support is a symmetric simplex. In this case we have equivalence, and we can find the minimizer of such a polynomial or exponential sum.

2. Our second result departs from the known results that the zero set of a SAGE exponential constitutes a subspace and is therefore convex and that every SONC polynomial with a finite number of zeroes has at most one zero in the positive orthant. In sharp contrast to this, and somewhat surprisingly, we show that even for the rather structured class of SONC polynomials and SAGE exponentials, the minimal solutions of symmetric optimization problems are in general not symmetric. We say that these functions have the minimum *outside the diagonal*, see Theorem 4.5 and Corollary 4.6. Our constructions depart from constructions of non-symmetric SONC polynomials and SAGE exponentials f with non-convex set of minimizers (where the minimal value of f is different from zero).

3. For Newton polytopes which have a single interior support point, the cone of SONC polynomials or SAGE exponentials coincides with the cone of all non-negative polynomials or exponentials with that Newton polytope. It is of general interest to understand under which conditions (such as symmetries), the cone of SONC polynomials or SAGE exponentials coincides with the cone of all non-negative polynomials or exponentials. We prove that already in a very restricted setting of quartic polynomials with two interior support points in the Newton polytope, the cone of symmetric SONC polynomials differs from the cone of all symmetric polynomials with that support. See Theorem 5.1.

4. We evaluate the structural results in the paper in terms of computations. In situations with strong symmetry structure, the number of variables and the number of equations and inequalities becomes substantially smaller. Accordingly, the interior-point solvers underlying the computation of SAGE bounds then show strong reductions of computation time. In various cases, the symmetry-adapted computation succeeds when the conventional SAGE computation fails.

We mostly concentrate on the unconstrained optimization, but the techniques can generally also be extended to the constrained case. See, for example, Theorem 3.9.

The paper is structured as follows. After collecting relevant notions and concepts in Section 2, we study in Section 3 how to characterize and to decide whether a G-symmetric exponential sum is contained in the SAGE cone. This is connected with a specific way of writing sums of arithmetic-geometric exponentials in the presence of a group symmetry. Section 4 addresses the set of minimizers of SONC polynomials and SAGE exponentials, and Section 5 gives some results on the comparison between the symmetric SONC (respectively SAGE) cone and the cone of all non-negative polynomials (respectively exponential sums). Section 6 provides experimental results of an implementation of the symmetry reduction techniques. We conclude the paper in Section 7.

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2. Preliminaries

Throughout the article, we use the notation $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For a finite subset $\mathcal{A} \subset \mathbb{R}^n$, let $\mathbb{R}^{\mathcal{A}}$ be the set of $|\mathcal{A}|$-tuples whose components are indexed by the set $\mathcal{A}$.

The SAGE cone. For a given non-empty finite set $\mathcal{A}$, we consider exponential sums supported on $\mathcal{A}$ as defined in the Introduction. For finite $\mathcal{A} \subset \mathbb{R}^n$, the SAGE cone $C_{\mathrm{SAGE}}(\mathcal{A})$ is defined as

$$C_{\mathrm{SAGE}}(\mathcal{A}) := \sum_{\beta \in \mathcal{A}} C_{\mathrm{AGE}}(\mathcal{A} \setminus \{\beta\}, \beta),$$

where for $\mathcal{A}' := \mathcal{A} \setminus \{\beta\}$

$$C_{\mathrm{AGE}}(\mathcal{A}', \beta) := \Big\{ f = \sum_{\alpha \in \mathcal{A}'} c_\alpha e^{\alpha^T x} + c_\beta e^{\beta^T x} \ : \ c_\alpha \geqslant 0 \text{ for } \alpha \in \mathcal{A}',\ c_\beta \in \mathbb{R},\ f(x) \geqslant 0 \text{ on } \mathbb{R}^n \Big\}$$

denotes the non-negative exponential sums which may only have a negative coefficient in the term indexed by β (see [5]). The elements in these cones are called SAGE signomials and AGE signomials, respectively. The cone $C_{\mathrm{SAGE}}(\mathcal{A})$ is a closed convex cone in $\mathbb{R}^{\mathcal{A}}$ (see [20, Proposition 2.10]).

Membership to this convex cone can be decided in terms of relative entropy programming. For a finite set $\emptyset \neq \mathcal{A} \subset \mathbb{R}^n$, denote by $D : \mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}^{\mathcal{A}}_{>0} \to \mathbb{R}$,

$$D(\nu, \gamma) \ = \ \sum_{\alpha \in \mathcal{A}} \nu_\alpha \ln\left(\frac{\nu_\alpha}{\gamma_\alpha}\right)$$

the *relative entropy function*, which can be extended to $\mathbb{R}^{\mathcal{A}}_+ \times \mathbb{R}^{\mathcal{A}}_+ \to \mathbb{R} \cup \{\infty\}$ via the conventions $0 \cdot \ln \frac{0}{y} = 0$ for $y \geqslant 0$ and $y \cdot \ln \frac{y}{0} = \infty$ for $y > 0$. To decide membership of a given signomial f supported on $\mathcal{A}$ to the SAGE cone, assume that f is written in the form

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T x) + \sum_{\beta \in \mathcal{B}} c_\beta \exp(\beta^T x)$$

with $c_\alpha > 0$ for $\alpha \in \mathcal{A}$ and $c_\beta < 0$ for $\beta \in \mathcal{B}$. In this notation, the overall support set of f is $\mathcal{A} \cup \mathcal{B}$. Accordingly, for disjoint sets $\emptyset \neq \mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, it is convenient to denote by

$$C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B}) := \sum_{\beta \in \mathcal{B}} C_{\mathrm{AGE}}(\mathcal{A} \cup \mathcal{B} \setminus \{\beta\}, \beta) \tag{2.1}$$

the *signed SAGE cone*, which allows negative coefficients only in a certain subset $\mathcal{B}$ of the support $\mathcal{A} \cup \mathcal{B}$. This a common notation in optimization viewpoints [10, 11, 18, 22, 23].

Proposition 2.1 ([22]). *A signomial f belongs to $C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ if and only if for every $\beta \in \mathcal{B}$ there exist $c^{(\beta)} \in \mathbb{R}^{\mathcal{A}}_+$ and $\nu^{(\beta)} \in \mathbb{R}^{\mathcal{A}}_+$ such that*

$$\begin{array}{rcll} \sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\beta)} \alpha & = & (\sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\beta)}) \beta & \text{for } \beta \in \mathcal{B}, \\ D(\nu^{(\beta)}, e \cdot c^{(\beta)}) & \leqslant & c_\beta & \text{for } \beta \in \mathcal{B}, \\ \sum_{\beta \in \mathcal{B}} c_\alpha^{(\beta)} & \leqslant & c_\alpha & \text{for } \alpha \in \mathcal{A}. \end{array}$$

Note that this proposition reflects the statement of Murray, Chandrasekaran and Wierman [22] that every SAGE signomial can be decomposed into AGE signomials in such a way that every term with a negative coefficient only appears in a single AGE signomial.

The SONC cone. Assume the non-empty finite set $\mathcal{A}$ to be contained in $\mathbb{N}^n$. A tuple (A, β) with $A \subset (2\mathbb{N})^n \cap \mathcal{A}$ and $\beta \in \mathbb{N}^n$ is called a *circuit supported on* $\mathcal{A}$ if A is an affinely independent set and $\beta \in \operatorname{relint}(\operatorname{conv} A) \cap \mathcal{A}$, where relint denotes the relative interior of a set. For singleton sets $A = \{\alpha\}$, the sets (A, β) are formally of the form $(\{\alpha\}, \alpha)$. By convention, we write these circuits simply as (α).

For a circuit (A, β), denote by $P_{A,\beta}$ the set of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ whose supports are contained in $A \cup \{\beta\}$ and which are non-negative on $\mathbb{R}^n$ ("non-negative circuit polynomials"). Note that, due to the circuit structure, $c_\alpha > 0$ for all $\alpha \in A$ (see, e.g., [12]). Then the SONC cone with support $\mathcal{A}$ is defined (see [1, 17]) as the Minkowski sum

$$C_{\mathrm{SONC}}(\mathcal{A}) \;=\; \sum_{\substack{(A,\beta) \text{ circuit} \\ \text{supported on } \mathcal{A}}} P_{A,\beta}.$$

A non-zero non-negative circuit polynomial supported on a one-element circuit is a *monomial square*, i.e., a polynomial of the form cx^α with $c > 0$ and $\alpha \in (2\mathbb{N})^n$.

The SAGE cone $C_{\mathrm{SAGE}}(\mathcal{A})$ as well can be represented using circuits (A, β), where the condition "$A \subset \mathcal{A} \cap (2\mathbb{N})^n$" is replaced by $A \subset \mathcal{A} \subset \mathbb{R}^n$ and "$\beta \in \mathbb{N}^n$" simply by $\beta \in \mathbb{R}^n$. Namely, it was shown in [22] that $C_{\mathrm{SAGE}}(\mathcal{A})$ is generated by its subset of signomials supported on circuits. See also [20] for an exact characterization of the extreme rays of $C_{\mathrm{SAGE}}(\mathcal{A})$.

The cone $C_{\mathrm{SONC}}(\mathcal{A})$ is a closed convex cone. Similar to Theorem 2.1, membership in the SONC cone can also be formulated in terms of a relative entropy program.

Remark 2.2. Note that for all statements in this section, there exist similar statements for SONC polynomials. In particular, for $\mathcal{A}, \mathcal{B} \subset (2\mathbb{N})^n$, consider the polynomial

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha + \sum_{\beta \in \mathcal{B}} c_\beta x^\beta,$$

where for all $\alpha \in \mathcal{A}$ we have $c_\alpha > 0$ and for all $\beta \in \mathcal{B}$ we have $\beta \in \mathbb{N}^n \setminus (2\mathbb{N})^n$ or $c_\beta < 0$. For the unconstrained case (and under the name "SAGE polynomial"), a polynomial f is a SONC polynomial if and only if

$$\tilde{f} = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T x) - \sum_{\beta \in \mathcal{B}} |c_\beta| \exp(\beta^T x)$$

is a SAGE exponential [22]. For the relative entropy program in Proposition 2.1, this means that the right hand side of the second condition needs to be replaced by "$-|c_\beta|$" to work as a SONC-certificate.

Optimizing over the SAGE and SONC cones. Since the SAGE cone is contained in the cone of non-negative signomials, relaxing to the SAGE cone gives an approximation of the global infimum f^* of a signomial f supported on $\mathcal{A}$:

$$f^{\mathrm{SAGE}} = \sup\{\lambda \in \mathbb{R} : \ f - \lambda \in C_{\mathrm{SAGE}}(\mathcal{A})\}$$

satisfying $f^{\mathrm{SAGE}} \leqslant f^*$.

We first record the following, which is closely related to the strong duality statement for the SAGE bound in [22, Proposition 2]:

Proposition 2.3. *Let*

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T x) + \sum_{\beta \in \mathcal{B}} c_\beta \exp(\beta^T x)$$

with $c_\alpha > 0$ *for* $\alpha \in \mathcal{A}$. *Assume* $\mathcal{B} \subset \mathrm{relint}(\mathrm{conv}(\mathcal{A} \cup \{(0, \ldots, 0)^T\}))$. *Then* $f^{\mathrm{SAGE}} > -\infty$.

Proof. The finiteness of $\mathcal{B}$ and Proposition 2.1 allow to assume that $|\mathcal{B}| = 1$. We may also assume that $\beta \neq (0, \ldots, 0)^T$. Then, the ray with initial point $(0, \ldots, 0)^T$ and passing through β meets a facet of the convex hull of $\mathcal{A}$ at a point γ. We can express γ as

$$\gamma = \sum_{\alpha \in \mathcal{A}'} \nu'_\alpha \alpha,$$

where $\nu'_\alpha \geqslant 0$ for $\alpha \in \mathcal{A}'$, $\sum_{\alpha \in \mathcal{A}'} \nu'_\alpha = 1$, and $\mathcal{A}' \subset \mathcal{A} \backslash \{(0, \ldots, 0)^T\}$. In turn, since $\beta \neq \gamma$, we can write

$$\beta = \sum_{\alpha \in \mathcal{A}' \cup \{(0,\ldots,0)\}} \nu_\alpha \alpha,$$

where $\nu_\alpha \geqslant 0$, for $\alpha \in \mathcal{A}'$, $\nu_{(0,\ldots,0)} > 0$ and $\sum_{\alpha \in \mathcal{A}' \cup \{(0,\ldots,0)\}} \nu_\alpha = 1$. To conclude, it is enough to verify that for λ small enough, the conditions of Proposition 2.1 are satisfied by $f - \lambda$: the first condition follows from the previous decomposition of β, while the third one is trivially satisfied when $|\mathcal{B}| = 1$. For the second one, we observe that the function

$$l(\lambda) = \sum_{\alpha \in \mathcal{A}'} \nu_\alpha \ln \frac{\nu_\alpha}{e \cdot c_\alpha} + \nu_{(0,\ldots,0)} \ln \frac{\nu_{(0,\ldots,0)}}{e \cdot (c_{(0,\ldots,0)} - \lambda)}$$

tends to $-\infty$ when $\lambda \to -\infty$. Hence, there exists λ such that $l(\lambda) < c_\beta$. □

Remark 2.4. When $\beta \notin \mathrm{conv}(\mathcal{A} \cup \{(0, \ldots, 0)^T\})$, the hyperplane separation theorem implies $\min f = -\infty$, forcing $f^{\mathrm{SAGE}} = -\infty$. If β is on the boundary of $\mathrm{conv}(\mathcal{A} \cup \{(0, \ldots, 0)^T\})$, then we cannot conclude in general. For example, consider the function

$$f(x, y) = \mu + e^{2x} + e^{2y} - \delta e^{x+y}.$$

Then $f^{\mathrm{SAGE}} = \mu$ when $\delta \leqslant 2$, while $f^{\mathrm{SAGE}} = -\infty$ when $\delta > 2$.

In the same spirit, in the corresponding setting for polynomials, we can define f^{SONC}:

$$f^{\mathrm{SONC}} \ = \ \sup\{\lambda \in \mathbb{R} : \ f - \lambda \in C_{\mathrm{SONC}}(\mathcal{A})\}.$$

According to Remark 2.2, f^{SONC} and f^{SAGE} coincide, after the appropriate transformation from the remark. For this reason, from now on we will only consider f^{SAGE}.

Remark 2.5. The finiteness of f^{SAGE} can be seen as an advantage with respect to the sum of squares analogue f^{SOS}. Indeed, the Motzkin polynomial $f = x^4+y^4+x^2+y^2-3x^2y^2+1$ satisfies $f^{\mathrm{SOS}} = -\infty$ while $f^{\mathrm{SAGE}} = f^* = 0$.

Constrained versions. While many aspects of this article are devoted to the unconstrained situation, we briefly collect the extension of SAGE certificates to the constrained situation. Let X be a convex and closed subset of $\mathbb{R}^n$. For a convex set $X \subset \mathbb{R}^n$ and a non-empty finite set $\mathcal{A} \subset \mathbb{R}^n$, the X-SAGE cone $C_X(\mathcal{A})$ is defined (see [23]) as

$$C_X(\mathcal{A}) := \sum_{\beta \in \mathcal{A}} C_X(\mathcal{A} \setminus \{\beta\}, \beta),$$

where for $\mathcal{A}' := \mathcal{A} \setminus \{\beta\}$,

$$C_X(\mathcal{A}', \beta) := \Big\{ f = \sum_{\alpha \in \mathcal{A}'} c_\alpha e^{\alpha^T x} + c_\beta e^{\beta^T x} \ : \ c_\alpha \geqslant 0 \text{ for } \alpha \in \mathcal{A}',\, c_\beta \in \mathbb{R},\, f(x) \geqslant 0 \text{ on } X \Big\}.$$

Moreover, (2.1) can be generalized by defining, for disjoint sets $\emptyset \neq \mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, the *signed X-SAGE cone*

$$C_X(\mathcal{A}, \mathcal{B}) := \sum_{\beta \in \mathcal{B}} C_X(\mathcal{A}, \beta).$$

This is the set of X-SAGE signomials, where negative coefficients are only possible in a certain subset $\mathcal{B}$ of the support $\mathcal{A} \cup \mathcal{B}$. The following decomposition result holds.

Theorem 2.6 ([23], Corollary 5)**.** *If $f \in C_X(\mathcal{A}, \mathcal{B})$ with $c_\alpha > 0$ for all $\alpha \in \mathcal{A}$ and $c_\beta < 0$ for all $\beta \in \mathcal{B} \neq \emptyset$, then there exist X-AGE signomials $f_\beta \in C_X(\mathcal{A}, \beta)$ for $\beta \in \mathcal{B}$ such that $f = \sum_{\beta \in \mathcal{B}} f_\beta$.*

For the constrained approach, a similar result to Proposition 1.1 is known.

Proposition 2.7 ([23])**.** *$f \in C_X(\mathcal{A} \cup \mathcal{B})$ if and only if for every $\beta \in \mathcal{B}$ there exist $c^{(\beta)} \in \mathbb{R}_+^{\mathcal{A}}$ and $\nu^{(\beta)} \in \mathbb{R}_+^{\mathcal{A}}$ such that*

$$\begin{aligned} D(\nu^{(\beta)}, e \cdot c^{(\beta)}) + \sup_{x \in X} \Big(- \sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\beta)} (\alpha - \beta) \Big)^T x \ &\leqslant \ c_\beta \quad \text{for } \beta \in \mathcal{B}, \\ \sum_{\beta \in \mathcal{B}} c_\alpha^{(\beta)} \ &\leqslant \ c_\alpha \quad \text{for } \alpha \in \mathcal{A}. \end{aligned}$$

3. Symmetric decompositions and symmetric relative entropy formulation

In this section, we provide a structural result on the decomposition of symmetric SAGE exponentials as sums of orbits of (non-symmetric) AGE exponentials. Building upon this, we provide a symmetry-adapted relative entropy formulation for containment in the SAGE cone or its variants. These symmetric decompositions and symmetric entropy formulations carry over to SONC polynomials. As an application of the decomposition results, we answer a specific question on symmetric signomials or polynomials which have only one orbit of interior points in the Newton polytope. We show that if the positive support is a symmetric simplex, then, for this support set, the class of SAGE exponentials coincides with the class of all non-negative signomials (and similarly for SONC polynomials).

Let G be a permutation group on the set $[n] := \{1, \ldots, n\}$. Every $\sigma \in G$ acts on $x = (x_1, \ldots, x_n)$ through $\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. On a signomial $f(x) = \sum_\alpha c_\alpha \exp(\alpha^T x)$, this induces the action

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \tag{3.1}$$

Since $\sigma f(x) = \sum_\alpha c_\alpha \exp((\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})^T x)$, the induced action on exponent vectors α is

$$\sigma(\alpha) = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)}). \tag{3.2}$$

Remark 3.1. All these actions are left actions. To verify this for (3.2), observe that $(\sigma\tau)(\alpha) = \sigma(\tau(\alpha)) = \sigma(\alpha_{\tau^{-1}(1)}, \ldots, \alpha_{\tau^{-1}(n)})$. For $\gamma := \tau(\alpha) = (\alpha_{\tau^{-1}(1)}, \ldots, \alpha_{\tau^{-1}(n)})$, we have $\gamma_i = \alpha_{\tau^{-1}(i)}$ and thus $\gamma_{\sigma^{-1}(i)} = \alpha_{\tau^{-1}(\sigma^{-1}(i))}$. Hence

$$\sigma(\tau(\alpha)) = (\alpha_{\tau^{-1}(\sigma^{-1}(1))}, \ldots, \alpha_{\tau^{-1}(\sigma^{-1}(n))}) = (\sigma\tau)(\alpha).$$

For a set $\mathcal{S} \subset \mathbb{R}^n$ of exponent vectors, the *orbit of* $\mathcal{S}$ under G is

$$G \cdot \mathcal{S} = \{\sigma(s) : s \in \mathcal{S}, \, \sigma \in G\}.$$

We call a subset $\hat{\mathcal{S}} \subset \mathcal{S}$ *a set of orbit representatives for* $\mathcal{S}$ if $\hat{\mathcal{S}}$ is an inclusion-minimal set with $(G \cdot \hat{\mathcal{S}}) = \mathcal{S}$. Moreover, let $\operatorname{Stab} \beta := \{\sigma \in G \ : \ \sigma(\beta) = \beta\}$ denote the *stabilizer* of an exponent vector β.

In the following statements, we consider G-invariant signomials f. It is convenient to write f here in the form

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T x) + \sum_{\beta \in \mathcal{B}} c_\beta \exp(\beta^T x) \tag{3.3}$$

with $c_\alpha > 0$ for $\alpha \in \mathcal{A}$ and $c_\beta < 0$ for $\beta \in \mathcal{B}$, i.e., f is an element of the signed SAGE cone $C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ introduced in Section 2. As already mentioned, in this notation, the overall support set of f is $\mathcal{A} \cup \mathcal{B}$.

Theorem 3.2. *Let $X \subset \mathbb{R}^n$ be convex and G-invariant, let f be a G-symmetric signomial of the form* (3.3) *and $\hat{\mathcal{B}}$ be a set of orbit representatives for $\mathcal{B}$. Then $f \in C_X(\mathcal{A}, \mathcal{B})$ if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$, there exist X-AGE signomials $h_{\hat{\beta}} \in C_X(\mathcal{A}, \hat{\beta})$ such that*

$$f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G/\operatorname{Stab}(\hat{\beta})} \rho h_{\hat{\beta}}. \tag{3.4}$$

The functions $h_{\hat{\beta}}$ can be chosen to be invariant under the action of $\operatorname{Stab}(\hat{\beta})$.

Here, $\rho \in G/\operatorname{Stab}(\hat{\beta})$ shortly denotes that ρ runs over a set of representatives of the left quotient space $G/\operatorname{Stab}(\hat{\beta})$, which is defined through the left cosets $\{\sigma \operatorname{Stab}(\hat{\beta}) \ : \ \sigma \in G\}$. We will also use the right quotient space, denoted by $\operatorname{Stab}(\hat{\beta})\backslash G$, further below.

Proof. Since it is clear that a signomial f of the form (3.4) is non-negative, we only have to show the converse direction. Let $f \in C_X(\mathcal{A}, \mathcal{B})$. By Theorem 2.6, there exist X-AGE signomials $f_\beta \in C_X(\mathcal{A}, \beta)$ for $\beta \in \mathcal{B}$, such that $f = \sum_{\beta \in \mathcal{B}} f_\beta$. The G-symmetry of f gives

$$f = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\beta \in \mathcal{B}} \sigma f_\beta. \tag{3.5}$$

The idea is to group in this sum all the σf_β that have the same "possibly negative" term. Since the group actions σf on signomials f and $\sigma\beta$ on exponent vectors β are both left actions by Remark 3.1, we have

$$\sigma \exp(\beta^T x) = \exp((\sigma\beta)^T x),$$

so that the possibly negative term of σf_β is given by $\sigma\beta$. For any $\beta \in \mathcal{B}$, the signomial

$$h_\beta = \frac{1}{|G|} \sum_{\sigma \in G} \sigma f_{\sigma^{-1}\beta}$$

is a sum of X-AGE signomials in $C_X(\mathcal{A}, \beta)$, hence it is contained in $C_X(\mathcal{A}, \beta)$ as well. Moreover, (3.5) can be expressed as

$$f = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\beta \in \mathcal{B}} \sigma f_\beta = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\gamma \in \mathcal{B}} \sigma f_{\sigma^{-1}\gamma} = \sum_{\gamma \in \mathcal{B}} h_\gamma.$$

Let $\beta \in \mathcal{B}$ and $\hat{\beta} \in \hat{\mathcal{B}}$ be the representative of its orbit in $\hat{\mathcal{B}}$. If $\sigma, \tau \in G$ are such that $\sigma(\hat{\beta}) = \tau(\hat{\beta}) = \beta$, then $\tau^{-1}\sigma \in \operatorname{Stab}(\hat{\beta})$ and $\tau = \sigma$ in $G/\operatorname{Stab}(\hat{\beta})$. Hence,

$$f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G/\operatorname{Stab}\hat{\beta}} h_{\rho\hat{\beta}}. \tag{3.6}$$

Now observe that $h_{\rho\beta} = \rho h_\beta$ for every $\beta \in \mathcal{B}$ and $\rho \in G$, because

$$|G|\rho h_\beta = \sum_{\sigma \in G} \rho\sigma f_{\sigma^{-1}\beta} = \sum_{\tau \in G} \tau f_{\tau^{-1}\rho\beta} = |G| h_{\rho\beta}. \tag{3.7}$$

Substituting (3.7) into (3.6) gives $f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G/\operatorname{Stab}\hat{\beta}} \rho h_{\hat{\beta}}$ as desired. Moreover, the $\operatorname{Stab}(\hat{\beta})$-invariance of $h_{\hat{\beta}}$ for $\hat{\beta} \in \hat{\mathcal{B}}$ follows from (3.7). □

Building upon Theorem 3.2, the next theorem gives a symmetry-adapted relative entropy program that certifies non-negativity.

Theorem 3.3. *Let $\hat{\mathcal{B}}$ be a set of orbit representatives for $\mathcal{B}$. A G-symmetric signomial f of the form* (3.3) *is contained in $C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$ there exist*

$c^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}}$ *and* $\nu^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}}$ *, invariant under the action of* $\mathrm{Stab}(\hat{\beta})$*, such that*

$$\sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\hat{\beta})}(\alpha - \hat{\beta}) = 0 \quad \textit{for every } \hat{\beta} \in \hat{\mathcal{B}}, \tag{3.8}$$

$$D(\nu^{(\hat{\beta})}, e \cdot c^{(\hat{\beta})}) \leqslant c_{\hat{\beta}} \quad \textit{for every } \hat{\beta} \in \hat{\mathcal{B}}, \tag{3.9}$$

$$\sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\sigma \in \mathrm{Stab}\,(\hat{\beta}) \backslash G} c_{\sigma(\alpha)}^{(\hat{\beta})} \leqslant c_\alpha \quad \textit{for every } \alpha \in \mathcal{A}. \tag{3.10}$$

Remark 3.4. The right coset condition (3.10) can equivalently be expressed in terms of the left cosets,

$$\sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\sigma \in G / \mathrm{Stab}\,\hat{\beta}} c_{\sigma^{-1}(\alpha)}^{(\hat{\beta})} \leqslant c_\alpha \quad \text{for every } \alpha \in \mathcal{A}.$$

Namely, if $\beta \in \mathcal{B}$, $\hat{\beta} \in \hat{\mathcal{B}}$ and $\sigma, \tau \in G$ are such that $\sigma^{-1}(\hat{\beta}) = \tau^{-1}(\hat{\beta}) = \beta$, then $\tau\sigma^{-1} \in \mathrm{Stab}(\hat{\beta})$ and $\tau = \sigma$ in the right quotient space $\mathrm{Stab}(\hat{\beta}) \backslash G$.

Proof of Theorem 3.3. If f is G-symmetric, then, by Theorem 3.2, there exist $\mathrm{Stab}(\hat{\beta})$-invariant AGE signomials $h_{\hat{\beta}} \in C_X(\mathcal{A}, \hat{\beta})$ for every $\hat{\beta} \in \hat{\mathcal{B}}$ such that

$$f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G / \mathrm{Stab}(\hat{\beta})} \rho h_{\hat{\beta}}.$$

Writing $h_{\hat{\beta}}$ in the form

$$h_{\hat{\beta}} = \sum_{\alpha \in \mathcal{A}} c_\alpha^{(\hat{\beta})} \exp(\alpha^T x) + c_{\hat{\beta}} \exp(\hat{\beta}^T x)$$

with coefficients $c_\alpha^{(\hat{\beta})}$ and $c_{\hat{\beta}}$ for $\alpha \in \mathcal{A}$ and $\hat{\beta} \in \hat{\mathcal{B}}$, the two conditions (3.8) and (3.9) follow from the property $h_{\hat{\beta}} \in C_X(\mathcal{A}, \hat{\beta})$. For (3.10), we observe that for $\alpha \in \mathcal{A}$, the coefficient of $\exp(\alpha^T x)$ in $\rho h_{\hat{\beta}}$ is $c_{\rho^{-1}(\alpha)}^{(\hat{\beta})}$. We obtain inequality (3.10), even with equality, by setting $\sigma := \rho^{-1}$ and summing over $\hat{\beta} \in \hat{\mathcal{B}}$ and over $\sigma \in \mathrm{Stab}(\hat{\beta}) \backslash G$, following Remark 3.4. Moreover, the $\mathrm{Stab}(h_{\hat{\beta}})$-invariance of $h_{\hat{\beta}}$ implies the $\mathrm{Stab}(\hat{\beta})$-invariance of $c^{(\hat{\beta})}$. In order to make $\nu^{(\hat{\beta})}$ invariant under $\mathrm{Stab}(\hat{\beta})$, we can replace it by

$$\mu_\alpha^{(\hat{\beta})} = \frac{1}{|\,\mathrm{Stab}(\hat{\beta})|} \sum_{\sigma \in \mathrm{Stab}(\hat{\beta})} \nu_{\sigma(\alpha)}^{(\hat{\beta})}.$$

Obviously, this has no influence on (3.10). For (3.8), we have

$$
\begin{aligned}
|\operatorname{Stab}(\hat\beta)|\sum_{\alpha\in\mathcal{A}}\mu_\alpha^{(\hat\beta)}(\alpha-\hat\beta) &= \sum_{\alpha\in\mathcal{A}}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\nu_{\sigma(\alpha)}^{(\hat\beta)}(\alpha-\hat\beta)\\
&= \sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\sigma^{-1}\sum_{\alpha\in\mathcal{A}}\nu_{\sigma(\alpha)}^{(\hat\beta)}(\sigma(\alpha)-\sigma(\hat\beta))\\
&= \sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\sigma^{-1}\sum_{\alpha\in\mathcal{A}}\nu_{\alpha}^{(\hat\beta)}(\alpha-\hat\beta)) = 0.
\end{aligned}
$$

Finally, for (3.9), using $c_\alpha^{(\hat\beta)} = c_{\sigma(\alpha)}^{(\hat\beta)}$ for $\sigma\in\operatorname{Stab}(\hat\beta)$ and applying Jensen's inequality on the convex function $x\mapsto x\ln x$ gives, for all $\alpha\in\mathcal{A}$,

$$
\begin{aligned}
\mu_\alpha^{(\hat\beta)}\ln\frac{\mu_\alpha^{(\hat\beta)}}{c_\alpha^{(\hat\beta)}} &= \left(\frac{1}{|\operatorname{Stab}(\hat\beta)|}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\nu_{\sigma(\alpha)}^{(\hat\beta)}\right)\ln\frac{\frac{1}{|\operatorname{Stab}(\hat\beta)|}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\nu_{\sigma(\alpha)}^{(\hat\beta)}}{c_\alpha^{(\hat\beta)}}\\
&= c_\alpha^{(\hat\beta)}\left(\frac{\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\nu_{\sigma(\alpha)}^{(\hat\beta)}/c_{\sigma(\alpha)}^{(\hat\beta)}}{|\operatorname{Stab}(\hat\beta)|}\ln\frac{\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\nu_{\sigma(\alpha)}^{(\hat\beta)}/c_{\sigma(\alpha)}^{(\hat\beta)}}{|\operatorname{Stab}(\hat\beta)|}\right)\\
&\leqslant c_\alpha^{(\hat\beta)}\left(\frac{1}{|\operatorname{Stab}(\hat\beta)|}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\frac{\nu_{\sigma(\alpha)}^{(\hat\beta)}}{c_{\sigma(\alpha)}^{(\hat\beta)}}\ln\frac{\nu_{\sigma(\alpha)}^{(\hat\beta)}}{c_{\sigma(\alpha)}^{(\hat\beta)}}\right).
\end{aligned}
$$

Using again the $\operatorname{Stab}(\hat\beta)$-invariance of $c^{(\hat\beta)}$ and the precondition then yields

$$
\sum_{\alpha\in\mathcal{A}}\mu_\alpha^{(\hat\beta)}\ln\frac{\mu_\alpha^{(\hat\beta)}}{ec_\alpha^{(\hat\beta)}} \leqslant \frac{1}{|\operatorname{Stab}(\hat\beta)|}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}\sum_{\alpha\in\mathcal{A}}\nu_{\sigma(\alpha)}^{(\hat\beta)}\ln\frac{\nu_{\sigma(\alpha)}^{(\hat\beta)}}{ec_{\sigma(\alpha)}^{(\hat\beta)}} \leqslant \frac{1}{|\operatorname{Stab}(\hat\beta)|}\sum_{\sigma\in\operatorname{Stab}(\hat\beta)}c_{\hat\beta} = c_{\hat\beta}.
$$

Conversely, assume that $c^{(\hat\beta)}$ and $\nu^{(\hat\beta)}$, invariant under the action of $\operatorname{Stab}(\hat\beta)$, satisfy (3.8)–(3.10). Let $\beta\in\mathcal{B}$ and $\hat\beta\in\hat{\mathcal{B}}$ be the representative of its orbit in $\hat{\mathcal{B}}$. If $\sigma,\tau\in G$ are such that $\sigma(\beta)=\tau(\beta)=\hat\beta$, then $\tau\sigma^{-1}\in\operatorname{Stab}(\hat\beta)$ and $\tau=\sigma$ in $\operatorname{Stab}(\hat\beta)\backslash G$. Since $c^{(\hat\beta)}$ and $\nu^{(\hat\beta)}$ are invariant under $\operatorname{Stab}(\hat\beta)$, we have

$$
c_{\tau(\alpha)}^{(\hat\beta)} = c_{\sigma(\alpha)}^{(\hat\beta)},\quad \nu_{\tau(\alpha)}^{(\hat\beta)} = \nu_{\sigma(\alpha)}^{(\hat\beta)}\quad \text{for } \alpha\in\mathcal{A}.
$$

Thus we can define

$$
c_{\alpha}^{(\beta)} = c_{\sigma(\alpha)}^{(\hat\beta)},\quad \nu_{\alpha}^{(\beta)} = \nu_{\sigma(\alpha)}^{(\hat\beta)}\quad \text{for } \alpha\in\mathcal{A},
$$

which is independent of σ such that $\sigma(\beta)=\hat\beta$. As a consequence, if $\tau\in\operatorname{Stab}(\hat\beta)\backslash G$, then $c_\alpha^{(\tau^{-1}(\hat\beta))} = c_{\tau(\alpha)}^{(\hat\beta)}$ is well defined.

To see that the first conditions of Proposition 2.1 are satisfied, let $\beta \in \mathcal{B}$ and $\sigma \in G$ such that $\sigma(\beta) = \hat{\beta}$. Then

$$\begin{aligned}\sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\beta)}(\alpha - \beta) &= \sum_{\alpha \in \mathcal{A}} \nu_{\sigma(\alpha)}^{(\hat{\beta})}(\alpha - \sigma^{-1}\hat{\beta}) \\ &= \sigma^{-1} \sum_{\alpha \in \mathcal{A}} \nu_{\sigma(\alpha)}^{(\hat{\beta})}(\sigma(\alpha) - \hat{\beta}) = \sigma^{-1} \sum_{\alpha \in \mathcal{A}} \nu_\alpha^{(\hat{\beta})}(\alpha - \hat{\beta}) = 0\end{aligned}$$

$$\text{and} \quad D(\nu^{(\beta)}, ec^{(\beta)}) = D(\nu^{(\hat{\beta})}, ec^{(\hat{\beta})}) \leqslant c_{\hat{\beta}} = c_\beta.$$

For the third condition of Proposition 2.1, we obtain

$$\sum_{\beta \in \mathcal{B}} c_\alpha^{(\beta)} = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\tau \in \operatorname{Stab}(\hat{\beta}) \backslash G} c_\alpha^{(\tau^{-1}\hat{\beta})} = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\tau \in \operatorname{Stab}(\hat{\beta}) \backslash G} c_{\tau(\alpha)}^{(\hat{\beta})} \leqslant c_\alpha,$$

which altogether shows that $f \in C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$. □

The following consequence of Theorem 3.3 further reduces the number of variables in the relative entropy program, since a certain number of $c_\alpha^{(\hat{\beta})}$ and $\nu_\alpha^{(\hat{\beta})}$ are actually equal, and we can take each $c^{(\hat{\beta})}, \nu^{(\hat{\beta})}$ in the ground set $\mathbb{R}_+^{\mathcal{A}/\operatorname{Stab}(\hat{\beta})}$.

Corollary 3.5. *Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be a set of orbit representatives for $\mathcal{A}$ and $\mathcal{B}$. A G-symmetric signomial f of the form* (3.3) *is contained in $C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$ there exist $c^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}/\operatorname{Stab}(\hat{\beta})}$ and $\nu^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}/\operatorname{Stab}(\hat{\beta})}$ such that*

$$\sum_{\alpha \in \mathcal{A}/\operatorname{Stab}(\hat{\beta})} \nu_\alpha^{(\hat{\beta})} \sum_{\alpha' \in \operatorname{Stab}(\hat{\beta}) \cdot \alpha} (\alpha' - \hat{\beta}) \;=\; 0 \quad \textit{for every } \hat{\beta} \in \hat{\mathcal{B}}, \tag{3.11}$$

$$\sum_{\alpha \in \mathcal{A}/\operatorname{Stab}(\hat{\beta})} \left|\operatorname{Stab}(\hat{\beta}) \cdot \alpha\right| \nu_\alpha^{(\hat{\beta})} \ln \frac{\nu_\alpha^{(\hat{\beta})}}{e c_\alpha^{(\hat{\beta})}} \;\leqslant\; c_{\hat{\beta}} \quad \textit{for every } \hat{\beta} \in \hat{\mathcal{B}}, \tag{3.12}$$

$$\sum_{\hat{\beta} \in \hat{\mathcal{B}}} \frac{|\operatorname{Stab}(\alpha)|}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\gamma \in (G\cdot\alpha)/\operatorname{Stab}(\hat{\beta})} \left|\operatorname{Stab}(\hat{\beta}) \cdot \gamma\right| c_\gamma^{(\hat{\beta})} \;\leqslant\; c_\alpha \quad \textit{for every } \alpha \in \hat{\mathcal{A}}. \tag{3.13}$$

Proof. For (3.11) and (3.12), equivalence to their versions in Theorem 3.3 is straightforward to check. For (3.13), equivalence to (3.10) follows by observing that for every $\alpha \in \mathcal{A}$

$$\begin{aligned}\sum_{\sigma \in \operatorname{Stab}(\hat{\beta}) \backslash G} c_{\sigma(\alpha)}^{(\hat{\beta})} &= \sum_{\sigma \in \operatorname{Stab}(\hat{\beta}) \backslash G} \frac{1}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\tau \in \operatorname{Stab}(\hat{\beta})} c_{\tau(\sigma(\alpha))}^{(\hat{\beta})} = \frac{1}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\rho \in G} c_{\rho(\alpha)}^{(\hat{\beta})} \\ &= \frac{|\operatorname{Stab}(\alpha)|}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\gamma \in G\cdot\alpha} c_\gamma^{(\hat{\beta})} = \frac{|\operatorname{Stab}(\alpha)|}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\gamma \in (G\cdot\alpha)/\operatorname{Stab}(\hat{\beta})} \left|\operatorname{Stab}(\hat{\beta}) \cdot \gamma\right| c_\gamma^{(\hat{\beta})},\end{aligned}$$

and the last expression only depends on the orbit $G \cdot \alpha$ rather than on α itself. □

Example 3.6. Consider the support set $\{\alpha_0, \ldots, \alpha_7\} = \{(0,0,0)^T, (7,0,0)^T, (0,7,0)^T, (0,0,7)^T, (1,1,2)^T, (1,2,1)^T, (2,1,1)^T, (2,2,2)^T\}$ and let $G := \mathcal{S}_3$ be the symmetric group on three elements. In order to avoid too heavy notation, we will write $c_j^{(i)}$ instead of $c_{\alpha_j}^{(\alpha_i)}$ and $\nu_j^{(i)}$ instead of $\nu_{\alpha_j}^{(\alpha_i)}$. Consider a signomial

$$f(x_1, x_2, x_3) = \sum_{i=0}^{7} c_i e^{\alpha_i^T(x_1,x_2,x_3)},$$

with $c_0, c_1, c_2, c_3 > 0$ and $c_4, c_5, c_6, c_7 < 0$, i.e., set $\mathcal{A} = \{\alpha_0, \ldots, \alpha_3\}$, $\mathcal{B} = \{\alpha_4, \ldots, \alpha_7\}$. Then $\hat{\mathcal{A}} = \{\alpha_0, \alpha_1\}$ and $\hat{\mathcal{B}} = \{\alpha_4, \alpha_7\}$ are sets of orbit representatives. By Corollary 3.5, $f \in C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ if and only there exist $c^{(4)} = (c_0^{(4)}, c_1^{(4)}, c_3^{(4)})$, $\nu^{(4)} = (\nu_0^{(4)}, \nu_1^{(4)}, \nu_3^{(4)})$, $c^{(7)} = (c_0^{(7)}, c_1^{(7)})$ and $\nu^{(7)} = (\nu_0^{(7)}, \nu_1^{(7)})$ which satisfy

$$\begin{aligned}
\nu_0^{(4)}(\alpha_0 - \alpha_4) + \nu_1^{(4)}(\alpha_1 + \alpha_2 - 2\alpha_4) + \nu_3^{(4)}(\alpha_3 - \alpha_4) &= 0, \\
\nu_0^{(7)}(\alpha_0 - \alpha_7) + \nu_1^{(7)}(\alpha_1 + \alpha_2 + \alpha_3 - 3\alpha_7) &= 0, \\
\nu_0^{(4)} \ln \frac{\nu_0^{(4)}}{c_0^{(4)}} + 2\nu_1^{(4)} \ln \frac{\nu_1^{(4)}}{c_1^{(4)}} + \nu_3^{(4)} \ln \frac{\nu_3^{(4)}}{c_3^{(4)}} &\leqslant c_4, \\
\nu_0^{(7)} \ln \frac{\nu_0^{(7)}}{c_0^{(7)}} + 3\nu_1^{(7)} \ln \frac{\nu_1^{(7)}}{c_1^{(7)}} &\leqslant c_7, \\
\text{as well as } \ 3c_0^{(4)} + c_0^{(7)} \leqslant c_0, \quad 2c_1^{(4)} + c_3^{(4)} + c_1^{(7)} &\leqslant c_1.
\end{aligned}$$

For any non-trivial symmetry group, the symmetry-adapted formulation gives a smaller relative entropy program as the original formulation. Both the number of variables and the number of inequalities are smaller. The original relative entropy program has $2|\mathcal{B}||\mathcal{A}|$ variables and $|\mathcal{B}|n+|\mathcal{B}|+|\mathcal{A}|$ (in)equalities, since every vector equality in (3.11) brings n scalar equalities. In comparison, the symmetric relative entropy program in Corollary 3.5 has $2\sum_{\hat{\beta}\in\hat{\mathcal{B}}} |\mathcal{A}/\operatorname{Stab}(\hat{\beta})| \leqslant 2|\hat{\mathcal{B}}||\mathcal{A}|$ variables and at most $|\hat{\mathcal{B}}|n + |\hat{\mathcal{B}}| + |\hat{\mathcal{A}}|$ (in)equalities, since the invariance of (3.11) under $\operatorname{Stab}(\hat{\beta})$ might imply further redundancy in the individual scalar equations. In particular, in the situation of "large" stabilizers, the number of variables will become small. In the situation of "small" stabilizers (with not too small group sizes), we observe that Burnside's Lemma tells us $|\hat{\mathcal{A}}| = \frac{1}{|G|}\sum_{\sigma\in G} |\{\alpha \in \mathcal{A} \ : \ \sigma(\alpha) = \alpha\}|$, and hence

$$|\hat{\mathcal{A}}| = \frac{1}{|G|} \,|\{(\sigma, \alpha) \ : \ \sigma \in G,\, \alpha \in \mathcal{A},\, \sigma(\alpha) = \alpha\}| = \frac{1}{|G|} \sum_{\alpha\in\mathcal{A}} |\operatorname{Stab}(\alpha)|$$

(and analogously for $\hat{\mathcal{B}}$); hence, the number of (in)equalities will become small then.

Remark 3.7. Note that we cannot simply assume $c_\alpha^{(\beta)} = c_{\alpha'}^{(\beta)}$ for some $\alpha' \in G \cdot \alpha$ and, similarly, we cannot simply assume $\nu_\alpha^{(\beta)} = \nu_{\alpha'}^{(\beta)}$ for some $\alpha' \in G{\cdot}\alpha$, for instance due to (2.1). Namely, if an element β lies in $\operatorname{conv} \mathcal{A}$ with barycentric coordinates λ, say $\beta = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha \alpha$, then for any $\sigma \in G$, we have

$$\sigma(\beta) = \sigma\left(\sum_{\alpha\in\mathcal{A}} \lambda_\alpha \alpha\right) = \sum_{\alpha\in\mathcal{A}} \sigma(\lambda_\alpha \alpha) = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha \sigma(\alpha)$$

rather than $\sigma(\beta) = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha \alpha = \sum_{\alpha\in\mathcal{A}} \lambda_{\sigma(\alpha)} \sigma(\alpha)$. Of course, this caveat does not occur whenever there is a single inner term.

The following example shows the usefulness of Theorem 3.3 and Corollary 3.5 for a whole family of polynomials (or exponential sums).

Example 3.8. Using $\mathcal{S}_n$ for the symmetric group on the set $[n]$, consider the $\mathcal{S}_n$-symmetric polynomial

$$f(x_1, x_2, x_3) \ = \ 1 + \sum_{i=1}^{3} x_i^8 - \delta \sum_{(i,j,k)\in\mathcal{S}_3} x_i^3 x_j^2 x_k - \delta \sum_{(i,j,k)\in\mathcal{S}_3} x_i^2 x_j x_k,$$

and we ask for the largest δ for which f is a SONC polynomial. Let $e^{(i)}$ denote the i-th unit vector for $i \in \mathbb{N}$, and $\mathbf{0}$ denote the three-dimensional zero vector. Setting $\mathcal{A} = \{\mathbf{0}, 8e^{(1)}, 8e^{(2)}, 8e^{(3)}\}$, the conventional relative entropy program is

$$\begin{array}{rcll}
 & \min \delta & & \\
\text{s.t. } \sum_{\alpha\in\mathcal{A}} \nu_\alpha^{(\beta)}(\alpha - \beta) & = & 0 & \text{for } \beta \in S_3 \cdot (3,2,1)^T \cup S_3 \cdot (2,1,1)^T, \\
D(\nu^{(\beta)}, e \cdot c^{(\beta)}) & \leqslant & \delta & \text{for } \beta \in S_3 \cdot (3,2,1)^T \cup S_3 \cdot (2,1,1)^T, \\
\sum\limits_{\beta\in S_3((3,2,1)^T)\cup S_3((2,1,1)^T)} c_\alpha^{(\beta)} & \leqslant & 1 & \text{for } \alpha \in \mathcal{A}, \\
\delta \in \mathbb{R} \text{ and } c^{(\beta)}, \nu^{(\beta)} & \in & \mathbb{R}_+^4 & \text{for } \beta \in S_3(3,2,1)^T \cup S_3(2,1,1)^T.
\end{array}$$

This optimization problem has $73 = 2 \cdot 4 \cdot 9 + 1$ variables (including the δ-variable) and $40 = 3 \cdot 9 + 9 + 4$ equations or inequalities. We find

$$|\mathcal{A}/\operatorname{Stab}((3,2,1)^T)| = |\mathcal{A}| = 4, \quad |\mathcal{A}/\operatorname{Stab}((2,1,1)^T)| = |\{\mathbf{0}, 8e^{(1)}, 8e^{(2)}\}| = 3$$

and

$$|\hat{\mathcal{B}}| = |\{(3,2,1)^T, (2,1,1)^T\}| = 2, \quad |\hat{\mathcal{A}}| = |\{\mathbf{0}, 8e^{(1)}\}| = 2.$$

Therefore, the symmetric relative entropy program from 3.4 involves $17 = 2 \cdot (4+2) + 1$ variables and at most $10 = 2 \cdot 3 + 2 + 2$ equations or inequalities.

For symmetric constraint sets X, a constrained version of Theorem 3.3 (and similarly, of Corollary 3.5) can be given as well. The proof is similar.

Corollary 3.9. *Let $X \subset \mathbb{R}^n$ be convex and G-invariant. A G-symmetric signomial f of the form* (3.3) *is contained in $C_X(\mathcal{A}, \mathcal{B})$ if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$ there exist $c^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}}$ and $\nu^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}}$ such that*

$$\begin{array}{rcll}
D(\nu^{(\hat{\beta})}, e \cdot c^{(\hat{\beta})}) + \sup_{x\in X} \left(- \sum\limits_{\alpha\in\mathcal{A}} \nu_\alpha^{(\hat{\beta})}(\alpha - \hat{\beta})\right)^T x & \leqslant & c_{\hat{\beta}} & \textit{for every } \hat{\beta} \in \hat{\mathcal{B}}, \\
\sum\limits_{\hat{\beta}\in\hat{\mathcal{B}}} \sum\limits_{\sigma\in\operatorname{Stab}\hat{\beta}\backslash G} c_{\sigma(\alpha)}^{(\hat{\beta})} & \leqslant & c_\alpha & \textit{for every } \alpha \in \mathcal{A}.
\end{array}$$

We remark that Corollary 3.9 as well as Theorem 3.2 and 3.3 can be formulated in the polynomial setting as well.

Symmetric exponentials with one orbit of interior points. In general, for a given signomial f, the notions of being non-negative and being a SAGE signomial differ. However, in the non-symmetric setting, it is known that these notions coincide for some prominent subclasses. If f is supported on a circuit, i.e.,

$$f = \sum_{\alpha \in A} c_\alpha \exp(\alpha^T x) + c_\beta \exp(\beta^T x)$$

with a circuit support (A, β) and $c_\alpha > 0$ for $\alpha \in A$, then

$$f \text{ is non-negative } \iff f \in C_{\text{AGE}}(\mathcal{A}, \beta) \iff \prod_{\alpha \in \mathcal{A}} \left(\frac{c_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} \geqslant -c_\beta,$$

where $\lambda = (\lambda_\alpha)_{\alpha \in A}$ denotes the barycentric coordinates of β with respect to the points $\alpha \in A$ (see [20, Theorem 2.7]). This is known as the circuit number characterization (see [17]).

Moreover, in the setting of polynomials, Jie Wang has shown that a polynomial with a single interior point in the convex hull of the “positive monomials” is non-negative if and only if it is a SONC polynomial (see [31]).

We can extend these results to certain symmetric situations. Denote by $\mathbb{1}$ the all-ones vector and by $e^{(i)}$ the i-th unit vector. Let the set of “positive monomials” $\mathcal{A}$ be invariant with respect to the symmetric group and consist of $n+1$ affinely independent points in $\mathbb{R}^n$. Then $\mathcal{A} = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is the union of two orbits: an orbit made of the n points of the form

$$\alpha_i \;=\; a\mathbb{1} + (b-a)e^{(i)} \;=\; (a, \ldots, a, \underbrace{b}_{i}, a, \ldots, a), \quad 1 \leqslant i \leqslant n$$

with $a, b \in \mathbb{R}$, $a \neq b$ as well as a fully symmetric point $\alpha_0 = (c, \ldots, c)$ with $c \in \mathbb{R}$.

For $\hat{\beta}$ in the interior of $\mathcal{A}$, if $(\mathcal{S}_n \cdot \hat{\beta})$ denotes the orbit of $\hat{\beta}$ under the action of the symmetric group $\mathcal{S}_n$, we consider symmetric signomials of the form

$$\sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\alpha^T x) - c_{\hat{\beta}} \sum_{\beta \in (\mathcal{S}_n \cdot \hat{\beta})} \exp(\beta^T x) \tag{3.14}$$

with coefficients $c_\alpha \geqslant 0$ and where $c_{\hat{\beta}}$ is treated as a parameter. We will determine the optimal value $c_{\hat{\beta},\max}$ such that, for every $c_{\hat{\beta}} < c_{\hat{\beta},\max}$, the signomial (3.14) is a SAGE exponential.

Up to multiplication by a positive scalar, (3.14) can be written as

$$f_\delta(x) = \mu \exp(\alpha_0^T x) + \sum_{i=1}^{n} \exp(\alpha_i^T x) - \delta \sum_{\sigma \in G_\beta} \sigma \exp(\beta^T x), \tag{3.15}$$

with some β in the interior of $\mathcal{A}$ and where $G_\beta := \mathcal{S}_n / \operatorname{Stab} \beta$.

Using the notation $|v| = \sum_{i=1}^n v_i$ for a vector $v \in \mathbb{R}^n$ and writing $\beta = (\beta_1, \ldots, \beta_n)$, we set $d_0 := |\alpha_0| = nc$, $d_\beta := |\beta| = \sum_{i=1}^n \beta_i$ and $d := (n-1)a + b$, i.e., $d = |\alpha_i|$ for all i.

Note that we cannot have $d_0 = d$, otherwise $\mathcal{A}$ would not be affinely independent. The barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ of β with respect to $\mathcal{A}$ are given by

$$\lambda_i = \frac{(\beta_i - a) - \lambda_0(c-a)}{b-a}, \ 1 \leqslant i \leqslant n, \quad \text{and} \quad \lambda_0 = \frac{d_\beta - d}{d_0 - d}.$$

Theorem 3.10. *For an $\mathcal{S}_n$-symmetric signomial f_δ of the form* (3.15) *with $\delta \in \mathbb{R}$, the following are equivalent:*

(1) f_δ is non-negative.
(2) $f_\delta \in \mathrm{C}_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$.
(3) $\delta \ \leqslant \ \delta_{\max} := \dfrac{1}{|G_\beta|}\left(\dfrac{\mu}{\lambda_0}\right)^{\lambda_0}\left(\dfrac{n}{1-\lambda_0}\right)^{1-\lambda_0}$.

Hence, for the specific class under consideration, condition (3) in Theorem 3.10 can be viewed as an $\mathcal{S}_n$-symmetric analog of the circuit number condition.

Proof. Since there is only one orbit for $\mathcal{B}$, Theorem 3.2 guarantees a decomposition of the form $f_\delta \ = \sum_{\rho \in G_\beta} \rho h_\delta$, where $h_\delta \in C_{\mathrm{AGE}}(\mathcal{A} \setminus \{\beta\}, \beta)$ for some $\beta \in \mathcal{B}$. We provide an explicit decomposition of this form here, which then facilitates the exact non-negativity characterization. For $1 \leqslant i \leqslant n$, let $s_i = \frac{n}{|G_\beta|} \frac{\lambda_i}{1-\lambda_0}$, and

$$h_\delta(x) = \frac{\mu}{|G_\beta|} \exp(\alpha_0^T x) + \sum_{i=1}^n s_i \exp(\alpha_i^T x) - \delta \exp(\beta^T x).$$

To verify that

$$f_\delta = \sum_{\sigma \in G_\beta} \sigma h_\delta, \tag{3.16}$$

consider the sum $\sum_{\sigma \in G_\beta} \sigma h_\delta$. For the first term of h_δ, this sum gives

$$\sum_{\sigma \in G_\beta} \frac{\mu}{|G_\beta|} \ \exp(\alpha_0^T x) = \mu \exp(\alpha_0^T x).$$

For the second term of h_δ, the sum gives

$$\begin{aligned} \sum_{g \in G_\beta} \sigma \left(\sum_{i=1}^n s_i \exp(\alpha_i^T x) \right) \ &= \ \sum_{i=1}^n s_i \left(\sum_{\sigma \in G_\beta} \sigma(\exp(\alpha_i^T x)) \right) \ = \ \sum_{i=1}^n s_i \left(\frac{|G_\beta|}{n} \sum_{j=1}^n \exp(\alpha_j^T x) \right) \\ &= \ \sum_{j=1}^n \exp(\alpha_j^T x). \end{aligned}$$

For the third term of h_δ, the sum is simply $-\delta \sum_{\sigma \in G_\beta} \sigma \exp(\beta^T x)$, which altogether shows the decomposition (3.16).

We claim that the signomial h_δ is a non-negative if and only if we have $\delta \leqslant \delta_{\max}$. Using the circuit number characterization, h_δ is a non-negative circuit signomial if and only if

$$\begin{aligned}\delta &\leqslant \left(\frac{\mu}{|G_\beta|\lambda_0}\right)^{\lambda_0} \prod_{i=1}^{n}\left(\frac{s_i}{\lambda_i}\right)^{\lambda_i} = \left(\frac{\mu}{|G_\beta|\lambda_0}\right)^{\lambda_0} \prod_{i=1}^{n}\left(\frac{n}{|G_\beta|(1-\lambda_0)}\right)^{\lambda_i} \\ &= \left(\frac{\mu}{|G_\beta|\lambda_0}\right)^{\lambda_0}\left(\frac{n}{|G_\beta|(1-\lambda_0)}\right)^{1-\lambda_0} = \delta_{\max}.\end{aligned}$$

Therefore, if $\delta \leqslant \delta_{\max}$, then f_δ is a SAGE exponential, and hence non-negative. It remains to show the converse directions, i.e., that the bound is sharp for non-negativity and for membership of h_δ to $C_{\mathrm{SAGE}}(\mathcal{A},\mathcal{B})$. In order to show that the bound f_δ is sharp for both settings, we show that $f_{\delta_{\max}}$ has a zero. To this end, define

$$x = \frac{1}{d_0 - d} \ln\left(\frac{n}{\mu}\frac{\lambda_0}{1-\lambda_0}\right).$$

Since the argument of the logarithm is positive, this is a well defined positive number. We will show that

$$f_{\delta_{\max}}(x,\ldots,x) = 0.$$

This implies that for every $\delta > \delta_{\max}$,

$$f_\delta(x,\ldots,x) = f_{\delta_{\max}}(x,\ldots,x) - (\delta - \delta_{\max})|G_\beta| \exp(d_\beta x) < 0,$$

so that f is not non-negative, and thus f is not a SAGE either.

In order to show $f_{\delta_{\max}}(x,\ldots,x) = 0$, set $y = (d_0 - d)x = \ln\left(\frac{n}{\mu}\frac{\lambda_0}{1-\lambda_0}\right)$, which gives $\mu = \frac{n\lambda_0}{1-\lambda_0}e^{-y}$ and

$$f(x,\ldots,x) = \mu e^{\frac{d_0}{d_0-d}y} + n e^{\frac{d}{d_0-d}y} - \left(\frac{\mu}{\lambda_0}\right)^{\lambda_0}\left(\frac{n}{1-\lambda_0}\right)^{1-\lambda_0} e^{\frac{d_\beta}{d_0-d}y}. \tag{3.17}$$

Since $\frac{d_\beta}{d_0-d} = \lambda_0 + \frac{d}{d_0-d}$, the last term of (3.17) can be written as

$$-e^{\frac{d}{d_0-d}y}\left(\frac{\mu}{\lambda_0}\right)^{\lambda_0}\left(\frac{n}{1-\lambda_0}\right)^{1-\lambda_0} e^{\lambda_0 y} = -e^{\frac{d}{d_0-d}y}\frac{n}{1-\lambda_0},$$

which shows, by substituting μ,

$$f_{\delta_{\max}}(x,\ldots,x) = \frac{n\lambda_0}{1-\lambda_0} e^{\frac{d_0}{d_0-d}y-y} + \frac{n(1-\lambda_0)}{1-\lambda_0} e^{\frac{d}{d_0-d}y} - \frac{n}{1-\lambda_0} e^{\frac{d}{d_0-d}y} = 0.$$

This completes the proof. □

4. On the set of minimizers of SONC polynomials and SAGE exponentials

The structural considerations in the previous section, including the explicit criterion in Theorem 3.10, naturally encourages to study the symmetry of the minimizers. For example, in the class discussed in Theorem 3.10, we have exploited that the minimizer is fully symmetric, and hence located on the diagonal.

The goal of this section is to provide constructions of symmetric SAGE exponentials and SONC polynomials without minimizer on the diagonal. To prepare for this, it is useful to have a detailed look at the minimizers of SONC polynomials and SAGE exponentials in the non-symmetric case. Using the relative entropy formulation in Theorem 2.1 (and its dual), non-negativity of SONC polynomials and SAGE exponentials can be formulated in terms of a convex condition. Moreover, any SAGE exponential with a finite number of zeroes has at most one zero (see [12, Theorem 4.1]). Similarly, any SONC polynomial in $f \in \mathbb{R}[x_1, \ldots, x_n]$ with a finite number of zeroes has at most 2^n real zeroes in $(\mathbb{R}\setminus\{0\})^n$ (see [9, Corollary 4.1]), because it has at most one zero in each open orthant. In sharp contrast to this, we show that this does not hold for the set of minimizers of a SONC polynomial or a SAGE exponential if the minimal value is different from zero. These considerations will then form the point of departure for our constructions of symmetric polynomials with minimizers outside of the diagonal. We begin with the following theorem.

Theorem 4.1. *There exist SAGE signomials, even in one variable, which have several isolated minimizers. In particular, the set of minimizers is not convex for these signomials.*

As a preliminary of the proof, consider a signomial $f = \sum_{\alpha} c_{\alpha} \exp(\alpha^T x) - d \exp(\beta^T x)$ with $c_{\alpha} > 0$ and $d \in \mathbb{R}$. As already observed by Chandrasekaran and Shah [5, Proof of Lemma 2.2], the signomial function

$$\exp(-\beta^T x) f \ = \ \sum_{\alpha} c_{\alpha} \exp((\alpha^T - \beta^T)x) - d$$

is convex, since all non-constant summands are convex. However, in general the product of two convex functions is not convex.

Proof. In order to prove the theorem, we give a construction of a univariate signomial h as a sum of two AGE signomials f, g that has two isolated minimizers. Let

$$f = c_1 \exp(\alpha_1 x) + c_2 \exp(\alpha_2 x) + c_3 \exp(\alpha_3 x)$$

with support points $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 3$. We enforce $f(0) = 1$ by using the constraint

$$f(0) = c_1 + c_2 + c_3 \ = \ 1,$$

and we enforce that f has a double zero at 1 via the constraints

$$\begin{aligned} f(1) \ &= \ c_1 e^1 + c_2 e^2 + c_3 e^3 \ = \ 0, \\ f'(1) \ &= \ c_1 e^1 + 2c_2 e^2 + 3c_3 e^3 \ = \ 0. \end{aligned}$$

The linear system of three equations in c_1, c_2, c_3 has a unique solution:

$$c_1 = \frac{e^2 e^3}{D} (\approx 2.5027) \quad c_2 = -\frac{2e^1 e^3}{D} (\approx -1.8413), \quad c_3 = \frac{e^1 e^2}{D} (\approx 0.3387),$$

where $D := e^1 e^2 - 2e^1 e^3 + e^2 e^3$. Since c_1 and c_3 are positive, we have $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$. Hence, f is a non-negative signomial on $\mathbb{R}^n$ with only one negative coefficient. Moreover, $c_2 = -(c_1/\lambda_1)^{\lambda_1} \cdot (c_3/\lambda_3)^{\lambda_3}$ for barycentric coordinates $\lambda_1 = \lambda_3 = 1/2$ of the inner exponent of the circuit, i.e., the circuit number condition is satisfied with equality.

Let $g(x)$ be the signomial supported on -3, -2, -1 which satisfies $g(x) = f(-x)$. Then g has the same coefficients as f and is a non-negative signomial on $\mathbb{R}^n$ with only one negative coefficient, too.

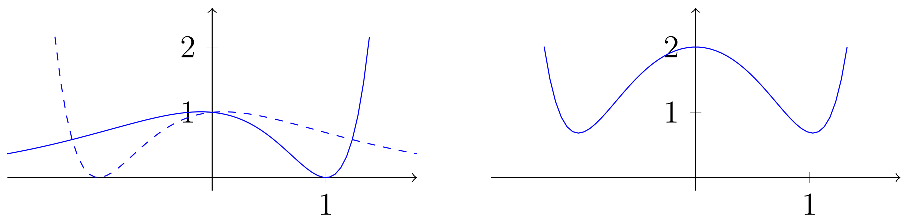

FIGURE 1. The exponential sums $f(x)$ (solid, left), $g(x) = (-x)$ (dashed, left) and $h(x)$ (right).

The exponential sum $h(x) := f(x)+g(x)$ is clearly non-negative. See Figure 1 for a visualization of the graph of h. We claim that h has two minimizers with identical function values. Indeed, we have $h(0) = 2$, $h(1) = h(-1) < 2$ and $\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} h(x) = \infty$. As h is symmetric, there exists $\tilde{x} \in (0, \infty)$ such that $\min_{x\in\mathbb{R}^n} h(x) = h(\tilde{x}) = h(-\tilde{x})$ but for $0 \in \text{relint}[-\tilde{x}, \tilde{x}]$ we have $h(0) = 2 > \min_{x\in\mathbb{R}} h(x)$.

Note that, by construction, the two minimizers are in the same orbit with respect to the group action $x \mapsto -x$. □

Remark 4.2. The construction shows the non-convexity phenomenon of the minimizers of SAGE signomials. We remark that the construction makes use of negative support points, namely in the definition of the function g. When varying the question to the class of SONC polynomials (where only non-negative integer vectors are allowed as exponent vectors), the construction given above does not work.

Example 4.3 (A symmetric SAGE signomial with several minima)**.** Building on the counterexample

$$f(x) = c_1(e^x + e^{-x}) + c_2(e^{2x} + e^{-2x}) + c_3(e^{3x} + e^{-3x})$$

with

$$c_1 + c_2 + c_3 - 1 = c_1 e + c_2 e^2 + c_3 e^3 = c_1 e + 2c_2 e^2 + 3c_3 e^3 = 0$$

and two minimal points, around 1 and -1, from Theorem 4.1, we build counterexamples of symmetric SAGE signomials with several minimizers. It is then easy to define a new function, namely

$$g(x, y) = f(x - y)$$

in two variables. f is a symmetric SAGE signomial for the same reasons as in the proof of Theorem 4.1. And of course, all the minimal points are on two lines parallel to the diagonal. This example is degenerated, since it has no isolated minimizers.

More generally, we can observe that for every univariate even exponential sum f (i.e., $f(-x) = f(x)$) which has 0 not as a minimal point, the exponential sum $g(x) := f(x - y)$ is symmetric and has its minimizers outside the diagonal.

Before we construct a symmetric SAGE exponential with several *isolated* minima, we turn to the case of symmetric SONC polynomials. Building upon this, we can then find a symmetric SAGE exponential with several isolated minima.

Minimizers of SONC polynomials. For SONC polynomials (which, in particular, have non-negative integer exponent vectors), by the specific type of symmetry with respect to the open orthants, it is clear that there can be several isolated minimizers in different orthants, and this occurs even in the case that the minimizers are zeroes.

Here, we provide a construction – different from the one above – which gives a SONC polynomial that has already several isolated minimizers in the positive orthant $\mathbb{R}^n_{>0}$.

Theorem 4.4. *There exist SONC polynomials, even in one variable, which have several isolated minimizers in the positive orthant. In particular, the set of minimizers is not convex for the polynomials, even if the set of minimizers is restricted to the positive orthant.*

Before the construction, let us recall the symmetry structure of the zeroes of SONC polynomials. Firstly, the roots come with a certain symmetry with respect to the orthants (see [9]). Secondly, the origin plays a particular role, since any polynomial f with vanishing constant term has a zero at the origin.

Proof. We consider univariate polynomials f without constant term. By choosing f in such a way that it also has a root at some positive x, we can achieve that f has two roots in the non-negative orthant. Specifically, we set

$$f \ := \ x^2 - x^3 + \frac{1}{4}x^4 = \frac{1}{4}x^2(x-2)^2.$$

Clearly, f is a SONC polynomial with a double root at 0 and a double root at 2. Hence, in the interval $[0, \infty)$, the polynomial f has the two roots 0 and 2, and thus they are the minimizers.

By adding a suitable SONC polynomial g, we perturb the situation and will arrive at a situation where both perturbed minimizers are contained in the positive orthant. To this end, let

$$g := 1 - \alpha x + \beta x^4$$

with constants $\alpha, \beta > 0$. The exact value of the constant term (here: 1) does not matter, it has just to be sufficiently large to ensure non-negativity. Since $g''(x) = 12\beta x^2$, the polynomial g is convex, and its minimizer x^* satisfies the condition $4\beta x^3 = \alpha$. Set $\beta := 1/50$, so that for small $|x|$, the monomial x^4 has just a small influence. Intuitively, since the minimizer of g is a positive number, we can choose $\alpha > 0$ such that in $h := f + g$ both minimizers will be in the positive orthant and still have the same function value. Namely, computing this value α (using subresultants, for example), yields $\alpha = \frac{100}{729} \approx 0.1372$. Then the minimizers of the quartic polynomial h are at $x_{+/-} := \frac{5}{27}(5 \pm \sqrt{21}) > 0$ with function value $\frac{19583}{19683} < 1$.

Since h is a sum of two non-negative circuit polynomials, it is a SONC polynomial. Altogether, this proves Theorem 4.4. □

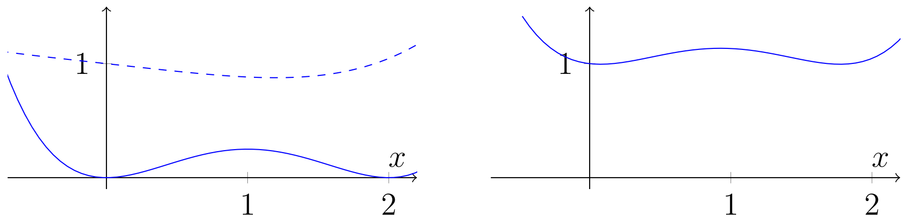

FIGURE 2. The polynomials $f(x)$ (solid, left), $g(x) = (-x)$ (dashed, left) and $h(x)$ (right).

For a visualization of the polynomials f, g and h, see Figure 2.

The following theorem gives an explicit class of symmetric SONC polynomials which have their minimizers not on the diagonal.

Theorem 4.5. *Let f be the symmetric polynomial*

$$f(x,y) = 1 + a(x^4 + y^4) + bx^2y^2 - d(x^2y + y^2x)$$

with $a, b, d > 0$. If $d^4 \leqslant 8ab^2$, then f is a SONC polynomial. If, moreover, $b > 22a$, then the global minimum of f restricted to the diagonal (i.e., of $f_{|y=x}$) is not a global minimum of f, and it is not even a local minimum of f.

Proof. f can be decomposed as $f = f_1 + f_2$ with

$$f_1 = \frac{1}{2} + ax^4 + \frac{b}{2}x^2y^2 - dx^2y, \quad f_2 = \frac{1}{2} + ay^4 + \frac{b}{2}x^2y^2 - dxy^2,$$

and, by the circuit number condition, f_1 and f_2 are non-negative circuit polynomials if and only if $d^4 \leqslant 8ab^2$. The restriction of $\frac{\partial f}{\partial x}$ to the diagonal $y = x$ is $\frac{\partial f(x,y)}{\partial x}\big|_{y=x} = x^2(4ax + 2bx - 3d)$, and of course this also coincides with $\frac{\partial f(x,y)}{\partial y}\big|_{y=x}$. Solving for x gives the solution set $S := \{0, \frac{3d}{2(2a+b)}\}$, where $x = 0$ has multiplicity 2. All points in the complement $\mathbb{R} \setminus S$ cannot even be local minimizers on the restriction to the diagonal. It remains to rule out the points in S.

To rule out the point $\overline{x} = 0$, it suffices to observe that $\frac{\partial f(x,y)}{\partial x}\big|_{y=x}$ is negative for small positive x.

For $\overline{x} = \frac{3d}{2(2a+b)}$, observe that evaluating the Hessian

$$H_f = \begin{pmatrix} 12ax^2 + 2by^2 - 2dy & 4bxy - d(2x + 2y) \\ 4bxy - d(2x + 2y) & 12ay^2 + 2bx^2 - 2dx \end{pmatrix}$$

at the point $(x, y) = (\overline{x}, \overline{x})$ gives

$$\frac{1}{2(2a+b)^2} \begin{pmatrix} 3d^2(14a + b) & -6d^2(4a - b) \\ -6d^2(4a - b) & 3d^2(14a + b) \end{pmatrix}.$$

Its determinant is

$$\frac{27d^4(22a - b)}{4(2a + b)^3}.$$

Hence, for $b > 22a$, the Hessian H_f is indefinite at $(\overline{x}, \overline{x})$. This shows that $(\overline{x}, \overline{x})$ cannot even be a local minimum of f. □

As a corollary, we obtain the following version of a symmetric SAGE exponential which does not have its minimizer on the diagonal.

Corollary 4.6. *Let f be the symmetric signomial $f(x, y) = 1 + a(\exp(4x) + \exp(4y)) + b(\exp(2x) + \exp(2y)) - d(\exp(2x) + \exp(2y))$ with $a, b, d > 0$. If $d^4 \leqslant 8ab^2$, then f is a SAGE signomial. If, moreover, $b > 22a$, then the global minimum of f restricted to the diagonal (i.e., of $f_{|y=x}$) is not a global minimum of f, and it is not even a local minimum of f.*

5. The symmetric SONC cone

As a motivation to the question posed in his seventeenth problem, Hilbert showed in 1888 that the cone of non-negative polynomials coincides with the cone of sums of squares (SOS) when $n = 1$, when $d = 2$ or when $n = 2$ and $d = 4$ [16]. He also showed that in all other cases, the non-negativity cone is strictly bigger than the SOS cone. Lately, Blekherman and Riener [4] showed that the symmetric SOS cone gives a rather good approximation of the symmetric non-negativity cone for a large number of variables. It is then natural to wonder whether an analogous property holds for the symmetric SONC cone. To attack this question, we adopt a slightly different point of view, that fits more the spirit of the previous sections. Given a symmetric polynomial f in n variables of even degree d having a finite minimum, when will the SONC (respectively, SAGE) bound provide us this minimum?

The first case of interest for symmetric SONC polynomials is the cone of polynomials with degree at most 4 in 2 variables. We will show that in this situation, we can already find polynomials that are non-negative without being SONC. We can even construct sequences of polynomials f_k such that the gap between the minimum f_k^* of f_k and the associated SONC bound f_k^{SONC} tends to infinity:

Theorem 5.1. *For $f \in \mathbb{R}[x, y]$, denote by $\|f\|$ the supremum of the absolute values of the coefficients of f. There exists a sequence of polynomials $(f_k(x, y))_k$ of degree 4 such that*

$$\lim_{k\to\infty} \frac{f_k^* - f_k^{\mathrm{SONC}}}{\|f_k\|} = \infty.$$

This result will be a consequence of a more general study of symmetric polynomials of degree 4 in two variables depending on their support. In our situation, the possible coefficients lie in the simplex whose vertices are $(0, 0), (4, 0)$ and $(0, 4)$. In particular, there are only three possible interior points $(1, 1)$, $(1, 2)$, $(2, 1)$. If the only interior point is $(1, 1)$, then we know that there is equivalence between being SONC and being non-negative [31]. The next case to consider is then when the support of our polynomial f contains the orbit made of $(1, 2)$ and $(2, 1)$. If the positive support is only $(0, 0)$, $(4, 0)$, $(0, 4)$, then we can apply Theorem 3.10: we also have equivalence in this case. Then the most natural next case is to add the diagonal point $(2,2)$ to the positive support, like in Theorem 4.5. In other words, we are considering polynomials of the form

$$f = a(x^4 + y^4) + 2bx^2y^2 - d(x^2y + xy^2)$$

and we want to understand and compare, depending on the non-negative coefficients a, b and d, the minimum f^* of f and the value f^{SONC}. As intuited by Theorem 4.5, these values do not always agree.

Theorem 5.2. *Let* $f = a(x^4 + y^4) + 2bx^2y^2 - d(x^2y + xy^2)$ *with* $a > 0$, $b \geqslant 0$ *and* $d \in \mathbb{R}$. *Then:*

- *If* $b \geqslant 11a$, *then* $f^* = -\dfrac{(2b^2 + 14ab + 11a^2 + (4ab + 5a^2)\delta)d^4}{512a(b-a)^3(a+b)}$, $f^{\mathrm{SONC}} = -\dfrac{d^4}{32ab^2}$, *where* $\delta = \sqrt{5 + 4\frac{b}{a}}$.
- *If* $2a \leqslant b \leqslant 11a$, *then* $f^* = -\dfrac{27d^4}{128(a+b)^3}$, $f^{\mathrm{SONC}} = -\dfrac{d^4}{32ab^2}$.
- *If* $b \leqslant 2a$, *then* $f^* = f^{\mathrm{SONC}} = -\dfrac{27d^4}{128(a+b)^3}$.

Note that for the two particular cases $b = 2a$ and $b = 11a$, the corresponding values agree.

Proof. To verify the claimed values for f^*, the idea is to decompose $f - f^*$ as a sum of squares, and show that $f - f^*$ attains 0 in some point. We treat the case $b \geqslant 11a$ first. Since $a, b, d \geqslant 0$ and $b \geqslant 11a > a$, we can easily check that

$$-f^* = \frac{(2b^2 + 14ab + 11a^2 + (4ab + 5a^2)\delta)d^4}{512a(b-a)^3(a+b)} \geqslant 0,$$

and we set $\mu = \sqrt{-f^*}$. Defining the polynomials

$$P_1 = \mu\left(1 + \frac{8a(3a\delta - 2b - 7a)}{d^2}(x+y)^2\right),$$

$$P_2 = \sqrt{\frac{a + 2b + a\delta}{16(b^2 - a^2)}}\left(d(x+y) + 2(a\delta - a - 2b)xy\right),$$

$$P_3 = a\sqrt{\frac{\delta - 1}{a+b}}\left(x^2 + y^2 + \frac{3-\delta}{2}xy\right),$$

it is then easy but tedious to check that $f - f^* = P_1^2 + P_2^2 + P_3^2$. Moreover, if

$$x_0 = \frac{d}{16(b-a)}\left(3 + \delta + \sqrt{\frac{4b^2 - 18a^2 - 22ab - (10a^2 + 2ab)\delta}{a(a+b)}}\right),$$

$$y_0 = \frac{d}{16(b-a)}\left(3 + \delta - \sqrt{\frac{4b^2 - 18a^2 - 22ab - (10a^2 + 2ab)\delta}{a(a+b)}}\right)$$

then $x_0, y_0 \in \mathbb{R}$ and $f(x_0, y_0) - f^* = f(y_0, x_0) - f^* = 0$.

We now treat the case $b < 11a$. Consider

$$g(x,y) = f(x,y) - \left(\sqrt{\frac{11a-b}{12}}(x^2-y^2)\right)^2.$$

Then

$$g(x,y) = a'(x^4+y^4) + 2b'x^2y^2 - dx^2y$$

with $a' = \frac{a+b}{12} > 0$ and $b' = 11a' \geqslant 0$. From the earlier case, we know that

$$g^* = -\frac{27d^4}{128(a'+b')^3} = -\frac{27d^4}{128(a+b)^3}.$$

In this case also, we have $x_0 = y_0 = \frac{3d}{4(a+b)}$, so that we have decomposed f into a sum of 4 squares, which attains a zero on the diagonal.

Now we look at f^{SONC}. From Theorem 3.2, $f - \lambda$ is a SONC polynomial if and only if there exists $0 \leqslant t \leqslant 1$ such that the polynomial

$$tax^4 + (1-t)ay^4 + bx^2y^2 - dx^2y - \frac{\lambda}{2}$$

is a SONC polynomial. And now, since we just have one interior point, this function is a SONC polynomial if and only if it is non-negative. Hence,

$$\begin{aligned} f^{\text{SONC}} &= \max\{\lambda \ : \ f-\lambda \text{ is SONC}\} \\ &= 2\max\{\rho \ : \ tax^4 + (1-t)ay^4 + bx^2y^2 - dx^2y - \rho \text{ is SONC}, 0 \leqslant t \leqslant 1\}. \end{aligned}$$

Let $j_t(x,y) = tax^4 + (1-t)ay^4 + bx^2y^2 - dx^2y - \frac{1}{2}f^{\text{SONC}}$. The strategy is to exhibit a $0 \leqslant t_0 \leqslant 1$ and a decomposition into sum of squares and circuit polynomials of j_{t_0}, such that it attains a zero, and such that for all other $0 \leqslant t \leqslant 1$, j_t has a negative value.

We start with the case $b \leqslant 2a$. Let $x_0 = \frac{3d}{4(a+b)}$. Then $j_t(x_0,x_0) = \frac{1}{2}(f(x_0,x_0) - f^*) = 0$ for every $0 \leqslant t \leqslant 1$. Let $t_0 = \frac{4a+b}{6a}$. Then we have

$$j_{t_0}(x,y) = \left(\frac{2a-b}{6}(x^2-y^2)^2\right) + \left(\frac{a+b}{3}x^4 + 2\frac{a+b}{3}x^2y^2 - dx^2y + \frac{27d^4}{256(a+b)^3}\right).$$

The second summand $\left(\frac{a+b}{3}x^4 + 2\frac{a+b}{3}x^2y^2 - dx^2y + \frac{27d^4}{256(a+b)^3}\right)$ is a circuit polynomial, and after turning it into a sum of exponentials, one can check that it is non-negative, thanks to the witnesses $\mu^{(0,0)} = d/4$, $\mu^{(4,0)} = d/4$ and $\mu^{(2,2)} = d/2$. As $b \leqslant 2a$, j_{t_0} is the sum of a square and a non-negative circuit polynomial, and is thus non-negative, and we have just seen that it attains a zero in (x_0,x_0). Now, let $t \in [0,1]\backslash\{t_0\}$. We show that (x_0,x_0) is not a local minimum by looking at its partial derivatives at that point. Namely,

$$\frac{\partial j_t}{\partial x}(x_0,x_0) = -\frac{\partial j_t}{\partial y}(x_0,x_0) = -\frac{9d^3(b+a(4-6t))}{32(a+b)^3}.$$

One of them is strictly negative as soon as $t \neq t_0$.

Consider now the case $b > 2a$. Setting $x_1 = \frac{d}{\sqrt{8ab}}$, $y_1 = \frac{d}{2b}$, we obtain

$$j_t(x_1,y_1) = -\frac{(b-2a)(2a+b)(1-t)}{64ab^4}.$$

Consequently, $j_t(x_1, y_1) = 0$ if and only if $t = 1$, and $j_t(x_1, y_1) < 0$ when $t < 1$. Moreover, the polynomial j_1 is a circuit polynomial. □

Our analysis is summarized in Figure 3.

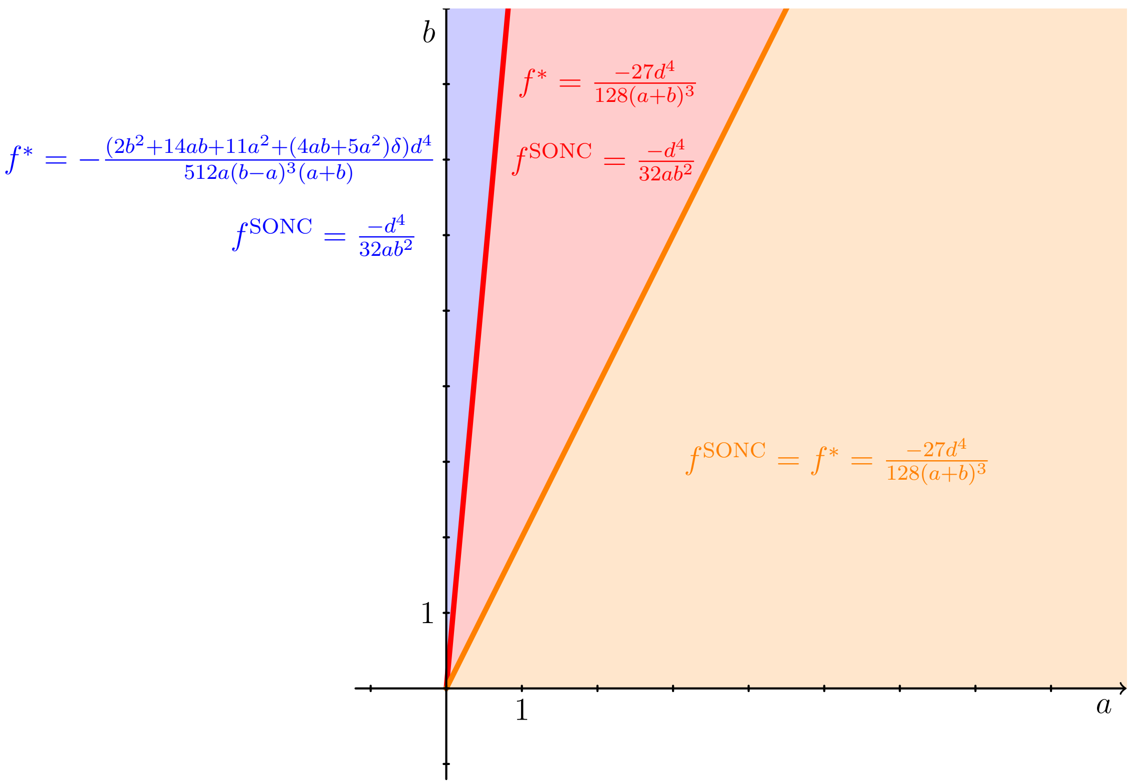

FIGURE 3. Comparison between f^* and f^{SONC} depending on the parameters a (horizontal axis) and b (vertical axis).

Theorem 5.1 is now a consequence of this study:

Proof of Theorem 5.1. Take the norm on the set of polynomials to be the maximum of the absolute values of the coefficients. Consider the polynomials

$$f_a(x, y) = 4a(x^4 + y^4) + 118ax^2y^2 - (x^2y + xy^2)$$

for $0 < a \leqslant \frac{1}{118}$. This ensures that $||f_a|| = 1$. Then from Theorem 5.2, we have

$$f_a^* = -\frac{1}{1185408a^3}, \qquad f_a^{\text{SONC}} = -\frac{1}{445568a^3},$$

and the gap becomes

$$\frac{1445}{1031601312a^3},$$

which becomes arbitrarily large when $a \to 0$. □

6. Symmetric computations

In Section 3, we have highlighted that the exploration of symmetry can possibly lead to a substantial reduction of the complexity of the relative entropy program, see the discussion before Remark 3.7. To complement these considerations, we present in this section classes of examples that spotlight the computational gains by the comparison of calculation times. For these computations, we used the ECOS solver and Python 3.7 on an Intel(R) Xeon(R) Platinum 8168 CPU with 2.7 GHz and 768 GB of RAM under CentOS Linux release 7.9.2009. Keeping the previous notation, for the standard method, that is the method that does not exploit the symmetries, the input consists of $\mathcal{A}$, $\mathcal{B}$ as well as the coefficients, while for the symmetry-adapted version, the input is $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ and the coefficients. This difference of input is mainly due to practical considerations and does not in itself influence the comparison of the time used by the solver. When both methods give an answer, the bounds coincide.

In all the tables in the sequel, dim is the dimension, n_v and n_c are the number of variables and constraints of the program, while t_s and t_r denote the solver time and the overall running time (including the building of the optimization program) in seconds. While it might happen that the standard method is slightly faster for very small instances, the size growth of the program in the standard method makes it quickly unsolvable. In that case this is represented by "$-$" in the table. The symmetric approach allows however to go further, and we give all the results until the solver warns about a possible inaccuracy. In this case, we mark the bound with "*".

Example 6.1. We first consider classes of signomials where $|\hat{\mathcal{A}}| = |\hat{\mathcal{B}}| = 1$. We have chosen four different classes of examples that show the influence of the sizes of the orbits on the solving time. These classes represent extremal situations, namely when the orbits are either very large or very small. In these situations, we can actually compute the exact number of variables and constraints in both cases according to Section 3. This analysis is given in the following table:

		Standard method		**Symmetric method**		
$\lvert\mathcal{S}_n \cdot \hat{\beta}\rvert$	$\lvert\mathcal{S}_n \cdot \hat{\alpha}\rvert$	n_v	n_c	n_v	n_c	table
1	$n!$	$2n! + 3$	$n! + n + 2$	5	4	2
$n!$	n	$2(n+1)n! + 1$	$(n+1)(n!+1)$	$2n+3$	$n+3$	3
$n!$	$n!$	$2(n!+1)n! + 1$	$n!(n+2) + 1$	$2n! + 3$	$n+3$	4
n	n	$2n(n+1) + 1$	$(n+1)^2$	7	5	5

Table 1. Comparison of the parameters when $|\hat{\mathcal{A}}| = |\hat{\mathcal{B}}| = 1$.

Now, let us give the numerical results for each of these classes. We have chosen the coefficients in a way that avoids numerical issues, namely preventing the bound to be either too small or too large.

Consider first the signomial

$$f_n^{(1)} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \sigma \exp(\alpha^T \cdot x) - \exp(\beta^T \cdot x),$$

where $\beta = (1,\dots,1)^T$ and $\alpha = (1,2,\dots,n)^T$. The numerical results are shown in Table 2.

		Standard method				**Symmetric method**			
dim	bound	n_v	n_c	t_s	t_r	n_v	n_c	t_s	t_r
2	-0.1481	7	6	0.0113	0.0121	5	4	0.0147	0.0158
3	-0.2499	15	11	0.0148	0.0160	5	4	0.0141	0.0149
4	-0.3257	51	30	0.0304	0.0337	5	4	0.0139	0.0147
5	-0.3849	243	127	–	–	5	4	0.0140	0.0147
6	-0.4327	1443	728	–	–	5	4	0.0136	0.0144
7	-0.4724*	10083	5049	–	–	5	4	0.0211	0.0222

TABLE 2. Numerical results for $f_n^{(1)}$.

Consider now the signomial

$$f_n^{(2)} = (n-1)!\sum_{i=1}^{n} \exp(n^2 x_i) - \sum_{\sigma\in\mathcal{S}_n} \sigma \exp(\beta^T \cdot x),$$

where $\beta = (1,2,\dots,n)^T$ (and $\alpha = (n^2,0,\dots,0)^T$). The numerical results are shown in Table 3.

		Standard method				**Symmetric method**			
dim	bound	n_v	n_c	t_s	t_r	n_v	n_c	t_s	t_r
2	-0.2109	13	9	0.0173	0.0185	7	5	0.0297	0.0311
3	-0.8888	49	28	0.0427	0.0454	9	6	0.0248	0.0264
4	-4.111	241	125	0.152	0.1701	11	7	0.0296	0.0318
5	-22.30	1441	726	0.7888	0.8433	13	8	0.0356	0.0384
6	-141.0	10081	5047	5.422	5.843	15	9	0.0423	0.0458
7	-1024	80641	40328	57.26	66.67	17	10	0.0491	0.0538
8	-8418	725761	362889	1514	2211	19	11	0.0568	0.0626
9	-77355	7257601	3628810	–	–	21	12	0.0661	0.0835
10		79833601	39916811	–	–	23	13	–	–

TABLE 3. Numerical results for $f_n^{(2)}$.

Next, we consider the case where both orbits are of maximal size. Let

$$f_n^{(3)} = \frac{1}{n}\sum_{\sigma\in\mathcal{S}_n} \exp(\alpha^T \cdot x) - \frac{1}{n}\sum_{\sigma\in\mathcal{S}_n} \sigma \exp(\beta^T \cdot x),$$

where $\beta = (1, 2, \ldots, n)^T$ and $\alpha = (2, 8, \ldots, 2n^2)^T$. The numerical results are shown in Table 4.

		Standard method				Symmetric method			
dim	bound	n_v	n_c	t_s	t_r	n_v	n_c	t_s	t_r
2	-0.4178	13	9	0.0301	0.0323	7	5	0.0431	0.0465
3	-1.0323	85	31	0.0558	0.0603	15	6	0.0531	0.0569
4	-3.494	1201	145	–	–	51	7	0.1212	0.1301
5	-15.13	29041	841	–	–	243	8	0.5750	0.6215
6		1038241	5761	–	–	1443	9	–	–

TABLE 4. Numerical results for $f_n^{(3)}$.

Finally, we consider the case where both orbits are small. Let

$$f_n^{(4)} = \frac{1}{n}\sum_{i=1}^{n} \exp(n^2 x_i) - \frac{1}{n}\sum_{i=1}^{n} \exp((n-1)(x_1 + \ldots + x_n) + x_i),$$

($\beta = (n, n-1, n-1, \ldots, n-1)^T$ and $\alpha = (n^2, 0, \ldots, 0)^T$). The numerical results are shown in Table 5. Note that this is a case treated in Theorem 3.10.

		Standard method				Symmetric method			
dim	bound	n_v	n_c	t_s	t_r	n_v	n_c	t_s	t_r
2	-0.1054	13	9	0.019 01	0.0204	7	5	0.0213	0.0229
3	-0.092	25	16	0.0268	0.0287	7	5	0.0205	0.0218
4	-0.076	41	25	0.0341	0.0367	7	5	0.0205	0.0218
68	-0.0053	9385	4761	–	–	7	5	0.0475	0.0519
95	-0.0038*	18241	9216	–	–	7	5	0.0267	0.0281

TABLE 5. Numerical results for $f_n^{(4)}$.

Example 6.2. Finally, we give an example where $\mathcal{A}$ and $\mathcal{B}$ consist of two orbits each:

$$\hat{\mathcal{A}} = \{(n^2, 0, \ldots, 0), (1, 4, \ldots, n^2)\} \quad \text{and} \quad \hat{\mathcal{B}} = \{(1, \ldots, 1), (1, 2, \ldots, n)\}.$$

In this case, we are still able to compute the number of constraints and the number of variables. With the standard approach,

$$n_v = 2(n! + n + 1)(n! + 1) + 1, \quad n_c = (n! + 1)(n + 2) + n,$$

while using symmetries,

$$n_v = 2n! + 2n + 9, \quad n_c = n + 6.$$

Table 6 shows the numerical results for the signomials

$$g_n = \frac{1}{n}\sum_{i=1}^{n} \exp(n^2 x_i) + \frac{1}{n}\sum_{\sigma \in \mathcal{S}_n} \sigma \exp(\alpha^T \cdot x) - \exp(x_1 + \ldots + x_n) - \frac{1}{n}\sum_{\sigma \in \mathcal{S}_n} \sigma \exp(\beta^T \cdot x)$$

for $\alpha = (1, 4, \ldots, n^2)^T$ and $\beta = (1, 2, \ldots, n)^T$.

		Standard method				Symmetric method			
dim	bound	n_v	n_c	t_s	t_r	n_v	n_c	t_s	t_r
2	-0.1918	31	14	0.0272	0.0292	17	8	0.0738	0.0776
3	-0.5223	141	38	0.0679	0.0727	27	9	0.0623	0.0663
4	-2.118	1451	154	–	–	65	10	0.1436	0.1539
5	-10.45	30493	852	–	–	259	11	0.5856	0.6320
6		1048335	5774	–	–	1461	12	–	–

Table 6. Numerical results for g_n.

7. Conclusion and open questions

We have developed techniques to exploit symmetries in AM/GM-based optimization and confirmed their benefit in terms of computational results. As parts of our structural results, we have initiated studying the symmetric SAGE and SONC cone and showed that the symmetric SONC cone differs from the non-negativity cone already for very restricted support sets. It remains open to capture further properties of the symmetric SAGE and SONC cone as well as their relations to the non-negativity cone, and to extend these considerations to more general setups.

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