# CELLULAR STRUCTURE OF THE POMMARET-SEILER RESOLUTION FOR QUASI-STABLE IDEALS. (EXTENDED ABSTRACT.)

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Abstract. We prove that the Pommaret-Seiler resolution for quasi-stable ideals is cellular and give a cellular structure for it. This shows that this resolution is a generalization of the well known Elaihou-Kervaire resolution for stable ideals in a deeper sense and also suggests an algorithm to construct the minimal free resolution of quasi-stable monomial ideals.

## 1. Pommaret-Seiler and Eliahou-Kervaire resolutions

1.1. **Pommaret bases and quasi-stable ideals.** Involutive bases were introduced in [11, 12]. They are a type of Gröbner bases with additional combinatorial properties. A survey of involutive divisions and their role in commutative algebra and the algebraic approach to partial differential equations can be seen in [20, 21, 22] where the particular case of Pommaret bases is studied deeply.

Let $R = \mathbf{k}[x_1, \ldots, x_n] = \mathbf{k}[\mathcal{X}]$ be the polynomial ring on $n$ variables over a field $\mathbf{k}$. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$. For the monomial $x^\mu \in R$ or a polynomial $f \in R$ such that $\mathrm{lt}(f) = x^\mu$, we say that the *class* of $\mu$ or $x^\mu$, resp. $f$ , denoted by $\mathrm{cls}(\mu) = \mathrm{cls}(x^\mu) = \mathrm{cls}(f)$, is equal to $\min\{i|\mu_i \neq 0\}$. We say that the *multiplicative variables* of $x^\mu$, resp. $f$, are $\mathcal{X}_P(x^\mu) = \mathcal{X}_P(f) = \{x_1, \ldots, x_{\mathrm{cls}(\mu)}\}$, we denote by $\overline{\mathcal{X}}_P(x^\mu)$ the set of *non-multiplicative variables* of $x^\mu$, given by $\mathcal{X} \setminus \mathcal{X}_P(x^\mu)$. We say that $x^\mu$ is an *involutive divisor* of $x^\nu$ if $x^\mu|x^\nu$ and $x^{\nu-\mu} \in \mathbf{k}[\mathcal{X}_P(\mu)]$.

**Definition 1.1.** Let $\mathcal{H}$ be a finite collection of monomials, $\mathcal{H} \subseteq R$. We say that $\mathcal{H}$ is a *Pommaret basis* of the monomial ideal $I = \langle \mathcal{H} \rangle$ if $I = \bigoplus_{h\in\mathcal{H}} h \cdot \mathbf{k}[\mathcal{X}_P(h)]$ as vector spaces.

A finite polynomial set $\mathcal{H}$ is a Pommaret basis of the polynomial ideal $I = \langle \mathcal{H} \rangle$ for the term order $\prec$, if all the elements of $\mathcal{H}$ possess distinct leading terms and these terms form a Pommaret basis of the leading ideal $\mathrm{lt}(I)$.

We call a monomial ideal $I$ *quasi-stable*, if it possesses a finite monomial Pommaret basis.

Quasi-stable monomial ideals can be characterized in several ways which are independent of the theory of Pommaret bases, see [22], Proposition 5.3.4. They have appeared in the literature under the names of *ideals of nested type* [5], *weakly stable ideals* [6] or *ideals of Borel-type* [14]. The name quasi-stable is due to the fact that these ideals are a generalization of the important class of *stable ideals*:

**Proposition 1.1** (cf. [22], Proposition 5.5.6)**.** *A monomial ideal $I$ is stable if and only if its minimal monomial generating set is also a Pommaret basis for $I$.*

1.2. **The Eliahou-Kervaire resolution for stable ideals.** A central object in the study of homogeneous (in particular monomial) ideals is the minimal free resolution. One of the main lines of research in minimal free resolutions is to study particular classes of ideals. In this vein, the result of Eliahou and Kervaire [9] that gives a closed form of the minimal free resolution of any stable monomial ideal is one of the most important ones.

Let us recall the definition of stable ideal.

**Definition 1.2.** A monomial ideal $I$ is stable if for every $x^\mu \in I$ it satisfies that for each index $i < \max(x^\mu)$ we have that $x^\mu \frac{x_i}{x_{\max(x^\mu)}} \in I$, where $\max(x^\mu)$ denotes the index of the last variable that divides $x^\mu$.

In order to describe the resolution of Eliahou and Kervaire, we need the following result and definition (we follow the notation in [17]).

**Proposition 1.2.** *Let $I$ be a monomial ideal and $x^\mu \in I$. Then there exists a unique generator $g$ and monomial $h$ such that $x^\mu = gh$ and for every $x_i$ dividing $h$ we have that $i \geq \max(g)$.*

*We say that $g$ is the* beginning *of $x^\mu$ and that $h$ is the* end *of $x^\mu$. We denote them by* $\mathrm{beg}(x^\mu)$ *and* $\mathrm{end}(x^\mu)$ *respectively.*

**Definition 1.3.** Let $I$ be a monomial ideal. An $EK$-symbol for $I$ is a pair of the form $[f, u]$ where $f$ is a minimal generator of $I$ and $u$ is a square-free monomial satisfying $\max(u) \lneq \max(f)$.

The Eliahou-Kervaire resolution is of the form

$$0 \longrightarrow \cdots \longrightarrow E_l \longrightarrow E_{l-1} \longrightarrow \cdots E_0 \longrightarrow I \longrightarrow 0,$$

where each of the modules $E_i$ is a free module generated by the set of $EK$-symbols $[f, u]$ such that $\deg(u) = i$. The differential of the resolution is given by

$$(1) \qquad d([f,u]) = \sum_{x_i|u} \mathrm{sgn}(x_i,u) x_i [f, \frac{u}{x_i}] - \sum_{x_i|u} \mathrm{sgn}(x_i,u)\mathrm{end}(x_i f) x_i [\mathrm{beg}(x_i f), \frac{u}{x_i}],$$

where $\mathrm{sgn}(x_i, u) = 1$ if the cardinality of the set $\{x_j | x_j \text{ divides } u \text{ and } j \leq i\}$ is odd, and $-1$ otherwise.

The Eliahou-Kervaire resolution raises as an iterated mapping cone [7, 15], in fact, this is a way to prove its minimality [18].

1.3. **The Pommaret-Seiler resolution.** For any quasi-stable ideal $I$ with Pommaret basis $\mathcal{H}$, Seiler gives in [21] an explicit description of a free resolution of $I$ that can be read from $\mathcal{H}$. In order to describe it we need the following notation.

For any generator $h_\alpha$ and any non-multiplicative variable $x_k \in \overline{\mathcal{X}}_P(h_\alpha)$ there exists a unique index $\Delta(\alpha, k)$ and a unique monomial $t_{\alpha;k} \in k[\mathcal{X}_P(h_{\Delta(\alpha,k)})]$ such that $x_k h_\alpha = t_{\alpha;k} h_{\Delta(\alpha,k)}$.

Let now denote by $\beta_0^{(k)}$ the number of generators in $\mathcal{H}$ of class $k$, and let $d = \min\{k | \beta_0^{(k)} > 0\}$. Then the Pommaret-Seiler resolution has the form

$$0 \longrightarrow R^{r_{n-d}} \longrightarrow \cdots \longrightarrow R^{r_1} \longrightarrow \cdots R^{r_0} \longrightarrow I \longrightarrow 0,$$

where the ranks of the free modules in the resolution are given by

$$r_i = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_0^{(k)}.$$

In order to describe the differential we use the above notations. The generators of the $i$-th free module in the Pommaret-Seiler resolution are given by pairs of the form $[h_\alpha, u]$ where $h_\alpha \in \mathcal{H}$ and $u$ is a is a degree $i$ square-free monomial satisfying $\min(u) \gneq \mathrm{cls}(h_\alpha)$.

$$\partial([h_\alpha, u]) = \sum_{j=1}^{i} (-1)^{i-j} \left( x_{u_j} [h_\alpha, \frac{u}{x_{u_j}}] - t_{\alpha,u_j} [h_{\Delta(\alpha,u_j)}, \frac{u}{x_{u_j}}] \right). \tag{2}$$

This resolution is not minimal in general. In fact, it is minimal if and only if $I$ is a stable ideal, in which case it coincides with the Eliahou-Kervaire resolution [1]. Despite its non-minimality, one can read most of the fundamental homological invariants from the Pommaret-Seiler resolution, like the projective dimension and Castelnuovo-Mumford regularity of $I$.

## 2. The Pommaret-Seiler resolution is cellular

The Pommaret-Seiler resolution of a quasi-stable ideal $I$ can be obtained as an iterated mapping cone for an adequate sorting of the generators of the Pommaret basis $\mathcal{H}$ of $I$ [1]. This sorting is known as a $P$-ordering, and sorts the elements of $\mathcal{H}$ first by class and within each class lexicographically. This indicates that this resolution inherits the mapping cone property from the Eliahou-Kervaire resolution, which gives a stronger meaning to the statement that the Pommaret-Seiler resolution generalizes the Eliahou and Kervaire's one. In this paper we intend to proof that this generalization also applies to the cellular character of the resolution.

We say that a resolution is *cellular* if it can be encoded by a regular cell complex. The concept originated in [3] and was extended in [4, 16]. J. Mermin proofs in [17] that the Eliahou-Kervaire resolution is cellular and gives an explicit cellular structure for it. Another cellular structure for the Eliahou-Kervaire resolution can be found in [2]. In their paper [8], Dochtermann and Mohammadi give a sufficient condition (possession of a regular decomposition function) for an iterated mapping cone resolution to be cellular and proof that the Eliahou-Kervaire resolution satisifies this condition. We slightly generalize here the argument in [8] by defining a regular decomposition function for the Pommaret-Seiler resolution and thus proving the main result of this section.

Let us recall the following definitions and notations from [8]. Let $I$ be a monomial ideal. We say that $I$ has *linear quotients* if there exists an ordering of the generators of $I$, $(m_1, \dots, m_r)$ such that for each $j \leq r$ the colon ideal $I_{j-1} : \langle m_j \rangle$ is generated by a subset of the variables, where $I_{j-1} = \langle m_1, \dots, m_{j-1} \rangle$ is the ideal generated by the first $j-1$ generators of $I$. In such case, we denote by $\mathrm{set}(m_j)$ the set of variables that generate $I_{j-1} : \langle m_j \rangle$. The iterated mapping cone of a monomial ideal that has linear quotients

[1]Observe that in the notation of [20, 21], which we follow here, the ordering of the variables is reversed with respect to the traditional one used for instance in [9].

with respect to an ordering of its minimal generating set is a minimal free resolution of $R/I$, cf. [8], Lemma 2.3 and [15], Lemma 1.5.

For any monomial ideal, let $M(I)$ be the set of monomials in $I$ and $G(I)$ a set of generators of $I$. A *decomposition function* for $I$ is an assignment $b : M(I) \to G(I)$. We say that the decomposition function $b$ is regular if for each $m \in G(I)$ and every $x_t \in \mathrm{set}(m)$ we have that $\mathrm{set}(b(x_t m)) \subseteq \mathrm{set}(m)$. Theorem 2.7 in [8] (see also Theorem 1.12 in [15]) gives a closed form formula for the differentials in the minimal free resolution (obtained as an iterated mapping cone) of any monomial ideal that has linear quotients with respect to an ordering of its minimal generating set and for which we can define a regular decomposition function. Furthermore, for any such ideal, the following result states that the minimal free resolution is supported on a regular $CW$-complex.

**Theorem 2.1** ([8], Theorem 3.11)**.** *Suppose $I$ has linear quotients with respect to some ordering $(m_1, \dots, m_r)$ of the minimal generators, and furthermore suppose that $I$ has a regular decomposition function. Then the minimal resolution of $I$ obtained as an iterated mapping cone is cellular and supported on a regular $CW$-complex.*

Any quasi-stable monomial ideal $I$ has linear quotients with respect to its $P$-ordered Pommaret basis, and the colon ideals are generated by the non-multiplicative variables of the corresponding generator:

**Proposition 2.1** ([1], Proposition 7.2 and [13], Proposition 26)**.** *Let $\mathcal{H} = \{h_1, \dots, h_r\}$ be a $P$-ordered monomial Pommaret basis of the quasi-stable monomial ideal $I$. Then $I$ possesses linear quotients with respect to the basis $\mathcal{H}$ and $\langle h_{\alpha+1}, \dots, h_s\rangle : h_\alpha = \langle \overline{\mathcal{X}}_p(h_\alpha)\rangle$ for all $\alpha = 1, \dots, s-1$.*

Observe that in this case, $\mathcal{H}$ is in general a non-minimal generating set of $I$, and hence the iterated mapping cone resolution obtained (i.e. the Pommaret-Seiler resolution of $I$) is not the minimal free resolution. Minimality is only obtained if $I$ is stable. Using the iterated mapping cone structure of the Pommaret-Seiler resolution we can proof that it is cellular by constructing a regular decomposition function for $I$ with respect to its Pommaret basis $\mathcal{H}$.

**Theorem 2.2.** *The Pommaret-Seiler resolution of a quasi-stable monomial ideal is cellular.*

*Proof.* Let $I$ be a quasi-stable ideal and $M(I)$ be the set of monomials in $I$. For every $h \in \mathcal{H}$ we define the involutive cone of $h$ with respect to the Pommaret division as $\mathcal{C}(h) = h \cdot \mathbf{k}[\mathcal{X}_P(h)]$. The fact that $\mathcal{H}$ is an involutive basis for $I$ means that $M(I) = \bigcup_{h\in\mathcal{H}} \mathcal{C}(h)$ where the union is disjoint, hence every monomial $M(I)$ has a unique involutive divisor in $\mathcal{H}$.

We now define the decomposition function $b : M(I) \to \mathcal{H}$ as $b(x^\mu) = h_\alpha$ where $h_\alpha$ is the unique element of $\mathcal{H}$ that is an involutive divisor of $x^\mu$. To see that $b$ is regular observe that $\mathrm{set}(h_\alpha) = \overline{\mathcal{X}}_P(h_\alpha)$. Now, for each $x_t \in \overline{\mathcal{X}}_P(h_\alpha)$ we have that $b(x_t h_\alpha) = h_{\Delta(\alpha,t)}$ and $\mathrm{cls}(h_{\Delta(\alpha,t)}) \geq \mathrm{cls}(h_\alpha)$, hence $\mathrm{set}(h_{\Delta(\alpha,t)}) \subseteq \mathrm{set}(h_\alpha)$ and $b$ is regular. The rest of the proof follows the lines of the proof of Theorem 3.11 in [8] except for the minimality of the resolution. $\square$

## 3. A cellular structure for the Pommaret-Seiler resolution

As we saw in the previos section, the fact that the Eliahou-Kervaire resolution for stable ideals and the minimal resolution of ideals with linear quotients arise as iterated mapping cones implies that these resolutions are cellular. This iterated mapping cone construction provides also with an explicit cellular structure for them, as seen in [17] and [8]. We extend this construction to the Pommaret-Seiler resolution by means of the $P$-graph of the Pommaret basis, which is defined in [21], see also [19].

Let $I$ be a quasi-stable ideal and $\mathcal{H}$ its Pommaret basis. We associate to it a directed graph, the $P$-graph of $\mathcal{H}$, which consists of a vertex for each $h_\alpha \in \mathcal{H}$ and a directed edge from $h_\alpha$ to $h_{\Delta(\alpha,t)}$ for each $h_\alpha$ in $\mathcal{H}$ and each $x_t \in \overline{\mathcal{X}}_P(h_\alpha)$. The 0-cells of the $CW$-structure for the Pommaret-Seiler resolution of $I$ are the vertices of the $P$-graph of $\mathcal{H}$ considered as points in $\mathbb{R}^n$.

Let now $h \in \mathcal{H}$, $\alpha = \{j_1, \ldots, j_p\} \subseteq \overline{\mathcal{X}}_P(h)$ with $j_1 < \cdots < j_p$ and $\sigma$ a permutation of $\alpha$. We define $\mathrm{ch}(h, \alpha, \sigma)$ to be the subset of $\mathbb{R}^n$ obtained as the convex hull of the elements of $\mathcal{H}$ that we reach by applying the decomposition function $b(h_i, t) = h_{\Delta(h_i,t)}$ in the order prescribed by $\sigma$. If there are no repetitions of elements of $\mathcal{H}$ involved in the description of $\mathrm{ch}(h, \alpha, \sigma)$ then $\mathrm{ch}(h, \alpha, \sigma)$ is a $p$-dimensional simplex, and we say that $\mathrm{ch}(h, \alpha, \sigma)$ is non-degenerate. Otherwise, we say that $\mathrm{ch}(h, \alpha, \sigma)$ is degenerate and it is in fact a face of $\mathrm{ch}(h, \alpha, \sigma')$ where $\sigma'$ is another permutation of $\alpha$ such that $\mathrm{ch}(h, \alpha, \sigma')$ is non-degenerate.

We define the cell $U(h, \alpha)$ as the union of the $\mathrm{ch}(h, \alpha, \sigma)$ over all permutations $\sigma$ of $\alpha$. For these cells we have a topological differential map

$$d(U(h,\alpha)) = \sum_i (-1)^i U(h, \alpha - j_i) - \sum_i (-1)^i U(h_{\Delta(h,j_i)}, \alpha - j_i).$$

Finally, by adding the monomials in these differentials we obtain the differential in (2) and have that the described structure is indeed the CW-structure that supports the Pommaret-Seiler resolution of $I$.

## 4. Reduction of the Pommaret-Seiler resolution

The Pommaret-Seiler resolution for quasi-stable ideals is known to be non-minimal, nevertheless, some of the homological invariants of this resolution can be read directly off it. In particular, we know that it is a resolution on minimal length [21]. If we denote by $\mathfrak{C}$ the CW-complex described in Section 3 then we have that $\dim(\mathfrak{C}) = \mathrm{pd}(I)$. In this section we show that $\mathfrak{C}$ can be reduced using Discrete Morse Theory [10], and since this complex supports the Pommaret-Seiler resolution, we can equivalently use the algebraic formulation in [16]. This fact suggests an algorithm for reducing the Pommaret-Seiler resolution so that we obtain the minimal one.

The first step consists on building an *annotated* P-graph of the quasi-stable ideal $I$ at the same time as we compute the Pommaret basis of $I$. Let us denote by $G_I$ the P-graph of $I$, and by $\mathbb{P}$ the Pommaret-Seiler resolution of $I$. As we build the Pommaret basis $\mathcal{H}$ we can store the information of $G_I$ assigning two pieces of information to each edge $e_{i,j}$, namely $k(e_{i,j})$ is the variable $k \in \overline{\mathcal{X}}_P(h_i)$ used to reach $h_j$ from $h_i$ and $t(e_{i,j})$ is the term $t_{i,k}$. This annotated graph allows us to directly read all the information in $\mathbb{P}$ from $G_I$.

The second step consists on building a Morse matching in $G_i$ or equivalently in $\Gamma_{\mathbb{P}}$, the graph associated to the Pommaret-Seiler resolution as described in [23]. For each directed path $p = (p_1, \dots, p_l) = (e_{i_1,j_1}, \dots, e_{i_l,j_l})$ in $G_I$ we say that the multidegree of the path is $\mathrm{md}(p) = \prod_{k=1}^{l} t(e_{i_k,j_k})$. Note that for any such path, the class of the variables $k(p_i)$ are strictly increasing. We say that a path between nodes $i$ and $j$ is a *valid path* if $\mathrm{md}(p) = 1$ . For any multidegree $\mu$ consider then the following set of vertices in $\Gamma_{\mathbb{P}}$ (equivalently in $\mathfrak{C}$): $V_\mu = \{\alpha : \{u\} \mid \mathrm{md}(\alpha : \{u\}) = \mu\}$ where $\alpha : \{u\}$ indicates the vertex in $\Gamma_{\mathbb{P}}$ corresponding to the generator indexed by $[h_\alpha, u]$ in the Pommaret-Seiler resolution $\mathbb{P}$. Consider now the following partial matching in $V_\mu$: $E_\mu = \{\alpha : \{u\} \to \beta : \{u/u_j\} \mid j = max(u)\}$. Then we have that

**Proposition 4.1.** *$\bigcup_{\mu\in\mathbb{N}^n} E_\mu$ is a Morse matching in $\Gamma_{\mathbb{P}}$.*

Proceeding iteratively by multidegree, we obtain a reduction of $\mathbb{P}$. If this reduction is already the minimal free resolution of $I$ then we stop the algorithm. Otherwise, we can proceed by further use of Morse matchings using those pairs of generators $[h_\alpha, u]$, $[h_\beta, u']$ in the reduced resolution such that the coefficient of $[h_\beta, u']$ in the differential of $[h_\alpha, u]$ is a nonzero scalar. These Morse mathings are always possible and they strictily reduce the number of such nonzero scalars in the differential of the resolution, hence the algorithm terminates after a finite number of steps and provides the minimal free resolution of $I$.

The cellular structure $\mathfrak{C}$ of $\mathbb{P}$ allows us to read this reduction in terms of the geometrical differential of $\mathfrak{C}$. Moreover, it can be used to obtain other geometrical Morse matchings that can eventually reduce the resolution completely. To describe one such reduction, we define the *skeleton* $P$-graph of $I$ as the graph that has a vertex for each minimal generator of $I$ and there is an edge from $m_i$ to $m_j \in G(I)$ only if the following conditions hold for $\mu = \mathrm{lcm}(m_i, m_j)$:

- $\mu/m_i \in \mathbf{k}[\overline{\mathcal{X}}_P(m_i)]$,
- $\mu \in \mathbf{k}[\mathcal{X}_P(m_j)]$,
- $m_k \nmid \mu$ for all other $m_k \in G(I)$.

With this definiton we have that

**Proposition 4.2.** *There is a Morse reduction from $\mathfrak{C}$ to a subcomplex of it whose* 1-*skeleton is the skeleton $P$-graph of $I$.*

**Example 4.1.** *To illustrate the above notions, let $I = \langle x^2, y^4, y^2z^2, z^3\rangle$ a quasi-stable ideal whose Pommaret basis is*

$$\mathcal{H} = \{x^2, x^2y, x^2z, x^2y^2, x^2y^3, x^2yz, x^2y^2z, x^2y^3z, x^2z, x^2z^2, x^2yz^2, y^4z\}.$$

*Figure 1 shows the $P$-graph of $I$ where each edge $e_{i,j}$ is labelled by $t(e_{i,j})$, and Figure 2 shows the critical cells of a Morse reduction on it applying the Morse-matching based on the sets $E_\mu$ described in Proposition 4.1. We can see that the reduced complex is exactly the skeleton $P$-graph of $I$, depicted in Figure 3. Observe that the skeleton $P$-graph is a cell complex that supports the minimal free resolution of $I$. Hence, the geometrical differential of this complex gives us the differentials in the minimal free resolution of $I$, which is given by*

$$0 \longrightarrow R^2 \xrightarrow{\partial_2} R^5 \xrightarrow{\partial_1} R^4 \xrightarrow{\partial_0} R \longrightarrow 0,$$

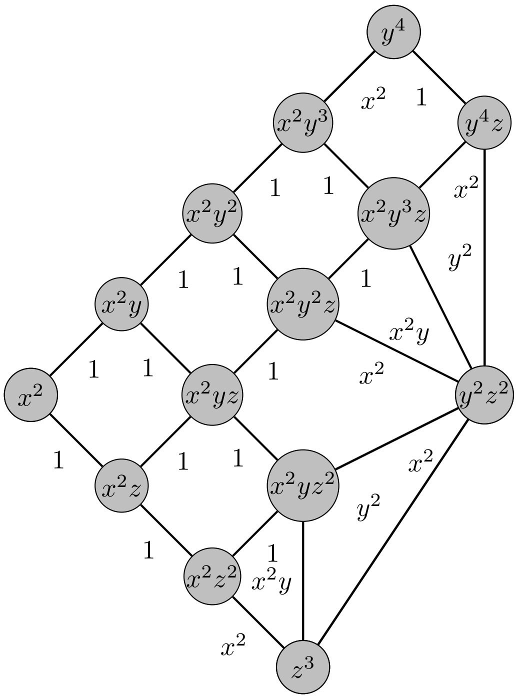


FIGURE 1. $P$-graph of $I = \langle x^2, y^4, y^2z^2, z^3\rangle$.

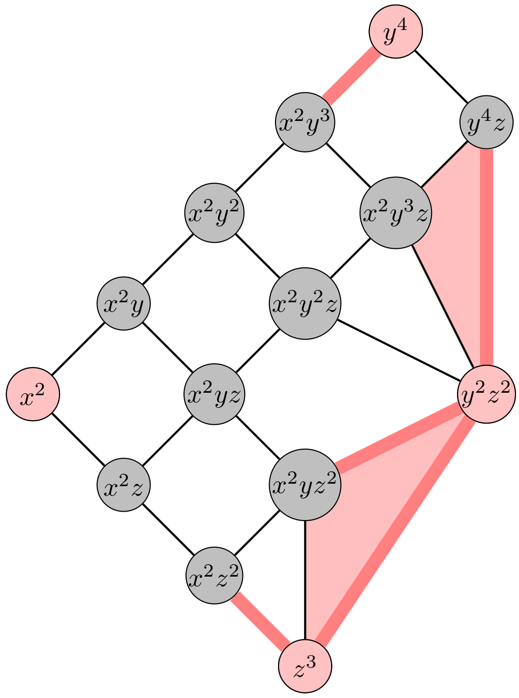


FIGURE 2. $P$-graph of $I = \langle x^2, y^4, y^2z^2, z^3\rangle$ with critical faces of Morse reduction.

*where*

$$\partial_0 = \begin{pmatrix} x^2 & y^4 & y^2z^2 & z^3 \end{pmatrix} \partial_1 = \begin{pmatrix} -y^4 & 0 & -y^2z^2 & -z^3 & 0 \\ -x^2 & -z^2 & 0 & 0 & 0 \\ 0 & y^2 & x^2 & 0 & -z \\ 0 & 0 & 0 & x^2 & y^2 \end{pmatrix} \partial_2 = \begin{pmatrix} z^2 & 0 \\ x^2 & 0 \\ -y^2 & z \\ 0 & -y^2 \\ 0 & x^2 \end{pmatrix}.$$

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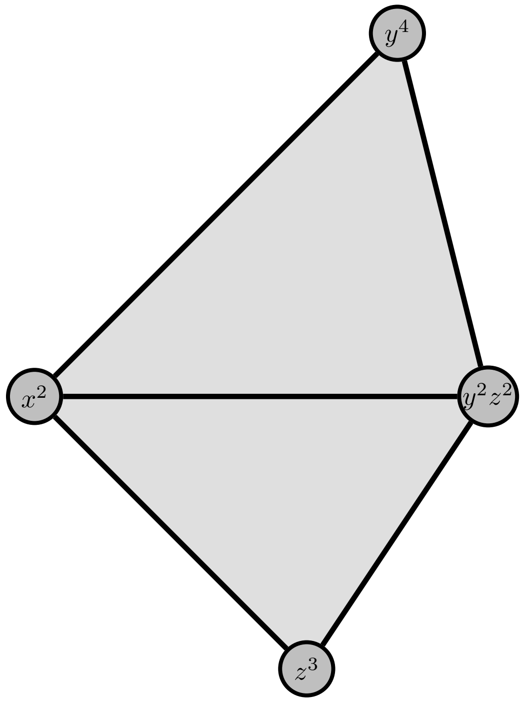


FIGURE 3. Skeleton $P$-graph of $I = \langle x^2, y^4, y^2z^2, z^3\rangle$ .

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