A POLYHEDRAL HOMOTOPY ALGORITHM FOR REAL ZEROS

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ABSTRACT. We design a homotopy continuation algorithm, that is based on numerically tracking Viro's patchworking method, for finding real zeros of sparse polynomial systems. The algorithm is targeted for polynomial systems with coefficients satisfying certain concavity conditions. It operates entirely over the real numbers and tracks the optimal number of solution paths. In more technical terms; we design an algorithm that correctly counts and finds the real zeros of polynomial systems that are located in the unbounded components of the complement of the underlying $\boldsymbol{A}$-discriminant amoeba.

1. INTRODUCTION

Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ be a system of sparse polynomials in $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$ with support sets $A_1, A_2, \ldots, A_n \subseteq \mathbb{Z}^n$. More precisely, let

$$p_i \ := \ \sum_{\alpha \in A_i} c_\alpha^{(i)} \boldsymbol{x}^\alpha \ , \text{ for } i = 1, 2, \ldots, n.$$

Bernstein's theorem from 1975 [Ber75] shows that for generic choice of coefficients of the p_i the number of zeros of $\boldsymbol{p}$ on $(\mathbb{C}^*)^n$ equals to the *mixed volume* $\mathcal{M}(Q_1, Q_2, Q_3, \ldots, Q_n)$ of the Newton polytopes $Q_i := \operatorname{conv}(A_i)$.

In the early 90's the *polyhedral homotopy* method was developed as an algorithmic counterpart of Bernstein's theorem [HS95, VVC94]. The main idea of the polyhedral homotopy method is to continuously deform a given polynomial system to another "easy" system, that can be solved by pure combinatorics, and then trace back the change in the solution set with numerical path trackers. This geometric idea is colloquially referred to as *toric deformation*, and the "easy" systems with combinatorial structure are referred to as the systems at the *toric limit*. Polyhedral homotopy method is currently implemented in several packages such as PHCPACK [Ver99], HOM4PS-3 [CLL14], PSS5 [Mal19], and HOMOTOPYCONTINUATION.JL [BT18], and it has remarkable practical success.

For most applications of polynomial system solving, and for certain questions in theoretical computer science one needs to count and find zeros of polynomial equations over real numbers, e.g. see [JS17, Koi10]. No general and efficient algorithm that counts real zeros of arbitrary sparse polynomial systems is known, and there are good complexity theoretic reasons to believe that at this level of generality the problem is intractable. Our aim is to locate a sufficiently general and tractable sub-case of real zero finding problem. Consider support sets $A_1, A_2, \ldots, A_n \subseteq \mathbb{Z}^n$ are given, can we find effectively checkable

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conditions on the coefficients of the equations that guarantee tractable solving over the reals? In other words: Where are the "easy" equations located in space of sparse real polynomial systems with n equations and n unknowns?

An important observation from real algebraic geometry suggests a map for "easy" polynomial systems: one can count real zeros by pure combinatorics if the polynomial system is at the "toric limit". We informally state this result (Sturmfels' generalization of *Viro's method*) to motivate our discussion; see Section 2.2 for a precise statement.

Theorem 1.1 (Viro's Patchworking Method for Finitely Many Zeros)**.** *Let $A_1, \ldots, A_n \subseteq \mathbb{Z}^n$, let $\omega_i : A_i \to \mathbb{R}$ be lifting functions, and consider the following family of equations parametrized by $t \geq 1$:*

$$p_i(t, \boldsymbol{x}) \ := \sum_{\boldsymbol{\alpha} \in A_i} c_{\boldsymbol{\alpha}}^{(i)} t^{\omega_i(\boldsymbol{\alpha})} \boldsymbol{x}^{\boldsymbol{\alpha}} \ i = 1, 2, \ldots, n.$$

Let $\varepsilon_i : A_i \to \{-1, +1\}$ be the sign functions defined by signs of the coefficients $c_{\boldsymbol{\alpha}}^{(i)} \in \mathbb{R}$. Then, for sufficiently large $t \gg 1$, the set of common zeros of $p_1(t, \boldsymbol{x}), p_2(t, \boldsymbol{x}), \ldots, p_n(t, \boldsymbol{x})$ on $\mathbb{R}_+^n$ is homeomorphic to

$$\mathrm{Trop}(A_1, \omega_1, \varepsilon_1) \cap \mathrm{Trop}(A_2, \omega_2, \varepsilon_2) \cap \ldots \cap \mathrm{Trop}(A_n, \omega_n, \varepsilon_n)$$

where $\mathrm{Trop}(A_i, \omega_i, \varepsilon_i)$ are the positive part of tropical varieties $\mathrm{Trop}(A_i, \omega_i)$ as defined in Section 2.2 .

Theorem 1.1 yields a polyhedral object that is homeomorphic to the common zero set of $p_1(t, \boldsymbol{x}), \ldots, p_n(t, \boldsymbol{x})$ on $\mathbb{R}_+^n$ for sufficiently large t, and it can also be used to handle the set of common zeros on $(\mathbb{R}^*)^n$. We have three immediate questions:

(1) How can we quantify precisely when t is "sufficiently large"?
(2) Given a polynomial system $p_1(t, \boldsymbol{x}), \ldots, p_n(t, \boldsymbol{x})$ with support sets $A_1, \ldots, A_n$ and coefficients $c_{\boldsymbol{\alpha}}^i$ for $\boldsymbol{\alpha} \in A_i$ (as in the theorem statement), can we guarantee that the number of common real zeros does not change as t goes from 1 to ∞?
(3) Can we use the technique in Theorem 1.1 for polynomial systems that are not necessarily at the "toric limit"?

The first two questions are interrelated, and they form the main difficulty with respect to developing an algorithmic version of Theorem 1.1. These questions were asked since 90's [Stu98]; to the best of our knowledge the current paper provides the first progress. We provide an explicit criterion to answer the second question (stated in Section 3). The criterion also furnishes a homotopy algorithm that operates entirely over the reals, which we call *real polyhedral homotopy algorithm (RPH)*.

The third question is due to Itenberg and Roy; they conjectured Viro's patchworking method provides an upper bound for the number of real zeros regardless of the polynomial system being at the toric limit or not [IR96]. Li and Wang provided a counterexample to the Itenberg-Roy Conjecture [LW98].

1.1. **Effective Patchworking.** Our development is based on an observation from the book [GKZ08] by Gelfand, Kapranov, and Zelevinsky (henceforth GKZ) which provides a link between Viro's patchworking method and A-discriminants. Using the GKZ observation for an algorithm is not straightforward. It requires to locate a query point against the A-discriminant variety, and this may be intractable: The defining equation of the discriminant locus is known to be *extremely* complicated; it obstructs the use of computational algebra methods. However, it is no obstruction against the use of amoeba theory. Discriminantal amoebas are proven to admit a certain parametric description, and it is easy to compute normal directions on their boundary; see Section 2.9. We exploit these special differential geometric properties of A-discriminant amoebas to develop an effective criterion for checking whether a given polynomial system is "easy".

Note that RPH relies on notions from discrete and tropical geometry, and on further notions from GKZ. Furthermore, we use an algorithm called *Tropical Homotopy* due to Jensen [Jen16b]. So, before reading the main statement in Section 3 we encourage the reader to check familiarity with the content of Section 2.1, Section 2.2, Section 2.6, Section 2.7, and Section 2.9.

1.2. **Complexity Aspects.** Our work is inspired by the *practical* efficiency of complex polyhedral homotopy algorithm. Complexity aspects of polyhedral homotopy remain elusive since more than two decades; early papers did not include any complexity analysis, later different authors approached the issue [MR04, Mal17, Mal16], certain technical obstacles still remain; see Section 5.4.

A complete complexity analysis of RPH will only become possible when the scientific community fully understands the complexity of numerical path tracking for sparse polynomial systems. We present our thoughts on the complexity of discrete computations, and touch upon the complexity of the numerical part of RPH in Section 5.

We point out here that the main parameters governing the complexity of RPH are different than its complex cousin: the overall complexity of RPH is controlled by the *number of mixed cells* (a combinatorial invariant), the complexity of complex polyhedral homotopy is, in contrast, controlled by the *mixed volume* (a geometric invariant).

1.3. **Connections to Fewnomial Theory.** A system of polynomials $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ is called a *patchworked polynomial system* if the real zero set of $\boldsymbol{p}$ is homeomorphic to a simplicial complex created by Viro's patchworking. For instance, every polynomial system that passes our test in Section 3 is a patchworked system. In Section 5.3 we prove the following result:

Theorem 1.2. *Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ be a patchworked polynomial system where every polynomial p_i has at most t terms. Then $\boldsymbol{p}$ can have at most $2^{n+1}e^n(t-1)^n$ many common zeros on $(\mathbb{R}^*)^n$.*

Theorem 1.2 is an application of McMullen's upper bound theorem. Things become geometrically more interesting when one tries to bound the number of mixed cells for support sets A_i with different cardinalities (mixed supports). In [Bih16] it is claimed that a patchworked polynomial system $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, where p_i has at most t_i many terms, can have at most $\prod_{i=1}^{n}(t_i - 1)$ many zeros on $\mathbb{R}^n_+$. We have learned from Bihan that the proof of this result is not correct, but the result still holds true. Bihan informed us that a new proof and an erratum will appear soon ([Bih20]).

1.4. **Structure of the paper.** Our aim is to write this paper as self contained as possible. The preliminaries section contains background information and results from discrete geometry, the theory of A-discriminants, symbolic computation, and numerical path trackers. Jensen's tropical homotopy algorithm and mixed cell cones are also introduced in this section. In the third section, we transform asymptotic and qualitative results from [GKZ08] to a more quantitative and checkable condition. In the fourth section we present our real polyhedral homotopy algorithm and an example. The fifth section is concerned with the complexity aspects. The last section contains a discussion of questions that were brought to our attention after the initial version of this paper appeared on ArXiv.

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2. Preliminaries

We denote $[n] := \{1, \ldots, n\}$, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. Let $\boldsymbol{e}_j$ denote the j-th coordinate vector in $\mathbb{R}^n$. To avoid redundancies later in the articles we set $\boldsymbol{e}_0 := \mathbf{0}$.

For a given convex set C, we denote its boundary by ∂C. For a given polytope P, we denote its *vertex set* as $\text{Vert}(P)$. For $\boldsymbol{v} \in \text{Vert}(P)$ we denote the corresponding *normal cone* as $\text{NC}(\boldsymbol{v})$ and the entire *normal fan* as $\text{NF}(P)$. We set the convention in this paper that normal vectors point in outer direction.

In what follows we consider finite sets $A := \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{Z}^n$ and $A_1, A_2, \ldots, A_k \subset \mathbb{Z}^n$, which are *support sets* of polynomials. We denote the *Minkowski sum* of the A_i as $\sum_{i=1}^k A_i$. Note that

$$\text{conv}\left(\sum_{i=1}^k A_i\right) \;=\; \sum_{i=1}^k \text{conv}(A_i).$$

For a polynomial $p \in \mathbb{C}[\boldsymbol{x}]$ with support A, the *Newton polytope* is given by $\text{New}(p) := \text{conv}(A)$. We denote the *variety* of a system of polynomials $\boldsymbol{p}$ as $\mathcal{V}(\boldsymbol{p})$, the *real locus* as $\mathcal{V}_{\mathbb{R}}(\boldsymbol{p}) := \mathcal{V}(\boldsymbol{p}) \cap \mathbb{R}^n$, and positive / nonzero real locus as $\mathcal{V}_{\mathbb{R}_{>0}}(\boldsymbol{p})$ and $\mathcal{V}_{\mathbb{R}^*}(\boldsymbol{p})$.

2.1. Polyhedral Subdivisions, Secondary Polytope and Cayley Configurations.

In this section we introduce polyhedral subdivisions, secondary polytopes and Cayley Configurations; for further details we refer the reader to [DLRS10].

Let $A \subset \mathbb{Z}^n$ be a set of lattice points and let $\omega : A \to \mathbb{R}$ be a function. The *lifting* of A induced by ω is defined as:

$$A^\omega \;:=\; \{(\boldsymbol{x}, \omega(\boldsymbol{x})) : \boldsymbol{x} \in A\}.$$

We call a facet F of $\text{conv}(A^\omega)$ an *upper face* if it is given by

$$F \;=\; \{\boldsymbol{x} \in \text{conv}(A^\omega) : \langle \boldsymbol{c}, \boldsymbol{x}\rangle \geq \langle \boldsymbol{c}, \boldsymbol{y}\rangle \text{ for all } \boldsymbol{y} \in \text{conv}(A^\omega)\}$$

where $\boldsymbol{c}$ is a vector with a positive last entry. Intuitively, upper faces are the faces that are "visible" from $(0,\dots,0,\infty)$. We project upper facets of $\operatorname{conv}(A^\omega)$ on the point set A:

$$\Delta_\omega \;:=\; \{\boldsymbol{x} \in A : (\boldsymbol{x}, \omega(\boldsymbol{x})) \text{ belongs to an upper face of } \operatorname{conv}(A^\omega)\}\,.$$

Δ_ω is a polyhedral subdivision of A. Polyhedral subdivisions obtained this way are called *coherent* or *regular*. Note that Δ_ω is a triangulation unless the lifted points A^ω have (particular) affine dependencies.

Now we define the *secondary polytope* of A, which encodes all coherent triangulations of A, and discuss its key properties; see [DLRS10, Section 5].

Definition 2.1. Let T be a triangulation of $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_m\}$, and let $\sigma_1, \dots, \sigma_s$ be the simplices in T. We define

$$\Phi_A(T) \;:=\; \sum_{j=1}^{m} \left(\sum_{\{\sigma \in T \,:\, \boldsymbol{a}_j \in \sigma\}} \operatorname{vol}(\sigma) \right) \boldsymbol{e}_j.$$

We define the *secondary polytope* of A as:

$$\Sigma(A) \;:=\; \operatorname{conv}\{\Phi_A(T) \;:\; T \text{ is a triangulation of } A\}\,.$$

The corresponding normal fan $\operatorname{NF}(\Sigma(A))$ is called the *secondary fan*. For its cones, the *secondary cones*, we use the abbreviated notation $\operatorname{NC}(T) := \operatorname{NC}(\Phi_A(T))$. ◇

We state a collection of key properties of the secondary polytope; see e.g., [DLRS10, Section 5].

Theorem 2.2. *The secondary polytope has the following properties:*

1. *The vertices of $\Sigma(A)$ are in one to one correspondence to the coherent triangulations of A.*
2. *The face lattice of $\Sigma(A)$ is isomorphic to a refinement poset of the coherent polyhedral subdivisions of A.*
3. *A lifting function $\omega : A \to \mathbb{R}$ induces the triangulation T if and only if $\omega \in \operatorname{int}(\operatorname{NC}(T))$.*
4. *Consider the support set A as a $n \times m$ integer matrix. Then every secondary cone $\operatorname{NC}(T)$ includes the $n+1$ dimensional linear space spanned by rows of A and all ones vector $(1,1,\dots,1)$. As a consequence, the secondary polytope $\Sigma(A)$ is $m-n-1$ dimensional.*

Let F be a cell in coherent polyhedral subdivision of $\sum_{i=1}^{k} A_i$ introduced by a lifting function ω. Then F corresponds to a face in $\sum_{i=1}^{n} \operatorname{conv}(A_i)^\omega$. Let $F = \sum_{i=i}^{k} F_i$ where F_i are the corresponding faces on $\operatorname{conv}(A_i)^\omega$.

Definition 2.3. A polyhedral subdivision Δ_ω of $A_1 + A_2 + \ldots + A_k$ for $A_i \subset \mathbb{Z}^n$ is called *fine mixed* if it satisfies the following conditions:

1. For all cells F in the subdivision, we have $\sum_{i=1}^{k} \dim(F_i) = n$, and
2. for all cells F in the subdivision we have $\sum_{i=1}^{k} (\#F_i - 1) = n$,

where $\#F_i$ denotes the number of vertices of F_i. ◇

Later, in Section 2.6, we introduce the concept of a mixed cell, which is related to a mixed subdivision. Note, however, that not every cell in a mixed subdivision is a mixed cell.

We also need to define Cayley configuration of point sets $A_1, A_2, \ldots, A_k$ and the corresponding Cayley polytope.

Definition 2.4. We define the *Cayley configuration* of $A_1, A_2, \ldots, A_k$ as

$$\mathbf{A} \;=\; A_1 * A_2 * \cdots * A_k \;:=\; \{(\boldsymbol{x}, \boldsymbol{e}_{i-1}) : \boldsymbol{x} \in A_i\} \subseteq \mathbb{R}^{n+k-1},$$

The *Cayley polytope* is defined as $\mathrm{conv}(\mathbf{A})$, denoted by $\mathrm{Cay}(\mathbf{A})$. ○

The following observation is implicit in most papers in literature: A natural slicing of the Cayley polytope $\mathrm{Cay}(\mathbf{A})$ is isomorphic to $\sum_{i=1}^{n} \mathrm{conv}(A_i)$. More precisely, consider the following set defined by the intersection of $\mathrm{Cay}(\mathbf{A})$ with several hyperplanes:

$$\widetilde{\mathrm{Cay}(\mathbf{A})} \;:=\; \left\{\boldsymbol{x} \in \mathrm{Cay}(\mathbf{A}) \;:\; x_{n+1} = x_{n+2} = \ldots = x_{n+k-1} = \frac{1}{k}\right\}.$$

Observe that a k-scaling of $\widetilde{\mathrm{Cay}(\mathbf{A})}$, i.e., $k \cdot \widetilde{\mathrm{Cay}(\mathbf{A})}$, is isomorphic to $\sum_{i=1}^{n} \mathrm{conv}(A_i)$. For a detailed explanation and a figure see e.g., [HRS00].

Suppose that T is a coherent triangulation of the Cayley configuration $\mathbf{A}$. First, note that $T \cap \widetilde{\mathrm{Cay}(\mathbf{A})}$ creates a polyhedral subdivision of $\widetilde{\mathrm{Cay}(\mathbf{A})}$. Via the isomorphism, this gives a polyhedral subdivision of $\sum_{i=1}^{n} \mathrm{conv}(A_i)$. Let σ be a simplex in T, then σ has $n+k$ vertices which split into sets of vertices σ_i that are induced by A_i. None of the σ_i are empty since otherwise σ can not be full-dimensional. Then, up to an isomorphism, $F_\sigma = \mathrm{conv}(\sigma_1) + \mathrm{conv}(\sigma_2) + \ldots + \mathrm{conv}(\sigma_k)$ yields a cell in the polyhedral subdivision of $\sum_{i=1}^{n} \mathrm{conv}(A_i)$, and all such cells yield a fine mixed subdivision of $\sum_{i=1}^{n} \mathrm{conv}(A_i)$. This correspondence gives a bijection between coherent triangulations of the Cayley polytope and coherent fine mixed subdivisions of the Minkowski sum $\sum_{i=1}^{k} \mathrm{conv}(A_i)$; see [Stu94a, Theorem 5.1].

In summary, coherent fine mixed subdivisions of $\sum_{i=1}^{k} \mathrm{conv}(A_i)$ can be understood by studying the vertices of the secondary polytope $\Sigma(A_1 * A_2 * \cdots * A_k)$ and the corresponding secondary cones.

Remark 2.5. *At various parts of this article (in our theorem statements and algorithms) we work with triangulations. For a generic lifting function ω the induced polyhedral subdivision Δ_ω is a triangulation. The proof of* [DLRS10, Proposition 2.2.4] *suggests an algorithm, albeit an inefficient one, to check whether a given lifting is generic. The question of finding an efficient algorithm to check genericity of a lifting is interesting, but it is beyond the scope of our paper.*

2.2. **Viro's Patchworking Method.** In this section we introduce Viro's patchworking method for complete intersections. For further details and relations to Hilbert's 16th problem, we kindly refer the reader to Viro's survey [Vir08]. For further background information on tropical geometry see e.g., [IMS09, MS15].

Definition 2.6. Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\} \subset \mathbb{Z}^n$ and Δ_ω be a coherent triangulation of A given by a lifting function $\omega : A \to \mathbb{R}$. We define the associated *tropical variety* as

$$\mathrm{Trop}(A, \omega) := \{\boldsymbol{x} \in \mathbb{R}^n : \max_i\{\langle \boldsymbol{x}, a_i\rangle + \omega(a_i)\} \text{ is attained at least twice}\}.$$

◯

Since we are interested in real varieties, we distinguish a positive and a negative part of $\mathrm{Trop}(A, \omega)$ depending on a given sign vector $\varepsilon : A \to \{-1, +1\}$. First, we observe that $\mathrm{Trop}(A, \omega)$ together with its complement creates a polyhedral decomposition of $\mathbb{R}^n$. Also, by definition, every full-dimensional cell in the complement of $\mathrm{Trop}(A, \omega)$ corresponds to a unique $\boldsymbol{a}_j \in A$ as it is given by the set:

$$\{\boldsymbol{x} \in \mathbb{R}^n \ : \ \langle \boldsymbol{x}, \boldsymbol{a}_j\rangle + \omega(a_j) > \langle \boldsymbol{x}, \boldsymbol{a}_i\rangle + \omega(a_i) \text{ for all } i \in [n] \setminus \{j\}\}.$$

We define the sign of this cell as $\varepsilon(\boldsymbol{a}_j)$. For every $(n-1)$-dimensional cell in $\mathrm{Trop}(A, \omega)$, there exist two adjacent n-dimensional cells with signs assigned by ε. This motivates the definition of the positive part of a tropical variety.

Definition 2.7. The *positive part* $\mathrm{Trop}(A, \omega, \varepsilon)$ of a given tropical variety $\mathrm{Trop}(A, \omega)$ is the subcomplex consisting of those $(n-1)$-dimensional cells that are adjacent to two n-cells with different signs. ◯

Now, we state Sturmfels' generalization of Viro's method to complete intersections [Stu94b].

Theorem 2.8 (Viro's Patchworking for Complete Intersections [Stu94b])**.** *Let $A_1, \ldots, A_k \subset \mathbb{Z}^n$, let $\omega : A_1 * A_2 * \ldots * A_k \to \mathbb{R}$ be a lifting function. Consider a system of polynomials $\boldsymbol{p} = (p_1, p_2, \ldots, p_k)$ defined as follows:*

$$p_i(t, \boldsymbol{x}) \ := \ \sum_{\alpha \in A_i} c_\alpha t^{\omega(\alpha)}\, \boldsymbol{x}^\alpha$$

with $c_\alpha \in \mathbb{R}$. Let $\varepsilon : A_1 * A_2 * \cdots * A_k \to \{-1, +1\}$ be the sign function defined by coefficients of $\boldsymbol{p}$. Then, for sufficiently large $t > 0$, the real algebraic set $\mathcal{V}_{\mathbb{R}_{>0}}(\boldsymbol{p})$ is homeomorphic to

$$\mathrm{Trop}(A_1, \omega_1, \varepsilon_1) \cap \mathrm{Trop}(A_2, \omega_2, \varepsilon_2) \cap \ldots \cap \mathrm{Trop}(A_k, \omega_k, \varepsilon_k)$$

where ω_i and ε_i are restrictions of ω and ε to A_i.

Remark 2.9. *For readers who are familiar with non-Archimedian tropical geometry the theorem statement here might look confusing. The only difference is that in non-Archimedian tropical geometry it is rather customary to use min notation, lower facets, and t tends to zero. In amoeba theory, however, it is customary to use max notation, upper facets, and t tends to ∞. We follow the amoeba theory convention.*

Theorem 2.8 generalizes to the set of zeros on $(\mathbb{R}^*)^n$ by applying the theorem on every one of the 2^n orthants separately, taking the signs of the variables $x_1, \ldots, x_n$ into account, and then gluing them together; see [Stu94b, Theorem 5]. We illustrate Theorem 2.8 on the most simple example possible.

Example 2.10. The set $A := \{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ represents the support set for linear forms (and hence its convex hull is the standard simplex). We consider positive solutions of an affine linear form $f = u_0 + \sum_{i=1}^n u_i x_i$, i.e., the solutions with $x_i > 0$. We define a variant of moment the from symplectic geometry called *algebraic moment map*:

$$\mu_A : \mathbb{R}_+^n \to \mathrm{conv}(A) \qquad \boldsymbol{x} \;\mapsto\; \frac{\sum_i x_i \boldsymbol{e}_i}{1 + \sum_i x_i}.$$

This map is a homeomorphism. The image of $\mathcal{V}_{\mathbb{R}_{>0}}(u_0 + \sum_{i=1}^n u_i x_i)$ under μ_A is given by:

$$\mu_A(\mathcal{V}_{\mathbb{R}_{>0}}(f)) \;=\; \left\{(y_1, y_2, \ldots, y_n) \in \mathrm{conv}(A) : u_0 \left(1 - \sum_{i=1}^n y_i\right) + \sum_{i=1}^n u_i y_i = 0\right\}.$$

Hence, $\mu_A(\mathcal{V}_{\mathbb{R}_{>0}}(f))$ is defined by the linear form $u_0 + u_1 x_1 + \ldots + u_n x_n$ on the simplex $\mathrm{conv}(A)$, and it separates those $\boldsymbol{e}_i$ with $u_i > 0$ from those $\boldsymbol{e}_j$ with $u_j < 0$. ⬡

To prove Theorem 2.8 above, one replaces the simplex with the triangulation, and the moment map with the moment map corresponding to the toric variety defined by $A_1 + A_2 + \ldots + A_k$ as explained in of [GKZ08, Chapter 11, Section 5, Subsections C and D]. We provide another example, first considered by Sturmfels [Stu94a, Page 382].

Example 2.11. Consider the two polynomials

$$\begin{aligned} f_t &= x_2^3 - t x_1 x_2^2 - t^5 x_1^2 x_2 + t^{12} x_1^3 - t x_2^2 + t^4 x_1 x_2 - t^9 x_1^2 - t^5 x_2 - t^9 x_1 + t^{12} \\ g_t &= t^8 x_2^2 - t^6 x_1 x_2 + t^6 x_1^2 - t^3 x_2 - t^2 x_1 + 1 \end{aligned}$$

We consider the lifting function ω introduced by the exponents of t, the sign function ε introduced by the coefficients of f_t and g_t, and compute the corresponding patchworking. We present the outcome in Figure 1. The computation was already carried out by Sturmfels in the original article [Stu94a] in '94. Here, we generate a plot using the Viro.sage package by O'Neill, Kwaakwah, and the second author [OKdW18]. ⬡

2.3. ***A*-Discriminants, a Theorem of Esterov, and Principal *A*-Determinants.** Given a set of lattice points $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\} \subset \mathbb{Z}^n$, we define

$$\mathbb{C}^A \;:=\; \left\{\sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}[\boldsymbol{x}] \;:\; c_{\boldsymbol{\alpha}} \in \mathbb{C} \text{ for all } \boldsymbol{\alpha} \in A\right\}$$

as the *space of polynomials supported on A*. Note that $\mathbb{C}^A$ is isomorphic to $\mathbb{C}^m$ with $m = \#A$. We define $(\mathbb{C}^*)^A$ analogously with $c_{\boldsymbol{\alpha}} \in \mathbb{C}^*$. Moreover, we define:

$$\nabla_A \;:=\; \left\{f \in (\mathbb{C}^*)^A \;:\; f \text{ has a singularity on } (\mathbb{C}^*)^n\right\}.$$

Except for particular special configurations A, which are called *defect*, the Zariski closure of this set is an irreducible hypersurface given by a polynomial with integral coefficients; [GKZ08, Chapter 9]. We are interested in the real part

$$\nabla_A(\mathbb{R}) \;:=\; \nabla_A \cap \mathbb{R}[\boldsymbol{x}]$$

of this hypersurface. Note that we de not require a polynomial in $\nabla_A(\mathbb{R})$ to posses a singularity in $(\mathbb{R}^*)^n$ instead of $(\mathbb{C}^*)^n$. $\nabla_A(\mathbb{R})$ represent polynomials with real coefficients

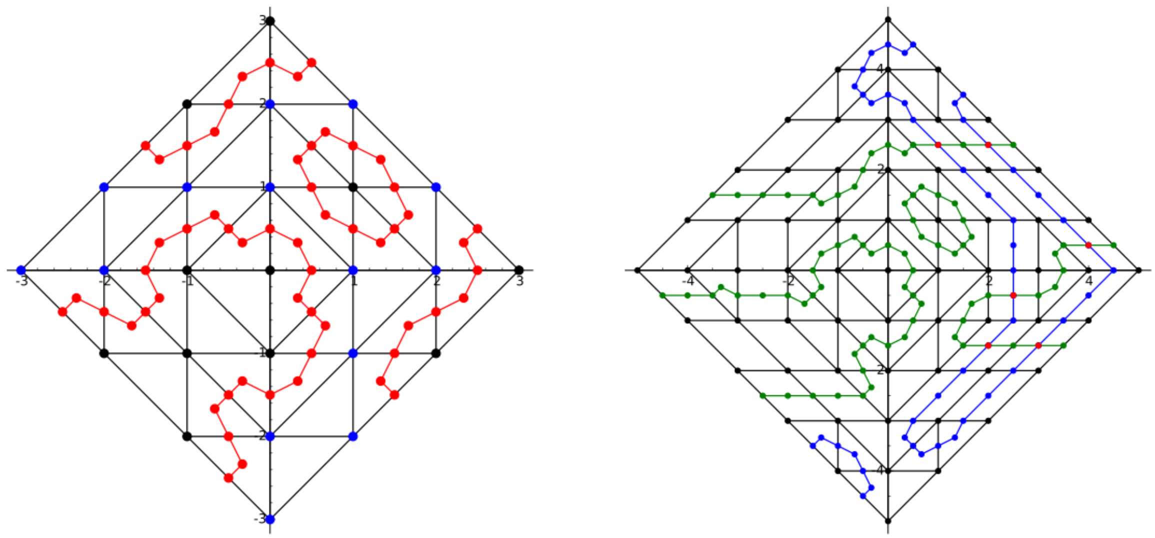

FIGURE 1. Viro Patchworking of f_t and the complete intersection of f_t and g_t for f_t, g_t defined as in Example 2.11.

and the complex roots of a real polynomial have a complex conjugated counterpart; thus the polynomials in $\nabla_A(\mathbb{R})$ that have a complex singularity form a higher codimension variety. For our purposes we are only interested in the codimension one part of $\nabla_A(\mathbb{R})$, so the definition of $\nabla_A(\mathbb{R})$ as above suffices for our purposes.

The hypersurface $\nabla_A(\mathbb{R})$ partitions the coefficient space $\mathbb{R}^A$ into connected components. If two polynomials $f, g \in \mathbb{R}^A$ lie in the same connected component of $\mathbb{R}^A - \nabla_A(\mathbb{R})$, then $\mathcal{V}_{\mathbb{R}^*}(f)$ and $\mathcal{V}_{\mathbb{R}^*}(g)$ are isotopic; see [GKZ08, page 380].

Let, as before, $A_1, A_2, \ldots, A_k \subset \mathbb{Z}^n$ be point sets, and let $\mathbf{A} = A_1 * A_2 * \ldots * A_k$ be the Cayley configuration. Also assume that the point sets A_i are full dimensional, i.e., $\dim(\mathrm{conv}(A_i)) = n$ for $1 \leq i \leq k$. For a tuple of *coefficient vectors* $\boldsymbol{C} = (\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_k)$ with $\boldsymbol{C}_i \in \mathbb{C}^{A_i}$, let $\boldsymbol{p_C}$ be the polynomial system $\boldsymbol{p_C} = (p_1, p_2, \ldots, p_k)$ with $p_i = \sum_{\boldsymbol{a}_{ij} \in A_i} \boldsymbol{C}_{ij}\, \boldsymbol{x}^{\boldsymbol{a}_{ij}}$. We define the *discriminantal locus* for systems of equations as follows:

$$\nabla_{A_1,A_2,\ldots,A_k} := \left\{(\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_k) \in \mathbb{C}^{A_1} \times \ldots \times \mathbb{C}^{A_k} : \boldsymbol{p_C} \text{ posses a singularity on } (\mathbb{C}^*)^n\right\}.$$

The discriminantal locus corresponding to hypersurfaces supported by the Cayley configuration $\mathbf{A} = A_1 * A_2 * \cdots * A_k$ is then given by

$$\nabla_{\mathbf{A}} := \left\{\boldsymbol{C} \in \mathbb{C}^{\mathbf{A}} : \sum_{a \in \mathbf{A}} c_i x^a \text{ posses a singularity on } (\mathbb{C}^*)^n\right\}.$$

If $\mathbf{A} = A_1 * A_2 * \cdots * A_k$ is not defect, then $\nabla_{\mathbf{A}}$ is an irreducible hypersurface. Also, by using the definition of singularity with the Jacobian matrix, it immediately follows that $\nabla_{\mathbf{A}} \subseteq \nabla_{A_1,A_2,\ldots,A_n}$. The following result of Esterov, proved by a simple perturbation argument, relates $\nabla_{\mathbf{A}}$ and $\nabla_{A_1,A_2,\ldots,A_k}$; see of [Est10, Lemma 3.36], and note that in Esterov's notation $\nabla_{A_1,A_2,\ldots,A_k}$ is denoted by $\Sigma_{A_0,A_1,\ldots,A_\ell}$.

Theorem 2.12 (Esterov)**.** *If* $\mathbf{A} = A_1 * A_2 * \cdots * A_k$ *is not defect, and* $\dim(\operatorname{conv}(A_i)) = n$ *for* $i = 1, 2, \ldots, k$*, then* $\nabla_{A_1, A_2, \ldots, A_k}$ *is irreducible of codimension one.*

Hence, if the assumptions of Esterov's theorem are satisfied, then $\nabla_{\mathbf{A}}$ and $\nabla_{A_1,A_2,\ldots,A_k}$ coincide. So, in order to control the changes in the topology for systems of equations supported with $A_1, A_2, \ldots, A_k$, we use the hypersurface $\nabla_{\mathbf{A}}(\mathbb{R})$. Our final object in this section is the principal A determinant.

Definition 2.13. For $A \subset \mathbb{Z}^n$, and a polynomial f supported by A the *principal A-determinant* E_A evaluated at the coefficients of f reads as follows

$$E_A(f) \ := \ R_A\left(f, x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}, \ldots, x_n\frac{\partial f}{\partial x_n}\right)$$

where R_A denotes the *sparse resultant*. ○

A nice exposition for sparse resultants with a computational focus can be found in [CE00]; for a more general development see [GKZ08, Chapter 8].

Theorem 2.14. [GKZ08] *The principal A-determinant E_A has the following properties:*

- *The Newton polytope of E_A is the secondary polytope $\Sigma(A)$.*
- *The hypersurface defined by E_A includes the irreducible hypersurface ∇_A.*

2.4. **A-Discriminant Amoeba.** In this section, we introduce the notion of amoeba following Gelfand, Kapranov, and Zelevinsky [GKZ08], and present special properties of discriminant amoebas. For a general overview of amoeba theory please see [Mik04, PT05].

Definition 2.15. We define the *Log-absolute value map* as

$$\operatorname{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, z_2, \ldots, z_n) \to (\log|z_1|, \log|z_2|, \ldots, \log|z_n|).$$

For a Laurent-polynomial $f \in \mathbb{C}[\boldsymbol{z}^{\pm 1}]$ and variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ we define the *amoeba* of f as $\mathcal{A}(f) := \operatorname{Log}|\mathcal{V}(f)| \subseteq \mathbb{R}^n$. ○

Note that the complement of an amoeba $\mathcal{A}(f)$ consists of at most $\mathbb{Z}^n \cap \operatorname{New}(f)$ many connected components and each of these components is convex.

Lemma 2.16. *Let $f = \sum_i c_i \boldsymbol{x}^{\boldsymbol{a}_i}$ be a polynomial with support $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$. Let $\boldsymbol{v}$ be a vertex of* $\operatorname{New}(f)$*. Suppose that $\boldsymbol{b} \in \operatorname{NC}(\boldsymbol{v})$ with*

$$\langle \boldsymbol{b}, \boldsymbol{v} - \boldsymbol{a}_i \rangle \ > \ \log\left(\frac{m \cdot |c_i|}{|c_{\boldsymbol{v}}|}\right)$$

for all $\boldsymbol{a}_i \neq \boldsymbol{v}$. Then, $\mathcal{A}(f) \cap (\boldsymbol{b} + \operatorname{NC}(\boldsymbol{v})) = \emptyset$.

The statement is well-known; see [GKZ08, Prop. 1.5, Page 195]. Here, we provide the main argument of the proof for the convenience of the reader.

Proof. We have

$$f(\boldsymbol{x}) \ = \ c_{\boldsymbol{v}}\, \boldsymbol{x}^{\boldsymbol{v}} \left(1 + \sum_{\boldsymbol{a}_i \neq \boldsymbol{v}} \frac{c_i}{c_{\boldsymbol{v}}} \boldsymbol{x}^{\boldsymbol{a}_i - \boldsymbol{v}}\right).$$

Set $g(\boldsymbol{x}) = \sum_{\boldsymbol{a}_i \neq \boldsymbol{v}} \frac{c_i}{c_{\boldsymbol{v}}} \boldsymbol{x}^{\boldsymbol{a}_i - \boldsymbol{v}}$. Then for a given $\boldsymbol{x} \in (\mathbb{C}^*)^n$ if $|g(\boldsymbol{x})| < 1$ this immediately implies $f(\boldsymbol{x}) \neq 0$ and hence $\operatorname{Log}|\boldsymbol{x}| \notin \mathcal{A}(f)$. The rest of the proof is straightforward. □

Using the property that $\nabla_{\mathbf{A}}$ is included in the zero set of $E_{\mathbf{A}}$ together with Theorem 2.14 and Lemma 2.16 we obtain the following statement.

Lemma 2.17. *Let $A_1, A_2, \ldots, A_k \subset \mathbb{Z}^n$ be point configurations with $\dim(A_i) = n$ for all $1 \leq i \leq k$. Let $\mathbf{A} = A_1 * A_2 * \ldots * A_k$, let T be a triangulation of $\mathbf{A}$, and let $\mathrm{NC}\,(T)$ be the secondary cone of T. Then for all $\boldsymbol{b} \in \operatorname{int}(\mathrm{NC}\,(T))$ there exist a $\lambda > 0$ with the following property: $\mathcal{A}(\nabla_{A_1,A_2,\ldots,A_k}) \cap (\lambda\boldsymbol{b} + \mathrm{NC}\,(T)) = \emptyset$.*

It suffices for λ in Lemma 2.17 to be big enough so that $\lambda\boldsymbol{b}$ satisfies the inequality in Lemma 2.16 for the polynomial $E_{\mathbf{A}}$. If we combine Lemma 2.17 with Theorem 2.8, then we obtain the following statement.

Proposition 2.18. *We use the same notation as Lemma 2.17 for A_i, $\mathbf{A} = A_1 * A_2 * \cdots * A_k$, T, $\boldsymbol{b}$, λ, and $\mathrm{NC}\,(T)$. Let $\varepsilon : \mathbf{A} \to \{+1, -1\}$ be a sign vector. We define:*

$$U(T,\varepsilon) \;:=\; \left\{\boldsymbol{C} \in (\mathbb{R}^*)^{\mathbf{A}} : \operatorname{Log}|\boldsymbol{C}| \in \lambda\boldsymbol{b} + \mathrm{NC}\,(T) \text{ and } \operatorname{sign}(\boldsymbol{C}) = \varepsilon\right\} \subseteq (\mathbb{R}^*)^{\mathbf{A}}.$$

Then for a system of polynomials $\boldsymbol{p}_{\boldsymbol{C}}$ with support $A_1, A_2, \ldots, A_k$ and a coefficient vector $\boldsymbol{C} \in U(T,\varepsilon)$, the real topology of $\boldsymbol{p}_{\boldsymbol{C}}$ is completely determined by the triangulation T and the sign vector ε.

The complement of the amoeba of $E_{\mathbf{A}}$ consists of convex regions corresponding to vertices $\phi_{\mathbf{A}}(T)$ of the secondary polytope $\Sigma_{\mathbf{A}}$; this is a general well-known fact from amoeba theory [PST05, Corollary 8]. Simply put, there is bijection between connected components in the complement of the amoeba of $E_{\mathbf{A}}$ and regular triangulations of $\mathbf{A}$.

Let $\boldsymbol{C} \in (\mathbb{R}^*)^{\mathbf{A}}$ be a vector of coefficients of a polynomial system $\boldsymbol{p}$; if $\operatorname{Log}|\boldsymbol{C}|$ is contained in a connected component in the complement of the amoeba of $E_{\mathbf{A}}$, then Lemma 2.17 and Proposition 2.18 show that the real topology of the polynomial system $\boldsymbol{p}_{\boldsymbol{C}}$ is completely determined by corresponding triangulation T and the signs of the coordinates of the vector $\boldsymbol{C}$. Hence, Proposition 2.18 is a more detailed, quantitative reformulation of Theorem 2.8.

2.5. **Solving Binomial Systems Over The Reals.** This section is about *binomial systems*, i.e., systems of polynomials where every polynomial has only two terms. This simple case is important for the construction in the next subsection. Consider the following system of binomials:

$$c_{11}\,\boldsymbol{x}^{\boldsymbol{a}_{11}} \;=\; c_{12}\,\boldsymbol{x}^{\boldsymbol{a}_{12}}, \quad c_{21}\,\boldsymbol{x}^{\boldsymbol{a}_{21}} \;=\; c_{22}\,\boldsymbol{x}^{a_{22}}, \;\ldots,\quad c_{n1}\,\boldsymbol{x}^{\boldsymbol{a}_{n1}} \;=\; c_{n2}\,\boldsymbol{x}^{\boldsymbol{a}_{n2}}$$

where $c_{ij} \in \mathbb{R}^*$ and $a_{ij} \in \mathbb{Z}^n$. This system is equivalent to the following system of equations:

$$\boldsymbol{x}^{\boldsymbol{a}_{11}-a_{12}} \;=\; \frac{c_{12}}{c_{11}}, \quad \boldsymbol{x}^{\boldsymbol{a}_{21}-a_{22}} \;=\; \frac{c_{22}}{c_{21}}, \;\ldots,\quad \boldsymbol{x}^{\boldsymbol{a}_{n1}-a_{n2}} \;=\; \frac{c_{n2}}{c_{n1}} \tag{2.1}$$

Set $\boldsymbol{D}_i = \boldsymbol{a}_{i1} - \boldsymbol{a}_{i2}$, and $B = [\boldsymbol{d}_1\boldsymbol{d}_2\ldots\boldsymbol{d}_n]$. To solve the system (2.1) over $(\mathbb{R}^*)^n$, it suffices to perform the elementary integer operations that reduce D into its Hermite normal form.

This operations can be done in strong polynomial time [KB79]. The result is a system of equations in the following format:

$$\boldsymbol{x}_1^{\boldsymbol{h}_{11}} \;=\; \lambda_1, \quad \boldsymbol{x}_1^{\boldsymbol{h}_{21}}\boldsymbol{x}_2^{\boldsymbol{h}_{22}} \;=\; \lambda_2, \;\ldots, \quad \boldsymbol{x}_1^{\boldsymbol{h}_{n1}}\ldots\boldsymbol{x}_n^{\boldsymbol{h}_{nn}} \;=\; \lambda_n, \tag{2.2}$$

where $\boldsymbol{h}_{ij} \neq 0$ and $\lambda_i \in \mathbb{R}^*$. The solutions of (2.2) are completely determined by the signs of λ_i and $\boldsymbol{h}_{ij}$ being even or odd. Hence, (2.2) either has no solution in $(\mathbb{R}^*)^n$, or there exist solutions differing only by their signs.

There is also a recent paper focusing on probabilistic analysis of numerical methods for binomial system solving [PPR19].

2.6. **Real Toric Deformation.** In this article we consider zero dimensional systems. Hence, we have n support sets $A_1, A_2, \ldots, A_n \subset \mathbb{Z}^n$. For this case, some cells in the regular triangulation Δ_ω (introduced by a lifting ω) of $\mathbf{A} = A_1 * A_2 * \cdots * A_n$ are of particular interest. These cells are called mixed cells. Formally, a cell $\sigma \in \Delta_\omega$ that has 2 elements from each A_i is called a *mixed cell*. Equivalently, in a fine mixed subdivision of $A_1 + A_2 + \ldots + A_n$, mixed cells σ are the cells that are given by the Minkowski sum of n edges.

On the dual side, when we consider the finite set of points in the intersection

$$\mathrm{Trop}(A_1, \omega_1, \varepsilon_1) \cap \mathrm{Trop}(A_2, \omega_2, \varepsilon_2) \cap \ldots \cap \mathrm{Trop}(A_n, \omega_n, \varepsilon_n)$$

each of these points correspond to a mixed cell in the triangulation of $\mathbf{A} = A_1 * A_2 * \cdots * A_n$ where every two vertices from each A_i have opposite signs. In the current literature, such a simplex is called an *alternating mixed cell* [Jen16b].

Since we repeat the Viro construction in every orthant of $(\mathbb{R}^*)^n$, the sign vector ε changes. However, the lifting function ω and the corresponding triangulation remains the same for all orthants. So, in order to count the number of real zeros with Viro's method, one needs to investigate the mixed cells and check how many times a mixed cell becomes an alternating one. Algorithmically, instead of going through Viro's construction 2^n times, it is more convenient to use binomial systems corresponding to mixed cells and just solve them over $\mathbb{R}^n$: this corresponds to finding all orthants that turn the mixed cell into an alternating one. This approach is much more effective.

Our discussion so far focused on understanding those coefficient vectors $\boldsymbol{C} \in (\mathbb{R}^*)^{\mathbf{A}}$ for which Viro's method gives the correct real zero count of $\boldsymbol{p}_{\boldsymbol{C}}$. Now, we intend to compute these real zeros. This is well understood in the complex case; see e.g., [HS95, VVC94]. The situation over the reals is very similar as we state in the following proposition.

Proposition 2.19. *Let $A_1, A_2, \ldots, A_n \subset \mathbb{Z}^n$ be point configurations with $\dim(A_i) = n$ for all $i \in [n]$, and let $\mathbf{A} = A_1 * A_2 * \cdots * A_n$ be the Cayley configuration. Suppose that $\boldsymbol{C}, \boldsymbol{v} = (\boldsymbol{v}_{\boldsymbol{a}})_{\{\boldsymbol{a} \in A_i \,:\, 1 \leq i \leq n\}} \in \mathbb{R}^{\mathbf{A}}$ are vectors with the following properties:*

(1) $\boldsymbol{v}$ is not on the boundary of any secondary cone of the point configuration $\mathbf{A}$.

(2) The ray $\mathrm{Log}\,|\boldsymbol{C}| + \lambda \boldsymbol{v}$ does not intersect the amoeba of $\nabla_{\mathbf{A}}(\mathbb{R})$ for any $\lambda \in [0, \infty)$.

We consider a system of equations $\boldsymbol{p}_{\boldsymbol{C}}(t, \boldsymbol{x}) = (p_1, p_2, \ldots, p_n)$:

$$p_i(t, \boldsymbol{x}) \;=\; \sum_{\boldsymbol{a} \in A_i} c_\alpha t^{-v_{\boldsymbol{a}}}\, \boldsymbol{x}^{\boldsymbol{a}} \quad \textit{for} \;\; i = 1, 2, \ldots, n. \tag{2.3}$$

Then the real Puiseux series

$$\boldsymbol{x}(t) = (x_1 t^{\zeta_1}, x_2 t^{\zeta_2}, \ldots, x_n t^{\zeta_n}) + \textit{ higher order terms} \tag{2.4}$$

is a solution to the system $\boldsymbol{p}_{\boldsymbol{C}}$ only if $(\boldsymbol{\zeta}, 1)$ is an outer normal to a lower facet of

$$\mathrm{conv}(A_1^{\boldsymbol{v}} + A_2^{\boldsymbol{v}} + \ldots + A_n^{\boldsymbol{v}})$$

where $A_i^{\boldsymbol{v}}$ stands for the lifting of A_i with respect to the coordinates of $\boldsymbol{v}$.

Proof. The statement follows from the proof of [HS95, Lemma 3.1], so we just list the main steps: Put (2.4) into (2.3), divide by the lowest degree term, and set $t = 0$. The obtained equation can only be solved if it is a binomial system of equations; see the previous section. On the one hand, the solutions of these binomial systems correspond to the points that are given by Viro's method via the Log-map. On the other hand, the points given by Viro's method correspond to the alternating mixed cells as explained above. □

2.7. **Mixed-Cell Cones and Jensen's Tropical Homotopy Algorithm.** We used the secondary cone in the statement of Proposition 2.19 for conceptual ease, but the statement can be extended to a larger cone.

The main observation is that Theorem 2.8 and Proposition 2.18 depend only on the mixed cells in a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$; we do not need to differentiate between two triangulations that have same collection of mixed cells. We formalize this as follows.

Definition 2.20 (Mixed-Cell Cone of a Triangulation)**.** Let T be a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$, and let $\sigma \in T$ be a mixed cell. For every lifting function $\omega : \mathbf{A} \to \mathbb{R}$ (represented by a vector in $\mathbb{R}^{\mathbf{A}}$) we denote the induced subdivision as Δ_ω. We define the *mixed cell cone of σ* as:

$$M(\sigma) \;:=\; \{\omega \in \mathbb{R}^{\mathbf{A}} : \sigma \text{ is a mixed-cell in } \Delta_\omega\}.$$

Moreover, we define the *mixed cell cone of T* as:

$$M(T) \;:=\; \bigcap_{\{\sigma \,:\, \sigma \text{ is mixed cell of } T\}} M(\sigma).$$

⬡

The mixed-cell cone includes the secondary cone: $\mathrm{NC}\,(T) \subseteq M(T)$. We state the following lemma related to their difference.

Lemma 2.21. *Let $T = \Delta_\omega$ for a lifting function ω, and assume that $\omega \in \mathrm{NC}\,(\boldsymbol{v})$ where $\boldsymbol{v}$ is a vertex of the Newton polytope of the $\mathbf{A}$-discriminant and $\mathrm{NC}\,(\boldsymbol{v})$ is its normal cone. Now let $\boldsymbol{v} \in M(T) \setminus \mathrm{NC}\,(T)$ and let $\omega' = \omega + \lambda \boldsymbol{v}$ for a $\lambda \in [0, \infty)$. Then, $\omega' \in \mathrm{NC}\,(\boldsymbol{v})$. This can also be stated as follows:*

$$\mathrm{NC}\,(T) \subseteq M(T) \subseteq \mathrm{NC}\,(\boldsymbol{v}).$$

In words; the secondary cone is included in the mixed cell cone, and the mixed cone is included in the corresponding normal cone of the $\mathbf{A}$-discriminant polytope.

Proof. Both of the cones $M(T)$ and $\mathrm{NC}\,(T)$ are described by inequalities supported on the circuits $Z \subset \mathbf{A}$. Let Z be a circuit that supports an inequality separating $\boldsymbol{v}$ from $\mathrm{NC}\,(T)$. We claim there exists $i \in [n]$ such that $|Z \cap A_i| = 1$. Assume otherwise, then the we have for some j: $|Z \cap A_i| = 2$ for all $i \neq j$, and $|Z \cap A_j| = 3$. Then, passing from one triangulation of Z to another involves a mixed cell change which contradicts with the assumption $\boldsymbol{v} \in M(T)$. Now without loss of generality assume $Z \cap A_1 = \boldsymbol{\alpha}$. Then $Z - \boldsymbol{\alpha}$ is a circuit lying in a face of $\mathbf{A}$, and the lattice distance from $\boldsymbol{\alpha}$ to affine hull of $Z - \boldsymbol{\alpha}$ is 1. This is precisely the case covered by [GKZ08, Chapter 11, section 3, subsection B, example 3.6 b) and Proposition 3.7], which completes the proof. □

Building on Lemma 2.21, and using Lemma 2.16 with $\mathbf{A}$-discriminant amoeba instead of the amoeba of $E_{\mathbf{A}}$, one can modify Proposition 2.18 as follows: the statement of Proposition 2.18 hold true if we replace $\mathrm{NC}\,(T)$ with $M(T)$ (more precise statement is in Section 3). So, we use $M(T)$ for the rest of the article because we can compute $M(T)$: Jensen's *Tropical Homotopy Algorithm*, see [Jen16b], computes for a given (generic) lifting function ω, and point configurations $A_1, A_2, \ldots, A_n$, the triangulation $T = \Delta_\omega$ of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$ and its mixed-cell cone $M(T)$.

The idea of the algorithm is to start from a lifting function τ yielding only one mixed cell. Then, one keeps track of the changes in the mixed-cell cone as one changes the lifting function linearly from τ to a target lifting ω. The algorithm updates the mixed-cell cone with the violated circuit inequalities, and halts whenever it arrives at a triangulation T with $\omega \in M(T)$. The correctness of the algorithm follows from the fact that changes in the regular triangulations always happen by a flip over a circuit, and every flip corresponds to one inequality being violated in the mixed-cell cone.

2.8. **Numerically Tracking a Solution from Toric Infinity.** The numerical part of our algorithm tracks real zeros of $\boldsymbol{p_C}(t, \boldsymbol{x})$, as in Proposition 2.19, from $\boldsymbol{p_C}(0, \boldsymbol{x})$ to $\boldsymbol{p_C}(1, \boldsymbol{x})$. This can be done in two ways:

(1) trace the solution curves $\boldsymbol{x}(t)$ numerically, or
(2) start a homotopy from $\boldsymbol{p_C}(0, \boldsymbol{x})$ with zeros given by alternating mixed cells and track the solution path from $t = 0$ to $t = 1$.

We refer to [AG12] for the former and to [BC13] for the later approach. The curve tracing approach a.k.a. standard numerical trackers have the advantage of being used by numerical analysts: it is fast, and it is used for many applications. However, to the best of our knowledge, the safeguards to control precision issues for standard path trackers only exist for specific cases. The homotopy method, i.e., the second approach, offers a well developed theory to control precision issues and also to conduct rigorous complexity analysis. Moreover, Malajovich recently developed a theory for polyhedral homotopy that allows to express complexity of numerical tracking with certain integrals of condition numbers [Mal16]. We briefly explain Malajovich's approach in Section 5.4.

Our algorithm can be implemented using any of the two ways. We refer the interested reader to [BS11a, Section 2.3 and 2.4] for a nice exposition on the comparison of homotopy continuation and curve tracing.

2.9. **An Entropy Type Formula for The Discriminant Locus.** In this section, we introduce useful facts about A-discriminants, mostly relying on [GKZ08, Chapter 9, Section 3, subsection C] and works of Passare and Tsikh [PT05].

Theorem 2.22 (Horn-Kapranov Uniformization)**.** *Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m]$ be a collection of lattice points in $\mathbb{Z}^n$, let ∇_A be the corresponding A-discriminant variety. We consider A as a $n \times m$ matrix, and define*

$$\Psi_A(\boldsymbol{u}, \boldsymbol{x}) \;:=\; [u_1\, \boldsymbol{x}^{a_1} : u_2\, \boldsymbol{x}^{a_2} : \ldots : u_m\, \boldsymbol{x}^{a_m}]\,.$$

Then ∇_A admits the following parametrization:

$$\nabla_A \;=\; \overline{\left\{ \Psi_A(\boldsymbol{u}, \boldsymbol{x}) \;:\; A\boldsymbol{u} = \boldsymbol{0}, \sum_{i=1}^{m} u_i = \boldsymbol{0}, \boldsymbol{x} \in (\mathbb{C}^*)^n \right\}}.$$

Now consider the amoeba of ∇_A:

$$\operatorname{Log}|\nabla_A| \;=\; \operatorname{Log}\left|\left\{ \boldsymbol{u} \;:\; A\boldsymbol{u} = \boldsymbol{0}, \sum_{i=1}^{m} u_i = \boldsymbol{0} \right\}\right| \;+\; (\operatorname{Log}|\,\boldsymbol{x}\,|)^T A$$

where $+$ denotes the Minkowski sum.

It is easy to observe that $(\operatorname{Log}|\,\boldsymbol{x}\,|)^T A$ corresponds to the row span of A. Moreover, for any $\boldsymbol{u}$ with $A\boldsymbol{u} = \boldsymbol{0}, \sum_{i=1}^{m} u_i = \boldsymbol{0}$ any scalar multiple of $\boldsymbol{u}$ satisfies the same equations. This n-dimensional row span and one dimensional linear space represents $n+1$ homogeneities that are present in the discriminant variety; the variety is invariant under torus action and scaling.

For a given hypersurface $\mathcal{V}(f) \subseteq (\mathbb{C}^*)^n$ consider all points which are critical under the $\operatorname{Log}|\cdot|$ map. The $\operatorname{Log}|\cdot|$-image of these points is called the *contour* of the corresponding amoeba $\mathcal{A}(f)$; see e.g., [PT05]. It is straightforward to show that the contour contains the boundary $\partial\mathcal{A}(f)$, but does not coincide with it in general; see e.g., [PT05]. Moreover, for a real polynomial f, the contour contains the amoeba of the smooth part of the real variety, i.e. $\mathcal{A}(\mathcal{V}_{\mathbb{R}^*}(f))$ [PT05].

Let B be a *Gale dual* of A, i.e, an $m \times (m-n-1)$ integer matrix that has all column sums to be 0 and satisfies $AB = \boldsymbol{0}$. Then, for any $\boldsymbol{u} \in (\mathbb{R}^*)^m$ with $A\boldsymbol{u} = \boldsymbol{0}$ and $\sum_i u_i = 0$ one can find a $\boldsymbol{\zeta} \in (\mathbb{R}^*)^{m-n-1}$ with $\boldsymbol{u} = B\boldsymbol{\zeta}$.

It follows from the discussion in [PT05] (see the section titled Discriminants and Real Contours, and specifically Theorem 4), that the parametrization of the contour of the reduced A-discriminant amoeba $B^T\mathcal{A}(\nabla_A(\mathbb{C}))$ is given as follows:

$$B^T \operatorname{Log}\left|\left\{ \boldsymbol{u} : \boldsymbol{u} \in (\mathbb{R}^*)^m, A\boldsymbol{u} = \boldsymbol{0}, \sum_i u_i = \boldsymbol{0} \right\}\right|. \tag{2.5}$$

Using (2.5) and the fact that contour includes the amoeba of the real part of the variety, one can concisely write

$$B^T\mathcal{A}(\nabla_A(\mathbb{R})) \;\subseteq\; \left\{ B^T \operatorname{Log}|\boldsymbol{u}| \;:\; \boldsymbol{u} \in (\mathbb{R}^*)^m, A\boldsymbol{u} = \boldsymbol{0}, \sum_{i=1}^{m} u_i = 0 \right\}. \tag{2.6}$$

Using the row space of B to parameterize the set $\{\boldsymbol{u} \in (\mathbb{R}^*)^m, A\boldsymbol{u} = \boldsymbol{0}, \sum_{i=1}^m u_i = 0\}$, this can also be written as follows:

$$B^T \mathcal{A}(\nabla_A(\mathbb{R})) \ \subseteq \ \left\{ \sum_{i=1}^m \boldsymbol{b(i)} \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \ : \ \boldsymbol{\zeta} \in (\mathbb{R}^*)^{m-n-1} \right\} \tag{2.7}$$

where the $\boldsymbol{b(i)}$ denote the rows of B. As a next step, we define the following map:

$$\phi_A : (\mathbb{R}^*)^{m-n-1} \to (\mathbb{R}^*)^{m-n-1} \quad , \quad \phi_A(\boldsymbol{\zeta}) = \sum_{i=1}^m \boldsymbol{b(i)} \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \, .$$

The facts listed below are given in [GKZ08, Chapter 9, Section 3, subsection C]:

(1) The map ϕ_A is 0-homogeneous, that is for every $\lambda \in (0, \infty)$ and $\boldsymbol{\zeta} \in (\mathbb{R}^*)^{m-n-1}$ we have

$$\phi_A(\lambda \boldsymbol{\zeta}) \ = \ \phi_A(\boldsymbol{\zeta}).$$

(2) The image of the map ϕ_A is a hypersurface, and if the *Gauss map* γ is defined at $\phi_A(\boldsymbol{\zeta})$ then we have

$$\gamma(\phi_A(\boldsymbol{\zeta})) = \boldsymbol{\zeta}.$$

The first property follows since the column sums of B equals 0. The second property is proved by Kapranov [Kap91]. Now assume that we have a $\boldsymbol{\zeta} \in (\mathbb{R}^*)^{m-n-1}$, and we would like to write down the equation of the tangent hyperplane $H_{\boldsymbol{\zeta}}$ at $\phi_A(\boldsymbol{\zeta})$.

Since we know the image under the Gauss map (i.e., the normal direction), we obtain:

$$H_{\boldsymbol{\zeta}} \ = \ \left\{ \boldsymbol{x} \in \mathbb{R}^{m-n-1} : \langle x, \boldsymbol{\zeta} \rangle = \langle \phi_A(\boldsymbol{\zeta}), \boldsymbol{\zeta} \rangle \right\}.$$

One can rewrite this as follows:

$$H_{\boldsymbol{\zeta}} \ = \ \left\{ \boldsymbol{x} \in \mathbb{R}^{m-n-1} : \langle \boldsymbol{\zeta}, \boldsymbol{x} \rangle = \sum_{i=1}^m \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \right\}. \tag{2.8}$$

3. Effective Viro's Patchworking

Consider a polynomial system $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ with support sets $A_1, A_2, \ldots, A_n$ and the coefficient vector $\boldsymbol{C} = (\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_n)$. To certify that $\boldsymbol{p}$ is a patchworked system, we search for a ray $\operatorname{Log}|\boldsymbol{C}| + \lambda \boldsymbol{v}$ that does not intersect the discriminant amoeba. Note that since we are checking the intersection with the $\mathbf{A}$-discriminant amoeba instead of the real discriminant variety, this is a relaxation.

We keep the notation from Section 2.9, in particular we use B for the Gale dual matrix. What follows is the main result of this section.

Proposition 3.1. *Let $\boldsymbol{p_C}$ be a system of sparse polynomials with coefficient vector $\boldsymbol{C}$ and support sets $A_1, A_2, \ldots, A_n \subset \mathbb{Z}^n$. Let T be the triangulation of the Cayley configuration $\mathbf{A} = A_1 * A_2 * \ldots * A_n$ that is introduced by using $\operatorname{Log}|\boldsymbol{C}|$ as lifting function. Let $M(T)$ be the corresponding mixed cell cone, and suppose that the dual cone $M(T)^\circ$ is generated by vectors $\boldsymbol{\zeta(1)}, \ldots, \boldsymbol{\zeta(L)}$. Then, if*

$$\langle \operatorname{Log}|\boldsymbol{C}|, \boldsymbol{\zeta(i)} \rangle \ > \ \log(\#\mathbf{A}) \, \|\boldsymbol{\zeta(i)}\|_1 \tag{3.1}$$

for all $i = 1, 2, \ldots, L$*, the system* $\boldsymbol{p_C}$ *is a patchworked polynomial system. Furthermore, for any* $\boldsymbol{v} \in M(T)$ *the ray* $\operatorname{Log}|\boldsymbol{C}| + \lambda \boldsymbol{v}$ *for* $\lambda \in [0, \infty)$ *does not intersect the* $\mathbf{A}$*-discriminant amoeba.*

We use Jensen's tropical homotopy algorithm for computing the triangulation T given by the lifting function $\operatorname{Log}|\boldsymbol{C}|$. Moreover, the tropical homotopy algorithm computes the generators of the dual mixed cell cone $M(T)^\circ$ along the way for computing the triangulation T. So, the criterion (3.1) in Proposition 3.1 yields a certificate for the system $\boldsymbol{p_C}$ to be patchworked without extra cost: we check these mixed cell inequalities while computing the triangulation T. If all inequalities are satisfied, then we certify that the system is patchworked.

Note that, for any coefficient vector $\boldsymbol{C}$ with corresponding triangulation T, if the generators of the dual mixed cell cone $M(T)^\circ$ are $\boldsymbol{\zeta(i)}$ for $i = 1, 2, \ldots, L$, then, by definition,

$$\langle \boldsymbol{\zeta(i)}, \operatorname{Log}|\boldsymbol{C}|\rangle \ > \ 0$$

for all $i = 1, 2, \ldots, L$. To apply Proposition 3.1, we need

$$\langle \boldsymbol{\zeta(i)}, \operatorname{Log}|\boldsymbol{C}|\rangle \ > \ \log(\#\mathbf{A}) \, \|\boldsymbol{\zeta(i)}\|_1 \, .$$

Here $\|\boldsymbol{\zeta(i)}\|_1$ is a normalization; one can just use normalized generators with unit ℓ_1-norm. So the loss in our relaxation is represented by the logarithmic term $\log(\#\mathbf{A})$.

We prove Proposition 3.1 in several steps using the facts and notation introduced in the preceding section. For simplicity, we let $m = \#\mathbf{A}$. Note that $\mathbf{A} \subset \mathbb{Z}^{2n-1}$. We start with the following lemma. Here we assume the reader is familiar with the basic facts from Section 2.4.

Lemma 3.2. *Let* $\boldsymbol{\eta}$ *be a vertex of the Newton polytope of* $\Delta_{\mathbf{A}}$*, and let* $K_{\boldsymbol{\eta}}$ *be the corresponding connected component in the complement of the* $\boldsymbol{A}$*-discriminant amoeba.*

1. *Let* $\boldsymbol{u} \in K_{\boldsymbol{\eta}}$ *and let* $\boldsymbol{v} \in \operatorname{NC}(\boldsymbol{\eta})$ *then the ray* $\boldsymbol{u} + \lambda \boldsymbol{v}$ *for* $\lambda \in [0, \infty)$ *does not intersect the* $\boldsymbol{A}$*-discriminant amoeba.*
2. *Let* $\Phi_{\mathbf{A}}$ *and* B *respectively be the map and the matrix defined in Section 2.9. Suppose* $\boldsymbol{\zeta} \in \mathbb{R}^{m-2n}$ *with* $\Phi_{\mathbf{A}}(\boldsymbol{\zeta}) \in \partial(B^T K_{\boldsymbol{\zeta}})$*, then* $\boldsymbol{\zeta} \in \left(B^T \operatorname{NC}(\boldsymbol{\eta})\right)^\circ$*.*

Proof. As $K_{\boldsymbol{\eta}}$ is a component of the complement of an amoeba, it is a convex set. Moreover, by Lemma 2.16, it includes a shifted copy of $\operatorname{NC}(\boldsymbol{\eta})$. Now let $H_{\boldsymbol{w}} := \{\langle \boldsymbol{w}, \boldsymbol{x}\rangle = c\}$ be a supporting hyperplane of $K_{\boldsymbol{\eta}}$ (i.e., for every $\boldsymbol{y} \in K_{\boldsymbol{\eta}}$ we have $\langle \boldsymbol{w}, \boldsymbol{y}\rangle \geq c$). We claim $\boldsymbol{w} \in \operatorname{NC}(\boldsymbol{\eta})^\circ$: otherwise the shifted copy of the cone $\operatorname{NC}(\boldsymbol{\eta})$, that is included in $K_{\boldsymbol{\eta}}$, would intersect the supporting hyperplane $H_{\boldsymbol{w}}$, which is a contradiction.

Let $\boldsymbol{u} \in K_{\boldsymbol{\eta}}$ and $\boldsymbol{v} \in \operatorname{NC}(\boldsymbol{\eta})$. Then we have for any $\boldsymbol{w} \in \operatorname{NC}(\boldsymbol{\eta})^\circ$ and $\lambda > 0$

$$\langle \boldsymbol{w}, \boldsymbol{u}\rangle \ \leq \ \langle \boldsymbol{w}, \boldsymbol{u} + \lambda \boldsymbol{v}\rangle.$$

Hence, the ray $\boldsymbol{u} + \lambda \boldsymbol{v}$ does not intersect any supporting hyperplane of $K_{\boldsymbol{\eta}}$, and in consequence does not intersect the convex set $K_{\boldsymbol{\eta}}$ itself.

Now suppose that we have a $\boldsymbol{\zeta} \in \mathbb{R}^{m-2n}$ with $\Phi_{\mathbf{A}}(\boldsymbol{\zeta}) \in \partial(B^T K_{\boldsymbol{\zeta}})$, then by the second property in Section 2.9 the supporting hyperplane at $\Phi_{\mathbf{A}}(\boldsymbol{\zeta})$ will be

$$H_{\boldsymbol{\zeta}} \ := \ \left\{ \boldsymbol{x} \in \mathbb{R}^{m-n-1} \ : \ \langle \boldsymbol{\zeta}, \boldsymbol{x} \rangle = \sum_{i}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \right\}.$$

Since there is a shifted copy of $B^T \, \mathrm{NC}\,(\boldsymbol{\eta})$ inside the convex set $B^T K_{\boldsymbol{\eta}}$, this shows that $\boldsymbol{\zeta} \in \left(B^T \, \mathrm{NC}\,(\boldsymbol{\eta})\right)^{\circ}$. □

The Gale dual matrix B in Lemma 3.2 is of size $m \times (m-2n)$. Thus, $B^T K_{\boldsymbol{\zeta}}$ is a projection of $K_{\boldsymbol{\eta}}$ from $\mathbb{R}^m$ to $\mathbb{R}^{m-2n}$. Observe that this is not an arbitrary projection: The kernel of the matrix B^T is included in every connected component $K_{\boldsymbol{\eta}}$ of the complement of the $\mathbf{A}$-discriminant amoeba and this projection creates no loss of generality. Basically, the kernel of B^T represents the homogeneities present in the $\boldsymbol{A}$-discriminant variety as explained in Section 2.9.

Given a point $\mathrm{Log}\,|\boldsymbol{C}|$, testing if $\mathrm{Log}\,|\boldsymbol{C}| \in K_{\boldsymbol{\eta}}$ is equivalent to testing if $B^T \, \mathrm{Log}\,|\boldsymbol{C}| \in B^T K_{\boldsymbol{\eta}}$; the kernel of B^T is included in $K_{\boldsymbol{\eta}}$. One can test whether $B^T \, \mathrm{Log}\,|\boldsymbol{C}| \in B^T K_{\boldsymbol{\eta}}$ by checking all the supporting hyperplanes of $B^T K_{\boldsymbol{\eta}}$ due to convexity. By Lemma 3.2 these supporting hyperplanes are of the form

$$H_{\boldsymbol{\zeta}} \ := \ \left\{ \boldsymbol{x} \in \mathbb{R}^{m-n-1} : \langle \boldsymbol{\zeta}, \boldsymbol{x} \rangle = \sum_{i}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \right\}$$

for some $\boldsymbol{\zeta} \in \left(B^T \, \mathrm{NC}\,(\boldsymbol{\eta})\right)^{\circ}$. Now let T be a triangulation of $\mathbf{A}$, and let $\boldsymbol{\eta}$ be a vertex in the Newton polytope of $\Delta_{\mathbf{A}}$ with the property

$$\mathrm{NC}\,(T) \ \subseteq \ M(T) \ \subseteq \ \mathrm{NC}\,(\boldsymbol{\eta}),$$

see Lemma 2.21. By linearity, this means

$$\begin{aligned} &B^T \, \mathrm{NC}\,(T) \ \subseteq \ B^T M(T) \ \subseteq \ B^T \, \mathrm{NC}\,(\boldsymbol{\eta}), \quad \text{and} \\ &\left(B^T \, \mathrm{NC}\,(\boldsymbol{\eta})\right)^{\circ} \ \subseteq \ \left(B^T M(T)\right)^{\circ} \ \subseteq \ \left(B^T \, \mathrm{NC}\,(T)\right)^{\circ}. \end{aligned}$$

Instead of checking hyperplanes defined by $\boldsymbol{\zeta} \in \left(B^T \, \mathrm{NC}\,(\boldsymbol{\eta})\right)^{\circ}$ we check inequalities given by the larger cone $\left(B^T M(T)\right)^{\circ}$, and we prove the following lemma. The reason is algorithmic convenience: we had already computed the generators of $M(T)^{\circ}$ along the way, these are the circuit inequalities computed by Jensen's tropical homotopy algorithm, and we will eventually connect our discussion to these generators.

Lemma 3.3. *Let T be a triangulation of $\mathbf{A}$, please keep the notation from Lemma 3.2 for $K_{\boldsymbol{\eta}}$ and B. If a given vector $\mathrm{Log}\,|\boldsymbol{C}|$ satisfies*

$$\left\langle \boldsymbol{\zeta}, B^T \, \mathrm{Log}\,|\boldsymbol{C}| \right\rangle \ > \ \log(m) \, \|B\boldsymbol{\zeta}\|_1$$

for all $\boldsymbol{\zeta} \in (B^T M(T))^{\circ}$, then we have $\mathrm{Log}\,|\boldsymbol{C}| \in K_{\boldsymbol{\eta}}$ for a vertex $\boldsymbol{\eta}$ of $\Delta_{\mathbf{A}}$ which satisfies $M(T) \subseteq \mathrm{NC}\,(\boldsymbol{\eta})$.

The proof of Lemma 3.3 will follow after we make some observations. We first note a basic observation on entropy type sums.

Lemma 3.4. *Let $\boldsymbol{x} \in \mathbb{R}^d_{\geq 0}$ be a vector with nonnegative entries. Then, we have*

$$\|\boldsymbol{x}\|_1 \log \|\boldsymbol{x}\|_1 - \log(d) \|\boldsymbol{x}\|_1 \leq \sum_{i=1}^{d} x_i \log(x_i) \leq \|\boldsymbol{x}\|_1 \log \|\boldsymbol{x}\|_1 ,$$

where $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{d} |x_i|$ represents the ℓ_1-norm of the vector $\boldsymbol{x}$.

Proof. Let $\boldsymbol{y} := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}$. Since $\|\boldsymbol{y}\|_1 = 1$, and it has nonnegative entries, we can see $\boldsymbol{y}$ as a discrete probability distribution supported on d strings. As usual $H(\boldsymbol{y}) = \sum_{i=1}^{d} -y_i \log(y_i)$ is the entropy of $\boldsymbol{y}$, and it is well-known that $H(\boldsymbol{y}) \leq \log(d)$ [VL71]. So, we have

$$H(\boldsymbol{y}) \ = \ \frac{1}{\|\boldsymbol{x}\|_1} \left(\sum_{i=1}^{d} x_i \log \|\boldsymbol{x}\|_1 - x_i \log(x_i) \right) \leq \log(d).$$

This gives us the following inequality

$$\log \|\boldsymbol{x}\|_1 \sum_{i=1}^{d} x_i \ \leq \ \log(d) \|\boldsymbol{x}\|_1 + \sum_{i=1}^{d} x_i \log(x_i),$$

which proves the left-hand side inequality in the claim. The right-hand side is obvious. □

Now we derive the following useful estimate based on Lemma 3.4.

Lemma 3.5. *Let $\mathbf{A}$ be the support set, and let B be the $m \times (m-n-1)$ Gale dual. Then, for every $\boldsymbol{\zeta} \in \mathbb{R}^{m-n-1}$ we have*

$$-\frac{1}{2} \|B\boldsymbol{\zeta}\|_1 \log(m) \ \leq \ \sum_{i=1}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \ \leq \ \frac{1}{2} \|B\boldsymbol{\zeta}\|_1 \log(m).$$

Proof. By construction, every element in the column space of B has the sum of its coordinates equal to zero. So, for every $\boldsymbol{\zeta} \in \mathbb{R}^{m-n-1}$ the sum of the entries of $B\boldsymbol{\zeta}$ is zero. That is,

$$\sum_{i=1}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \ = \ 0,$$

where $\boldsymbol{b(i)}$ represents rows of the matrix B. We write $B\boldsymbol{\zeta} = (\boldsymbol{x}, -\boldsymbol{y})$ for some $\boldsymbol{x}$ and $\boldsymbol{y}$ that are nonnegative in all coordinates, so we have $\|\boldsymbol{x}\|_1 = \|\boldsymbol{y}\|_1 = \frac{1}{2} \|B\boldsymbol{\zeta}\|_1$. We also observe

$$\sum_{i=1}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \ = \ \sum_{i=1}^{m_1} x_i \log(x_i) - \sum_{i=1}^{m_2} y_i \log(y_i).$$

Note that m_1 and m_2 in the above expression are both less than m. Using Lemma 3.4 and $\|\boldsymbol{x}\|_1 = \|\boldsymbol{y}\|_1 = \frac{1}{2} \|B\boldsymbol{\zeta}\|_1$ gives us the following estimate:

$$-\frac{1}{2} \|B\boldsymbol{\zeta}\|_1 \log(m) \ \leq \ \sum_{i=1}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle \log |\langle \boldsymbol{b(i)}, \boldsymbol{\zeta} \rangle| \ \leq \ \frac{1}{2} \|B\boldsymbol{\zeta}\|_1 \log(m). \tag{3.2}$$

□

Proof of Lemma 3.3. Using Lemma 3.5 and the hypothesis of Lemma 3.3 we have

$$\langle \boldsymbol{\zeta}, B^T \operatorname{Log}|\boldsymbol{C}|\rangle \;>\; \log(m)\,\|B\boldsymbol{\zeta}\|_1 \;>\; \sum_{i=1}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta}\rangle \log|\langle \boldsymbol{b(i)}, \boldsymbol{\zeta}\rangle|$$

for all $\boldsymbol{\zeta} \in (B^T M(T))^\circ$. Note that $(B^T \operatorname{NC}(\boldsymbol{\eta}))^\circ \subset (B^T M(T))^\circ$. By Lemma 3.2, we know that the supporting hyperplanes of $K_{\boldsymbol{\eta}}$ are of the form

$$H_{\boldsymbol{\zeta}} \;:=\; \left\{ \boldsymbol{x} \in \mathbb{R}^{m-n-1} : \langle \boldsymbol{\zeta}, \boldsymbol{x}\rangle = \sum_{i}^{m} \langle \boldsymbol{b(i)}, \boldsymbol{\zeta}\rangle \log|\langle \boldsymbol{b(i)}, \boldsymbol{\zeta}\rangle| \right\}$$

for some $\boldsymbol{\zeta} \in (B^T \operatorname{NC}(\boldsymbol{\eta}))^\circ$. So, these two facts together imply that $B^T \operatorname{Log}|\boldsymbol{C}|$ and the shifted copy of $\operatorname{NC}(\boldsymbol{\eta})$ are not separated by any supporting hyperplane of $B^T K_{\boldsymbol{\eta}}$. This means $B^T \operatorname{Log}|\boldsymbol{C}| \in B^T K_{\boldsymbol{\eta}}$. Since the kernel of B^T is included in $K_{\boldsymbol{\eta}}$ this also implies $\operatorname{Log}|\boldsymbol{C}| \in K_{\boldsymbol{\eta}}$. □

Now we complete the proof of Proposition 3.1. By definition we have

$$\langle \boldsymbol{\zeta}, B^T \operatorname{Log}|\boldsymbol{C}|\rangle \;=\; \langle B\boldsymbol{\zeta}, \operatorname{Log}|\boldsymbol{C}|\rangle.$$

We observe that $B(B^T M(T))^\circ \subseteq M(T)^\circ$ as follows:

$$\boldsymbol{x} \in (B^T M(T))^\circ \;\Rightarrow\; \langle B\,\boldsymbol{x}, \boldsymbol{y}\rangle \geq 0 \text{ for all } \boldsymbol{y} \in M(T).$$

In the rest of the proof, instead of checking the inequalities given by elements $B(B^T M(T))^\circ$ as in Lemma 3.3 we will guarantee that the inequalities given by the bigger set $M(T)^\circ$ are satisfied: We now know that if a given vector $\operatorname{Log}|\boldsymbol{C}|$ satisfies

$$\langle \boldsymbol{\zeta}, \operatorname{Log}|\boldsymbol{C}|\rangle > \log(m)\,\|\boldsymbol{\zeta}\|_1 \tag{3.3}$$

for all $\boldsymbol{\zeta} \in M(T)^\circ$, then by Lemma 3.3 we have that $\operatorname{Log}|\boldsymbol{C}| \in K_{\boldsymbol{\eta}}$ for a vertex $\boldsymbol{\eta}$ of $\Delta_{\mathbf{A}}$ which satisfies $M(T) \subseteq \operatorname{NC}(\boldsymbol{\eta})$.

Suppose $M(T)^\circ$ is generated by $\boldsymbol{\zeta(1)}, \ldots, \boldsymbol{\zeta(L)}$, and assume for a given vector $\operatorname{Log}|\boldsymbol{C}|$ we have

$$\langle \boldsymbol{\zeta(i)}, \operatorname{Log}|\boldsymbol{C}|\rangle > \log(m)\,\|\boldsymbol{\zeta(i)}\|_1$$

for all $i = 1, 2, \ldots, L$. Then for any $\boldsymbol{x} \in M(T)^\circ$ with $\boldsymbol{x} = \sum t_i \boldsymbol{\zeta(i)}$ with $t_i \geq 0$ one has the following inequality

$$\langle \operatorname{Log}|\boldsymbol{C}|, \boldsymbol{x}\rangle \;>\; \log(m) \sum t_i\,\|\boldsymbol{\zeta(i)}\|_1 \geq \log(m)\,\|\boldsymbol{x}\|_1$$

where the last inequality follows from the triangle inequality. Hence, checking the condition in Proposition 3.1 only for the generators of $M(T)^\circ$ suffices to guarantee $\langle \operatorname{Log}|\boldsymbol{C}|, \boldsymbol{x}\rangle > \log(m)\,\|\boldsymbol{x}\|_1$ for all $x \in M(T)^\circ$, and this completes the proof.

4. Real Polyhedral Homotopy

In this section we summarize the main steps of our real polyhedral homotopy algorithm.

Initialization: Let $p_1, p_2, \ldots, p_n$ be the input polynomial system. Let $A_i \subset \mathbb{Z}^n$ be the support sets of p_i, and let $\boldsymbol{C}_i$ be the corresponding coefficient vectors. We first create the Cayley configuration $\mathbf{A} := A_1 * A_2 * \cdots * A_n$. Then we concatenate the coefficient vectors $\boldsymbol{C} := (\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_n)$. The pair $(\mathbf{A}, \boldsymbol{C})$ is the initialization of the input polynomial system $p_1, p_2, \ldots, p_n$.

Computing the Triangulation and the Mixed-Cell Cone: We use

$$\operatorname{Log}|\boldsymbol{C}| \ := \ (\operatorname{Log}|\boldsymbol{C}_1|, \operatorname{Log}|\boldsymbol{C}_2|, \ldots, \operatorname{Log}|\boldsymbol{C}_n|)$$

as a lifting function on the Cayley configuration $\mathbf{A}$, and denote the induced triangulation of $\mathbf{A}$ with $\Delta_{\boldsymbol{C}}$. We compute $\Delta_{\boldsymbol{C}}$ and the corresponding mixed-cell cone $M(\Delta_{\boldsymbol{C}})$ using Jensen's algorithm; see Section 2.7.

Locating the Input Against the Discriminantal Locus: We take a vector $\boldsymbol{v}$ from the interior of $M(\Delta_{\boldsymbol{C}})$ and draw the ray $\operatorname{Log}|\boldsymbol{C}| + \lambda \boldsymbol{v}$ for $\lambda \in [0, \infty)$. Then using the process described in Section 3, we check whether the ray $\operatorname{Log}|\boldsymbol{C}| + \lambda \boldsymbol{v}$ intersects the real part of $\mathbf{A}$-discriminant amoeba. If the non-intersection can not be certified, then the algorithm terminates without providing a solution.

Real Homotopy Continuation: This is the numerical part of our algorithm. It follows the general framework of homotopy continuation algorithms but works entirely over the reals. We first solve all the binomial systems corresponding to mixed cells of $\Delta_{\boldsymbol{C}}$ over the reals. We do this step as described in Section 2.5. Then we start a numerical iteration from these solutions at toric infinity, and track the solution paths to our target system $\boldsymbol{C}$ as described in Section 2.8. In the previous step of our algorithm we made sure the ray $\operatorname{Log}|\boldsymbol{C}| + \lambda \boldsymbol{v}$ with $\lambda \in (0, \infty)$ does not intersect the real part of $\mathbf{A}$-discriminant amoeba. Therefore, a homotopy deformation path defined by

$$\phi(t) = \left(\boldsymbol{C}_i t^{-\boldsymbol{v}_i}\right)_i$$

where $1 \leq i \leq \#A$, $t \in (0, 1]$ with $t = e^{-\lambda}$, and $\lambda \in [0, \infty)$ does not intersect the discriminantal locus, hence one has a continuous deformation path from toric infinity ($t = 0$) to the target system ($t = 1$).

We give an example showing how the algorithm performs in practice. The computation was carried out using a preliminary implementation together with the Homotopy.JL software [BT18] by Breiding and Timme.

Example 4.1. We reconsider the polynomials presented in Example 2.11, but this time we fix the coefficients to be real numbers instead of using coefficients that are Puiseux series.

$$\begin{aligned} f_t \ &= \ x_2^3 - (0.45)x_1x_2^2 - (0.45)^5x_1^2x_2 + (0.45)^{12}x_1^3 - (0.45)x_2^2 + (0.45)^4x_1x_2 - (0.45)^9x_1^2 \\ &- (0.45)^5x_2 - (0.45)^9x_1 + (0.45)^{12}, \\ g_t \ &= \ (0.45)^8x_2^2 - (0.45)^6x_1x_2 + (0.45)^6x_1^2 - (0.45)^3x_2 - (0.45)^2x_1 + 1. \end{aligned}$$

This leads to the following support and, using log-absolute values of the coefficients, the following lifting vectors:

$$\texttt{Support f: } 2\times 10 \texttt{ Array\{Int64,2\}}: \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\texttt{Lifting f: } \begin{bmatrix} 0 & 1 & 5 & 12 & 1 & 4 & 9 & 5 & 9 & 12 \end{bmatrix}$$
$$\texttt{Support g: } 2\times 6 \texttt{ Array\{Int64,2\}}: \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\texttt{Lifting g: } \begin{bmatrix} 8 & 6 & 6 & 3 & 2 & 0 \end{bmatrix}.$$

We use the indicated lifting and compute the mixed cells. Indeed, there are six of these, which all have volume 1, as it is depicted in the right picture of Figure 1. For every one of these mixed cells, we obtain a binomial system, which we then solve by computing a Hermite normal form and then solving the triangular system. For example, the first mixed cell is represented by

$$\texttt{volume: } 1 \qquad \texttt{indices: Tuple\{Int64,Int64\}}[(2,1),(5,6)] \qquad \texttt{normal: } [-2.0,-1.0]$$

with a solution for the corresponding binomial system given by

$$[4.938271604938272, 2.2222222222222223].$$

Similarly, we obtain five further solutions for the five other binomial systems corresponding to the other mixed cells.

$$[4.938271604938272, -0.20249999999999999] \quad [4.938271604938272, -0.041006249999999994]$$
$$[24.386526444139612, 10.973936899862824] \quad [24.386526444139612, -1.0]$$
$$[24.386526444139612, 0.09112500000000004].$$

Finally, we track the six real solutions obtained back to the original system. Technically, the polyhedral homotopy continuation in HOMOTOPY.JL currently only works with an arithmetic over the complex numbers. However, since we track over a real space only, we practically use real arithmetic.

After a total runtime of roughly 0.0001 seconds[1] we obtain the six real solutions for the original real system:

$$[4.20818, 2.41707] \quad [7.12063, -0.138875] \quad [6.94337, -0.0383256]$$
$$[49.3211, 24.3919] \quad [15.9697, -0.517115] \quad [17.5735, 0.0244792].$$

⬡

[1]Carried out on a MacBook Pro, Intel i5-5257U, 2.70GHz, 8GB RAM.

5. Remarks on Complexity

In this section we discuss complexity aspects of the real polyhedral homotopy algorithm. The algorithm consists of three main steps:

(1) computing a triangulation and the mixed-cell cone corresponding to a given lifting function,
(2) certifying that the given polynomial system is located in the unbounded components in the complement of the discriminant amoeba,
(3) tracking real solution paths numerically.

For the first step we use Jensen's tropical homotopy algorithm. We discuss complexity aspects of Jensen's algorithm in a high level. We also touch upon the complexity of the second step. After that we provide an upper bound for the number of solution paths in real polyhedral homotopy; for any fixed n, this bound is reminiscent to Kushnirenko's conjecture from fewnomial theory. Finally, we discuss complexity aspects of the numerical tracking phase.

Our goal in this section is to identify key parameters that governs the complexity of the real polyhedral homotopy algorithm. Our main finding is that the complexity of the real polyhedral homotopy algorithm is controlled by the number of mixed cells in triangulation of the Cayley configuration $\mathbf{A} = A_1 * A_2 * \ldots * A_n$ that is introduced by using the coefficients as a lifting function. We also show that the number of mixed cells admits an $O(t^n)$ upper bound. So if the number of variables n is considered to be fixed, and the number of terms t is a variable, the discrete computations in RPH takes polynomial time. In general, the discrete part of the RPH corresponds to computing mixed cells of a polyhedral subdivision given that is induced by a fixed lifting; without worrying about the volumes the mixed cells. Hence, any complexity theoretic upper and lower bounds for computing mixed cells (without volumes) applies to discrete computations in our algorithm. For the numerical part; the number of paths tracked by RPH is dramatically smaller than of complex homotopy algorithms. However, as noted in the introduction we are not able to provide a rigorous complexity analysis for the numerical part of the algorithm for the time being.

5.1. **Tropical Homotopy Algorithm.**

We start this section with bounding the number of inequalities needed to describe a mixed cell.

Lemma 5.1. *Let $A_1, A_2, \ldots, A_n$ be point configurations with at most t elements, and let T be a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$. Then a mixed cell $\sigma \in T$ is determined in the mixed-cell cone $M(\sigma)$ by at most $n(t-2)$ inequalities.*

Proof sketch. The mixed cell cone describes the case where the simplex corresponding to the mixed cell is a facet of the lifted Cayley polytope. So, for every element $\boldsymbol{\alpha} \in \mathbf{A}$ we get a circuit inequality given by $2n$ many vertices of the mixed cell and $\boldsymbol{\alpha}$ that determines whether $\boldsymbol{\alpha}$ is contained in the mixed cell. In total we have at most $n(t-2)$ many such $\boldsymbol{\alpha}$, and at most that many corresponding circuit inequalities. □

This immediately yields the following corollary.

Corollary 5.2. *Let $A_1, A_2, \ldots, A_n$ be point configurations with at most t elements, and let T be a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$ with k mixed-cells. Then the mixed-cell cone $M(T)$ can be described by at most $kn(t-2)$ many linear inequalities all supported on circuits.*

The proof of Proposition 5.6 gives us an upper bound the number of mixed-cells. Using this rough upper bound we derive the following corollary.

Corollary 5.3. *Let $A_1, A_2, \ldots, A_n$ be point configurations with at most t elements, and let T be a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$. Then the mixed-cell cone $M(T)$ can be described by at most $2ne^n(t-1)^{n+1}$ many linear inequalities all supported on circuits.*

Corollary 5.3 gives an upper bound to the number of updates in the tropical homotopy algorithm: For a fixed number of variables n, it is polynomial in t. This shows that the complexity of a mixed-cell cone computation is controlled by the cardinality of the support sets; this aligns well with Kushnirenko's fewnomial philosophy.

Jensen wrote a paper on implementation details of his algorithm for the purpose of mixed volume computation [Jen16a]. Thanks to real geometry, we do not need volumes, but only the mixed cells. So Jensen's current implementation does not output precisely what we need in this paper. A new implementation that outputs our needs in this paper is currently worked on by Timme. Real polyhedral homotopy is planned to be incorporated into HOMOTOPY.JL [BT18].

5.2. **Effective Viro's Patchworking.** As explained in Jensen's paper [Jen16b] and [DLRS10, Lemma 5.1.13], every circuit inequality is written by a vector with $n+2$ non-zero entries and every entry is given by the volume of a simplex. Since we can compute the volume of a simplex in $O(n^3)$ cost, we can compute a generator of a circuit inequality by $O(n^4)$ cost. This gives us the following basic complexity estimate as a corollary of Lemma 5.1 and Corollary 5.2.

Corollary 5.4. *Let $A_1, A_2, \ldots, A_n$ be point configurations with at most t elements, and let T be a triangulation of $\mathbf{A} = A_1 * A_2 * \ldots * A_n$ with k mixed-cells. Then the criterion in Lemma 3.3 can be checked by $O(kn^5(t-2))$ many arithmetic operations.*

Using Proposition 5.6 one can provide upper bound for k and hence deduce a $O(e^n n^5 t^{n+1})$ upper bound for the number of arithmetic operations.

5.3. **A Fewnomial Bound for Patchworked Polynomial Systems.** We start this section by stating a special case of McMullen's Upper Bound Theorem [Zie12].

Theorem 5.5 (Upper Bound Theorem; special case)**.** *Let $Q \subset \mathbb{R}^{2n}$ be a polytope with t vertices. Then the number of facets of Q is bounded by $2\binom{t-n}{n}$.*

In the case of zero dimensional systems, Viro's method counts the number of common zeros in $(\mathbb{R}^*)^n$. The discussions in Section 2.2 and in Section 2.6 show that for a patchworked polynomial system supported with point sets $A_1, A_2, \ldots, A_n \subset \mathbb{Z}^n$, the number zeros in the positive orthant is bounded by the number of mixed cells in the corresponding coherent polyhedral subdivision of $A_1 + A_2 + \ldots + A_n$. This yields the following statement.

Proposition 5.6 (Few Zeros for Patchworked Systems)**.** *Let $A_1, A_2, \ldots, A_n \subset \mathbb{Z}^n$, and let $|A_1 * A_2 * \cdots * A_n| \leq tn$. Then for a patchworked polynomial system $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ supported with $A_1, A_2, \ldots, A_n$, the number of common zeros of $\boldsymbol{p}$ in $(\mathbb{R}^*)^n$ is at most*

$$2^{n+1} e^n (t-1)^n.$$

Proof. Let ω be a lifting function and let Δ_ω be the corresponding coherent fine mixed subdivision of $A_1 + A_2 + \ldots + A_n$. The number of mixed cells in Δ_ω is equivalent to the number of corresponding simplices in the triangulation of the Cayley configuration $\mathbf{A} = A_1 * A_2 * \cdots * A_n$; see Section 2.1. The simplices that correspond to mixed cells are the simplices with two vertices from each A_i. The number of all simplices in the triangulation is equivalent to the number of facets in the lifted Cayley polytope $\mathrm{Cay}(A^\omega) = \mathrm{conv}(\mathbf{A}^\omega)$. $\mathrm{Cay}(A^\omega)$ is contained in $\mathbb{R}^{2n}$, and it has the same number of vertices as $\mathbf{A}$. So, the number of facets of $\mathrm{Cay}(A^\omega)$ is bounded by Theorem 5.5. We multiply this bound with 2^n to cover all orthants of $(\mathbb{R}^*)^n$, and obtain the following upper bound

$$2^{n+1} \binom{tn-n}{n} \;\leq\; 2^{n+1} e^n (t-1)^n,$$

where the last inequality follows from Stirling's estimate. □

5.4. **Complexity of Numerical Path Tracking.** Homotopy continuation theory of polynomials uses condition numbers to give bounds for the complexity of numerical iterative solvers [BC13]. Malajovich noticed that the current theory, which considers solutions of homogeneous polynomials over the projective space, fails to address subtleties of sparse polynomial systems. He developed a theory of sparse Newton iterations [Mal16]: For a given sparse polynomial system f, Malajovich's theory uses two condition numbers $\mu(f, \boldsymbol{x})$ and $v(\boldsymbol{x})$ at a given point $\boldsymbol{x} \in (\mathbb{C}^*)^n$, and it provides tools to analyze the accuracy and complexity of sparse Newton iterations. Let us state the main result of Malajovich below.

Theorem 5.7 (Malajovich, [Mal16])**.** *As in Proposition 2.19, let $\boldsymbol{p}_{\boldsymbol{C}}(t, \boldsymbol{x})$ be the polynomial system. Assume that we track a solution path from $\boldsymbol{p}_{\boldsymbol{C}}(\varepsilon, \boldsymbol{x})$ to $\boldsymbol{p}(\boldsymbol{C})(1, \boldsymbol{x})$ where $\varepsilon > 0$ is a sufficiently small real number. Then, there exists an algorithm which takes*

$$\int_{\varepsilon}^{1} \mu(\boldsymbol{p}_{\boldsymbol{C}}(t, \boldsymbol{x}), \boldsymbol{z}_s)\; v(\boldsymbol{z}_s)\; \left(\left\| \dot{\boldsymbol{p}}_{\boldsymbol{C},s} \right\|^2_{\boldsymbol{p}_{\boldsymbol{C},s}} + \left\| \dot{\boldsymbol{z}}_s \right\|^2_{\boldsymbol{z}_s} \right)^{\frac{1}{2}} \; ds$$

many iteration steps where $\boldsymbol{z}_s$ represents the solution path, and $\|\cdot\|_{\boldsymbol{x}}$ represents the local norms defined as pull-back of the classical Fubini-Study metric under the Veronese map.

It is customary in the theory of homotopy continuation to go from an integral representation as above to a more comprehensible complexity estimate by considering average or smoothed analysis of the iteration process. This amounts to introduce a probability measure on $\boldsymbol{p}_{\boldsymbol{C}}$, the input space of polynomials, and to compute the expectation of the integral estimate over the input space. Malajovich notes in his paper [Mal16] that the non-existence of a unitary group action on the space of sparse polynomials makes the probabilistic analysis harder. In our opinion, $\mu(\boldsymbol{p}_{\boldsymbol{C}}(t, \boldsymbol{x}))$ can be analyzed for general measures without group invariance [EPR18a, EPR18b]. However, the second condition number $v(\boldsymbol{x})$ seems hard to analyze; therefore we refrain from a probabilistic analysis for the moment.

Remark 5.8. *In Summer 2020, Gregorio Malajovich submitted a monumental 84 pages article on the ArXiv that improves the state of the art* [Mal20]. *We have not had the time to digest the paper in detail, and the community seems to be looking for ways to simplify and understand the results proved by Malajovich. Our hope is that these new results will pave the way for a rigorous complexity analysis of RPH, but until then what is written here represents our views.*

6. Discussion and Outlook

We discuss some open questions related to this work that are mostly brought to our attention after the initial submission of the article on ArXiv:

(1) How successful is RPH on practical problems? For instance, how would it perform in problems concerning real polynomial systems coming from chemical reaction networks?
(2) Imagine the support sets $A_1, A_2, \ldots, A_n$ are fixed, and we pick uniform Gaussian coefficients to create a random polynomial system. Can one prove with high probability that this random polynomial system would pass your effective patchworking test?
(3) Is our algorithm better than the polyhedral homotopy algorithm?
(4) How is the comparison of real polyhedral homotopy with Khovanskii-Rolle continuation algorithm of Bates and Sottile?

Regarding the first question: So far, we have only done a preliminary implementation and computed a few examples. The purpose of this article is to provide the theoretical aspects of RPH. We believe that a rigorous implementation and practical testing of the algorithm is crucial, but, given the magnitude of the task, it requires a second, separate article.

Regarding the second question: In a special case there are explicit estimates that shows indeed with high probability a random polynomial system is a patchworked system [DRRS07]. In the general case, it is a very intriguing question: A high probability positive answer to the second question would show that Viro's patchworking method captures an essential combinatorial structure in randomly generated systems of equations.

Regarding the third question: This question was brought to our attention by some colleagues after our paper appeared on arxiv, but the comparison between our algorithm and the polyhedral homotopy algorithm does not seem to be meaningful. The goal of the two algorithms are different; RPH tracks only real zero paths, and polyhedral homotopy tracks all complex zeros. If one is interested in real roots only, then the advantage of our algorithm is to track correct number of real zero paths (sometimes called optimal path tracking), where most algorithms in the literature find all complex zeros and then filter the real ones.

For the last question we first need to explain a notable algorithm of Bates and Sottile called *Khovanskii-Rolle Continuation Algorithm (KR)* [BS11a]. KR admits a sparse polynomial system where every polynomial has at most t terms, and traces at most

$$\frac{e^4+3}{4}2^{\binom{(t-2)n}{2}}\binom{(t-2)n}{t-2,t-2,\ldots,t-2}\sim\exp\left(t^2n^2\right)$$

many solution curves that can lead to real solutions [BS11a, BS11b]. The number of paths are given by the best fewnomial bound in the literature, and to the best of our knowledge for mixed support the best bounds are in [BS11b].

On the one hand, RPH algorithm tracks polynomially many solution paths with respect to t, where else the KR algorithm traces exponentially many solution curves. For instance, if one needs to solve a system of two bivariate polynomials both with 8 different terms, the KR algorithm traces more than 2^{76} many curves, and RPH tracks less than 2^{12} many paths. On the other hand, we stress that the KR algorithm can solve *all* input instances where RPH can only solve polynomials that are located against the discriminant variety. So, in our view these two algorithms are complementary to each other: for a given sparse systems one should use KR when RPH fails to admit the input.

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