Resolution of Algebraic Curves via Geometric Invariants

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February 14, 2021

Abstract

The aim of this note is to present a geometrically flavoured resolution algorithm for unibranch singular algebraic curves. The key players in this algorithm are (*higher*) *algebraic curvatures*, a family of generators of so-called *geometric invariants*, representing "higher order tangent spaces". Using them, our algorithm constructs for each singular curve a suitable height function such that the Zariski closure of its graph gives a resolution of singularities of the curve.

1 Introduction and State of the Art in Resolution of Curve Singularities

The history of resolution of singularities of algebraic curves goes back more than 150 years to the work of M. Noether [Noe71, Noe75] who used it in order to find a formula for the genus of plane algebraic curves. More on the analytic side, at that time, the concept of Puiseux parametrizations was known as well — first discovered by I. Newton [Ne36, pp. 191-209] and later rediscovered by V. A. Puiseux [Pu50] while studying the solution space of $f(x, y(x)) = 0$ — yielding an analytic form of resolution. Nowadays, several methods for resolution of singular algebraic curves are available (see J. Kollár's book [Ko07, Chapter 1]): Successive blowups of the singular points eventually resolve all singularities since it can be shown that certain well chosen local invariants improve under each blowup. By induction on these invariants — usually lexicographically ordered string of integers — one is done after finitely many steps. A one step resolution is obtained by normalization but here, the geometric intuition is hidden behind commutative algebra machinery, see [MZ39]. As the resolution of each algebraic curve is unique up to isomorphisms and the normalization is a finite map, it can also be used together with the result of A. Nobile [No75] saying the Nash modification is an isomorphism on whole curve X if and only if X

*Supported by Austrian Academy of Sciences, ÖAW, Doc Stipendium FA506081 and by START-Prize Y-966 of the Austrian Science Fund

is smooth, to prove that performing successively the Nash modification yields resolution after finitely many steps. See also the PhD thesis by V. Rebasso [Re77], or the work by H. Hironaka [Hi83].

Let us now fix the setting: Let $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ be an algebraic space curve with a singularity at the origin $0 \in X$ defined by polynomials $f_1, \dots, f_r \in \mathbb{C}[x, y_1, \dots, y_n]$. Assume that X is analytically irreducible at the origin, i.e., that it admits only one analytic branch at the origin. Let us for each $a \in X$ denote by $[a]$ its corresponding projective point in $\mathbb{P}^n_{\mathbb{C}}$. A comparison of the standard blowup of X at 0 with its Nash modification shows immediately that the Nash modification is a more refined approach to improve singularities. With the standard blowup of X, one associates to each smooth point a on X the slope of the secant going through a and the origin:

$$X\backslash\{0\} \to \mathbb{P}^n_{\mathbb{C}},\ a \mapsto [a]$$

and finally, one takes the Zariski closure of the graph of this map in $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$. The Nash modification looks into the local geometry of X at a point more closely. One associates to each smooth point $a \in X$ the tangent line $s(a) = T_a X$ of X at a as an element of the projective space $\mathbb{P}^n_{\mathbb{C}}$ and takes then the Zariski closure of the graph of the map

$$X\backslash\{0\} \to \mathbb{P}^n_{\mathbb{C}},\ a \mapsto s(a)$$

in $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$. This corresponds, for plane curves, to the blowup of the curve in the Jacobian ideal of the defining equation of the curve. Thus, the Nash modification represents already a more geometric treatment of curve singularities. However, at the same time, appart from the fact that in general many repetitions are necessary to achieve the resolution, already the simple example of two cusps explains why the study of tangent lines is not sufficient for understanding the singularities: Consider two cusps defined by equations $y^3 = x^2$ and $y^5 = x^2$, where each of them has only one singular point at the origin. Intuitively, the singularity of the cusp defined by $y^5 = x^2$ is worse than the singularity of the other cusp, and this can also be made precise and proven rigorously. On the other hand, the limiting position of the tangent lines coincide.

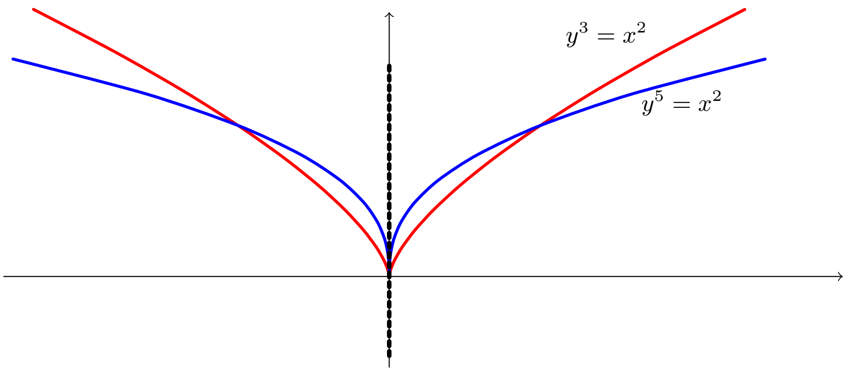

Figure 1: The limiting position of the tangent lines (black and dashed) of the nodes $y^3 = x^2$ and $y^5 = x^2$ at the origin (singular point) coincides.

Another disadvantage of the Nash modification is also the fact that, although it can also be defined and applied to varieties of any dimension, only for curves one knows that a sequence of Nash modifications resolves singularities.

In the present paper we, therefore, wish to present a more refined procedure based on the consideration of algebraic curvatures — a variation of the classical curvature known from differential geometry — which captures more accurately than the tangent lines how the curve runs into a singular point. Actually, there is not just one algebraic curvature but a whole sequence of them, each looking more closely into the local geometry of the curve at the point. In fact, these curvatures determine the curve locally at a smooth point, see [Me20b, Corollaries 2.11 & 3.5]. Now, as with the Nash modification, we may consider again the Zariski closure of the graphs of the maps induced by associating the individual curvatures, considered now as points in the projective space $\mathbb{P}^1_{\mathbb{C}}$, to smooth points on the curve together with the projection maps induced by the first projection $X \times \mathbb{P}^1_{\mathbb{C}} \to X$.

Above the singular point, the fiber will consist of all limiting values of the curvatures nearby. This is thus a very fine measure of the geometry of the singularity. And indeed, this Zariski closure improves the singularity in a much faster fashion than the classical blowup or the Nash modification. A precise description and analysis of this observation is given in the body of this paper.

The main point of these constructions is their geometric flavour. The crucial trick used in this paper to establish a resolution of X with one blowing up in a center reflecting the geometry of the curve at its singular point is to use local parametrizations of X at the origin providing very precise information about the complexity of the singularity itself: Look at X at 0 from the perspective of parametrizations. Let

$$\gamma\colon t \mapsto (x(t), y_1(t), \ldots, y_n(t))$$

be an analytic parametrization of X at 0. The goal is to construct a rational expression $z(t) = \frac{z_1(t)}{z_2(t)}$ in $x(t), y_1(t), \ldots, y_n(t)$ and their derivatives such that:

(i) $z(t)$ is a power series of order one,

(ii) $z(t)$ admits a rational expression as a formula in the polynomials defining X (and their partial derivatives), i.e., there exists

$$\widetilde{z} = \frac{\widetilde{z}_1}{\widetilde{z}_2} \in \mathbb{C}[\partial^i f_j : i \in \mathbb{N}^{n+1}, j = 1, \ldots, r] \subseteq \mathbb{C}[x, y_1, \ldots, y_n],$$

such that the equality

$$z(t) = \widetilde{z}(\gamma(t))$$

is fulfilled. Here, for $i = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$, by ∂^i we denote $\partial_x^{i_0} \partial_{y_1}^{i_1} \cdots \partial_{y_n}^{i_n}$.

Then, as proven in this paper, the Zariski closure $\widetilde{X}$ of the graph of the "height function"

$$\phi_z : X \backslash \{0\} \to \mathbb{P}^1_{\mathbb{C}},\ x \mapsto (\widetilde{z}_1(x) \,:\, \widetilde{z}_2(x))$$

together with the morphism $\widetilde{X} \to X$ induced by the projection onto the first $n+1$ components $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^{n+1}_{\mathbb{C}}$ is the blowup of X in the ideal $(\widetilde{z}_1, \widetilde{z}_2)$ and moreover, it defines already a resolution of singularities of X.

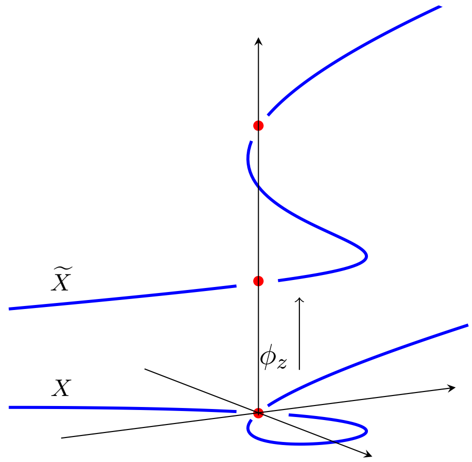

Figure 2: Resolution of singularities of the node given by the equation $y^2 - y^3 = x^3$.

The key players in the construction of the rational expression $z(t)$ are the algebraic curvatures. Let us now explain what our curvatures are.

Inspired by the differential geometric notion of the slope of the tangent vector, curvature and torsion of space curves in $\mathbb{A}^3_{\mathbb{R}}$, we define

$$\kappa_{0,j}(t) := \frac{y_j'(t)}{x'(t)}, \qquad \text{for } j = 1, \ldots, n,$$

to be the 0-*th algebraic curvatures* of X. Further, we define the *first* and *higher algebraic curvatures* of X via

$$\kappa_{1,j}(t) := \frac{y_j''(t)x'(t) - y_j'(t)x''(t)}{x'(t)} \text{ and } \kappa_{i,j}(t) := \frac{\partial_t \kappa_{i-1,j}(t)}{x'(t)} \text{ for } i \geq 2, j = 1, \ldots, n,$$

respectively. The observation now is that each of the algebraic curvatures is equivariant under reparametrizations and thus defines a quantity of the curve which does not depend on a choice of parametrization. As such, intuitively, they should admit also an implicit description as a rational function in the implicit equations $f_1, \ldots, f_r$ of X and their partial derivatives. In fact, this intuition is confirmed by a rigorous proof in my PhD thesis [Me20a]. Even more general statements can be found in [Me20b] (see Theorems 2.5, 2.10, 3.2 and 3.4).

In general, the algebraic curvatures themselves do not yield resolution of X immediately. Nevertheless, as their orders decrease with the index of the curvatures, notice that for $n = \operatorname{ord}(x(t)), m_j = \operatorname{ord}(y_j(t))$ we have

$$\operatorname{ord}(\kappa_{i,j}(t)) = m_j - (i+1)n,$$

they allow us to drop the multiplicity of the curve at its singularity. More precisely, there always exists a curvature

$$\kappa_{i,j}(t) = \frac{\kappa_{i,j}(t)_1}{\kappa_{i,j}(t)_2},$$

for some i, j, so that the multiplicity of the curve obtained as the Zariski closure of the graph of the height function $\phi_{\kappa_{i,j}}$ corresponding to $\kappa_{i,j}(t)$ — notice that this curve is then parametrized by the vector

$$(x(t), y_1(t), \ldots, y_n(t)) \times (\kappa_{i,j}(t)_1, \kappa_{i,j}(t)_2)$$

— is strictly smaller than the multiplicity of the curve X that is parametrized by $(x(t), y_1(t), \ldots, y_n(t))$. This fact will allow us to apply induction in order to resolve X.

At this point, one may object — and this is correct — that the multiplicity of the singularities of the curve drops also under the standard blowups. But this is not our punch line: our purpose here is to construct geometrically inspired modifications of varieties which may be able to advance in the resolution of higher dimensional singular varieties (geometric invariants of higher dimensional varieties were already defined in during my PhD study, see [Me20a]) and to provide a geometric approach complementing the known more algebraic algorithms. Moreover, the theory of geometric invariants has its own interest. This paper is also aimed to demonstrate the elegance and the beauty of the application of this theory towards resolution of singularities.

Let me finally mention, that all results of this paper were obtained during my PhD program under the supervision of Herwig Hauser and can be found with their complete proofs in my PhD thesis [Me20a].

Structure of the Paper

In Section 2, we introduce the concept of algebraic curvatures and geometric invariants by means of symbolic derivatives and symbolic chain rule in a differential field. They are the key players in our resolution algorithms for plane and space curves presented in the later sections.

Resolution of plane curves with only one singular point is discussed in Section 3. In this section, we list with Lemma 3.3 the most important properties of the height functions induced by geometric invariants. As a consequence of this lemma, we conclude with Corollary 3.4 that a geometric invariant of order one already defines a resolving height function. Finally, we present the algorithm PLANECURVATURE for the construction of a geometric invariant of order one, a resolving geometric invariant.

The generalization of the algorithm PLANECURVATURE to the algorithm SPACECURVATURE resolving space curves with only one singularity is provided in Section 4.

Finally, in Section 5, we treat also algebraic curves with more than one singularity. Using our knowledge and algorithms obtained in the previous sections, we construct a finite family of geometric invariants and define a corresponding height function that resolves all singularities simultaneously, see Theorem 5.3.

All our techniques, however, apply only to unibranch algebraic curves. Our algorithms are not able to resolve all singular algebraic curves that have more than one branches locally at their singular points. This problem is discussed in Section 6.

We also provide an appendix where the basic facts and properties of Puiseux parametrization, which we use in this paper very frequently, are collected. See Appendix A.

2 Geometric Invariants

In this section we recap briefly the abstract definition of geometric invariants and their basic properties (for a more detailed exposition see [Me20b]).

Algebraic Curvatures of Plane Curves

In order to make the definition of algebraic curvatures independent of a chosen curve, we introduce the field of rational functions in countably many variables $x^{(i)}$ and $y^{(i)}, i \in \mathbb{N}$ (we think of $x^{(i)}$ as a symbolic derivative of $x^{(i-1)}$):

$$F := \mathbb{C}(x^{(i)}, y^{(i)} \ : \ i \in \mathbb{N}).$$

On F, we simulate the classical derivative ∂_t with respect to t, that we have in the power series ring $\mathbb{C}[\![t]\!]$, by the $\mathbb{C}$-derivation

$$\begin{aligned} \partial : F &\to F \\ x^{(i)} &\mapsto x^{(i+1)} \\ y^{(i)} &\mapsto y^{(i+1)}. \end{aligned}$$

Thus, F becomes in this way a differential field (F, ∂).

Definition. We call the elements

$$\kappa_0 := \frac{y^{(1)}}{x^{(1)}} \quad \text{and} \quad \kappa_1 := \frac{y^{(2)}x^{(1)} - y^{(1)}x^{(2)}}{(x^{(1)})^3}$$

the *slope* (*of the tangent vector*) and the *first algebraic curvature of plane curves*, respectively. Further, we define the *higher algebraic curvatures of plane curves* iteratively by

$$\kappa_i := \frac{\partial(\kappa_{i-1})}{x^{(1)}}.$$

We set

$$I_F := \mathbb{C}(x^{(0)}, y^{(0)}, \kappa_i \ : \ i \in \mathbb{N})$$

and call the field I_F the *field of geometric invariants of plane curves* and its elements *geometric invariants of plane curves*. We will from now on use the notation

$$\underline{x} = (x^{(0)}, x^{(1)}, \dots) \quad \text{and} \quad \underline{y} = (y^{(0)}, y^{(1)}, \dots),$$

and will denote a rational function $p(x^{(i)}, y^{(i)} \ : \ i \in \mathbb{N}) \in F$ shortly by $p(\underline{x}, \underline{y})$. At the same time, for $\gamma(t) = (x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$ a parametrization of a plane algebraic curve, we will use the following notation for the vectors of higher derivatives of its components:

$$\underline{x(t)} = (x(t), x'(t), \dots) \quad \text{and} \quad \underline{y(t)} = (y(t), y'(t), \dots).$$

Further, we denote by

$$p(\underline{\gamma(t)}) = p(\underline{x(t)}, \underline{y(t)})$$

the evaluation of an element $p \in F$ at $\underline{\gamma(t)} = (\underline{x(t)}, \underline{y(t)})$, i.e., p after the substitution

$$x^{(i)} \mapsto \partial_t^i x(t),\ y^{(i)} \mapsto \partial_t^i y(t).$$

Further, it can be shown that each geometric invariant admits an implicit expression as a rational function in the variables x and y:

Theorem. *Let $X = V(f) = \{f = 0\}$, $f \in \mathbb{C}[x,y]$, be a plane algebraic curve. For each geometric invariant of plane curves $p = \frac{p_1}{p_2} \in I_F$, satisfying $p_2(\underline{\eta(t)}) \neq 0$ for a parametrization $\eta(t)$ of X, there exists a rational function*

$$p\big(X\big) = \frac{p\big(X\big)_1}{p\big(X\big)_2} \in \mathbb{C}(x,y)$$

so that

$$p\big(X\big)(\gamma(t)) = p(\underline{\gamma(t)})$$

is fulfilled for any parametrization $\gamma(t) \in \mathbb{C}[\![t]\!]^2$ of X. Moreover, if $p \in \mathbb{C}(\kappa_i\ :\ i \in \mathbb{N})$, so is the implicit expression $p\big(X\big)$ a rational function in f and its higher partial derivatives, i.e.,

$$p\big(X\big) \in \mathbb{C}(\partial_x^i \partial_y^j f\ :\ i,j \in \mathbb{N}) \subseteq \mathbb{C}(x,y).$$

Proof. See [Me20b, Theorem 2.10.] for a constructive proof. □

We call $p\big(X\big)$, as in the theorem, an *implicit expression* of p (w.r.t. X).

Algebraic Curvatures of Space Curves

Consider for a positive integer $n \in \mathbb{N}$ the set of variables $x^{(i)}, y_j^{(i)}$ for $i,j \in \mathbb{N}, 1 \leq j \leq n$, and the field

$$F_n := \mathbb{C}(x^{(i)}, y_j^{(i)}\ :\ i,j \in \mathbb{N}, 1 \leq j \leq n).$$

We extend the derivation ∂ to F_n by $\partial(y_j^{(i)}) = y_j^{(i+1)}$ and thus, make F_n a differential field (F_n, ∂), and define the algebraic curvatures of space curves:

Definition. We call the expressions

$$\kappa_{i,j} := \kappa_i(\underline{x}, \underline{y_j}), \ \text{ for } 1 \leq j \leq n,$$

the *slopes* (*of the tangent vector*) in the case $i = 0$ and (the *first* if $i = 1$ and the *higher* for $i \geq 2$) *algebraic curvatures of space curves* (*of embedding dimension* $n+1$) otherwise.

We define the *field of geometric invariants of space curves* by

$$I_{F_n} := \mathbb{C}(x^{(0)}, y_j^{(0)}, \kappa_{i,j}\ :\ i,j \in \mathbb{N}, 1 \leq j \leq n)$$

and call its elements *geometric invariants of space curves* (*of embedding dimension* $n+1$).

As in the plane curve case, each algebraic curvature of space curves can be expressed w.r.t. an algebraic space curve X as a rational function in the variables $x, y_1, \ldots, y_n$. More precisely, let $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ be a space algebraic curve and $I = \mathcal{I}(Y)$ its vanishing ideal. Further, let $\gamma(t) \in \mathbb{C}[\![t]\!]^{n+1}$ be a parametrization of X. Let us denote $\underline{\gamma(t)} = (\underline{x(t)}, \underline{y_1(t)}, \ldots, \underline{y_n(t)}) = (\partial_t^i x(t), \partial_t^i y_1(t), \ldots, \partial_t^i x_n(t),\ :\ i \in \mathbb{N})$.

Theorem. *Let $X = V(I)$ be an algebraic space curve and $p = \frac{p_1}{p_2} \in I_{F_n}$ a geometric invariant of space curves satisfying $p_2(\underline{\eta(t)}) \neq 0$ for a parametrization $\eta(t)$ of X. Then there exists a rational function*

$$p\big(X\big) \in \mathbb{C}(x, y_1 \dots, y_n)$$

with

$$p\big(X\big)(\gamma(t)) = p(\underline{\gamma(t)}) \tag{1}$$

for any parametrization $\gamma(t) \in \mathbb{C}[\![t]\!]^{n+1}$ of X. Moreover, if $p \in \mathbb{C}(\kappa_{i,j} \,:\, i \in \mathbb{N}, j = 1, \dots, n+1)$, then

$$p\big(X\big) \in \mathbb{C}(\partial_x^{i_0}\partial_{y_1}^{i_1} \dots \partial_{y_n}^{i_n} f \,:\, f \in I, i_j \in \mathbb{N} \text{ for } j = 0, 1, \dots, n) \subseteq \mathbb{C}(x, y_1 \dots, y_n).$$

Proof. See [Me20b, Theorem 3.4.]. □

We call a rational function $p\big(X\big)$ satisfying the equality (1) an *implicit expression of p* (w.r.t. X).

3 Resolution of analytically irreducible singular plane curves with only one singularity

In this section we study the behaviour of singularities of plane algebraic curves under blowups in ideals generated by the numerator and denominator of an implicit expression of an algebraic curvature of plane curves. We will show that given a unibranch plane algebraic curve with one singularity, in general, there exist several geometric invariants, even several algebraic curvatures, whose corresponding blowup improves the singularity of the curve. Moreover, there is always one among them, which yields already a resolution. The construction of a resolving geometric invariant is the goal of this section.

Let us fix a plane algebraic curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ with only one singularity at the origin and let $f \in \mathbb{C}[x, y]$ be its defining polynomial. Suppose f to be irreducible. Let us further assume that X is analytically irreducible at 0, i.e., it has only one analytic branch at the origin. In this section we prove the existence of a resolution of singularities of X by establishing an algorithm which constructs a geometric invariant

$$\widetilde{\kappa} = \frac{\widetilde{\kappa}_1}{\widetilde{\kappa}_2} \in I_F$$

with the property that the blowup of X in the ideal $(\widetilde{\kappa}\big(X\big)_1, \widetilde{\kappa}\big(X\big)_2)$ gives a smooth curve $\widetilde{X}_{\widetilde{\kappa}}$. Or equivalently (see [Ha14, §4]), with the property that the Zariski closure of the graph of the height function induced by $\widetilde{\kappa}$:

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ a &\mapsto \big(\widetilde{\kappa}\big(X\big)_1(a) : \widetilde{\kappa}\big(X\big)_2(a)\big) \end{aligned} \tag{2}$$

is smooth. Here, $\widetilde{\kappa}\big(X\big)_1$ and $\widetilde{\kappa}\big(X\big)_2$ denote the numerator and denominator of an implicit expression of $\widetilde{\kappa}$ w.r.t. X, respectively, and $Z = V(f, \widetilde{\kappa}\big(X\big)_1, \widetilde{\kappa}\big(X\big)_2)$ the vanishing set of the ideal $(f, \widetilde{\kappa}\big(X\big)_1, \widetilde{\kappa}\big(X\big)_2)$.

Definition 3.1. We call a geometric invariant $\widetilde{\kappa}$ that satisfies the above mentioned properties, a *crucial height* of X.

Remark that the crucial height is not unique:

Example 3.2. It is not hard to show that the singularity of the cusp defined by the polynomial $f = x^3 - y^2$ can be resolved by both, the standard blowup, i.e., the monomial blowup in the ideal (x, y), and by the Nash modification which is defined by the blowup in the ideal $(f_x, f_y) = (x^2, y)$. These two ideals correspond to the crucial heights

$$\widetilde{\kappa} = \frac{y^{(0)}}{x^{(0)}} \quad \text{and} \quad \hat{\kappa} = \frac{y^{(1)}}{x^{(1)}},$$

respectively. However, they are not equal.

At this point, it is very instructive to look at X from the perspective of parametrization. So let us consider a parametrization $\gamma(t) = (x(t), y(t)) \in \mathbb{C}\{t\}^2$ of X at 0 (one can always construct a convergent parametrization according to the Newton-Puiseux algorithm, see Appendix A). The evaluation $\widetilde{\kappa}(\underline{\gamma(t)})$ of a crucial height at $\underline{\gamma(t)}$ gives us the pair $\big(\widetilde{\kappa}_1(\underline{\gamma(t)}), \widetilde{\kappa}_2(\underline{\gamma(t)})\big) \in \mathbb{C}\{t\}^2$ of power series in t. The vector of power series

$$\gamma_{\widetilde{\kappa}}(t) = \gamma(t) \times \big(\widetilde{\kappa}_1(\underline{\gamma(t)}) : \widetilde{\kappa}_2(\underline{\gamma(t)})\big)$$

defines then a parametrization of $\widetilde{X}_{\widetilde{\kappa}}$.

Let us list now few important facts about maps of type (2) induced by geometric invariants. The following lemma will serve as the most important indicator for recognizing crucial heights:

Lemma 3.3. *Let $Y \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a plane algebraic curve and defined by a polynomial $f \in \mathbb{C}[x, y]$. Assume that Y is analytically irreducible at each point. Further, let $p = \frac{p_1}{p_2} \in I_F$ be a geometric invariant satisfying $p_i(\underline{\gamma(t)}) \neq 0$, for $i = 1, 2$, and for $\gamma(t)$ a parametrization of Y. Let further $p\big(Y\big) = \frac{p(Y)_1}{p(Y)_2}$ be an implicit expression of p w.r.t. Y. Consider the by p induced map*

$$\begin{aligned} \phi_p \colon Y \backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ a &\mapsto (p\big(Y\big)_1(a) : p\big(Y\big)_2(a)), \end{aligned}$$

where $Z = V(f, p\big(Y\big)_1, p\big(Y\big)_2)$ is the vanishing set of the ideal generated by f and the numerator and denominator of $p\big(Y\big)$. The Zariski closure $\widetilde{Y_p} \subseteq \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of the graph of ϕ_p then satisfies:

(i) The projection map $\pi : \widetilde{Y_p} \to Y$ induced by the first projection $\pi : \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ is a proper birational morphism which is an isomorphism $\pi : \widetilde{Y} \backslash E \to Y \backslash Z$ outside $E = \pi^{-1}(Z)$.

(ii) The projection π is injective.

(iii) $\widetilde{Y_p}$ is analytically irreducible at each point.

(iv) We have the inclusion $\mathrm{Sing}(\widetilde{Y}_p) \subseteq \pi^{-1}(\mathrm{Sing}(Y))$. *In other words, all singular points of* $\widetilde{Y}_p$ *lie over the singular points of* Y.

(v) $|\,\mathrm{Sing}(\widetilde{Y}_p)| \leq |\,\mathrm{Sing}(Y)|$.

Proof. (i): That π is a proper birational morphism follows from the fact that $\widetilde{Y}_p$ together with the projection $\pi : \widetilde{Y}_p \to Y$ is the blowup of Y in the ideal $(f, p(Y)_1, p(Y)_2)$ (see [Ha14, Definition 4.9.]) and from the properties of blowups, see e.g. [Ha77, Chapter II, Proposition 7.16]. That its restriction $\pi : \widetilde{Y} \backslash E \to Y \backslash Z$ is an isomorphism outside $E = \pi^{-1}(Z)$, follows now immediately.
(ii): As π is an isomorphism outside $E = \pi^{-1}(Z)$, we only have to prove that π is injective on E. Let us indirectly assume that there are two distinct points a, b in E with $\pi(a) = \pi(b)$. Let B_a and B_b be the branches of $\widetilde{Y}_p$ at the points a and b, respectively. Notice that as the points a and b are distinct, the branches B_a and B_b are distinct as well. As $B_a \setminus \{a\}$ and $B_b \setminus \{b\}$ are contained in $\widetilde{Y}_p \setminus E$, the projection π is an isomorphism on them and thus, $\overline{\pi(B_a \setminus \{a\})} = \pi(B_a \setminus \{a\}) \cup \{\pi(a)\}$ and $\overline{\pi(B_b \setminus \{b\})} = \pi(B_b \setminus \{b\}) \cup \{\pi(b)\}$ define two distinct branches of Y at the point $\pi(a) = \pi(b)$. This is a contradiction to the fact that Y is analytically irreducible at each point.
(iii): As π is a birational morphism, and even an isomorphism outside the finite set $\pi^{-1}(Z)$, the analytic branches of $\widetilde{Y}_p$ at a point b are uniquely determined by the images of the analytic branches of Y at $\pi(b)$ under the map

$$Y \backslash Z \to \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}},\ a \mapsto a \times \phi_p(a).$$

As Y is unibranch at $\pi(b)$, so is $\widetilde{Y}_p$ at b. This proves the analytical irreducibility of $\widetilde{Y}_p$.
(iv): As the restriction

$$\pi : \widetilde{Y}_p \backslash E \to Y \backslash Z$$

is an isomorphism, for each point $a \notin Z$ we have:

$$\pi^{-1}(a) \in \mathrm{Sing}(\widetilde{Y}_p) \text{ if and only if } a \in \mathrm{Sing}(Y).$$

Hence, it remains to discuss whether in the case that $a \in Z$, the point $\pi^{-1}(a)$ can be singular although a is not. Let us assume that a is a smooth point of Y. Then, Y is locally at a a manifold and as such biholomorphic to $\mathbb{C}$ via a parametrization $\gamma_a(t) = (x_a(t), y_a(t))$. As $\widetilde{Y}_p$ is parametrized at $\pi^{-1}(a)$ by the vector $\gamma_a(t) \times \big(p_1(\underline{\gamma_a(t)}), p_1(\underline{\gamma_a(t)})\big)$ and as such biholomorphic at $\pi^{-1}(a)$ to $\mathbb{C}$ as well, the point $\pi^{-1}(a)$ is non-singular.
(v): This is a consequence of items (ii) and (iv). $\square$

With Lemma 3.3, we obtain the following sufficient on a crucial height $\widetilde{\kappa}$:

Corollary 3.4. *If for a geometric invariant of plane curves* $p \in I_F$ *a parametrization* $\gamma(t)$ *of* X *at* 0 *exists with*

$$\mathrm{ord}\big(p(\underline{\gamma(t)})\big) = 1,$$

then p *is a crucial height of* X *at* 0.

Proof. According to (iv) and (v) of Lemma 3.3, $\widetilde{X}_p$ has at most one singular point $\widetilde{x}$ and it is lying over the origin. Now, if $\operatorname{ord}\big(p(\underline{\gamma(t)})\big) = 1$, so is $\tilde{x}$ visible only in one affine chart of $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and equals the origin at this chart. The affine expression of $\widetilde{X}_p$ in this chart is parametrized by the triple $\big(\gamma(t), p(\underline{\gamma(t)})\big) \in \mathbb{C}\{t\}^3$, which is a regular parametrization and thus the smoothness follows. □

Thus, to resolve X is equivalent to construct a geometric invariant $\widetilde{\kappa} \in I_F$ which satisfies

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1$$

for a suitably chosen parametrization $\gamma(t)$ of X. The construction of a crucial height $\widetilde{\kappa}$ is the objective of the remaining part of this section.

Let us now discuss how the orders of algebraic curvatures evaluated at a parametrization $\gamma(t) = (x(t), y(t))$, with $n = \operatorname{ord}(x(t)), m = \operatorname{ord}(y(t))$, behave. This will be of crucial meaning for our final algorithm constructing a crucial height (as the algebraic curvatures build a generating system of geometric invariants):

Remark 3.5. By induction, one can show the following equalities:

$$\begin{aligned}
\operatorname{ord}\big(\kappa_0(\underline{\gamma(t)})\big) &= \operatorname{ord}\left(\frac{y'(t)}{x'(t)}\right) = m - n,\\
\operatorname{ord}\big(\kappa_i(\underline{\gamma(t)})\big) &= \operatorname{ord}\left(\frac{\partial_t \kappa_{i-1}(\underline{\eta^X(t)})}{x'(t)}\right) = m - (i+1)n, \text{ for all } i \geq 1.
\end{aligned}$$

So for each higher algebraic curvature, the order of its evaluation at $\underline{\gamma(t)}$ drops by n compared to the order of the previous one. We can iteratively even construct a geometric invariant $\hat{\kappa}$ with

$$\operatorname{ord}\big(\hat{\kappa}(\underline{\gamma(t)})\big) = \gcd(n, m).$$

From now on, given a geometric invariant $p \in I_F$, we will use the notation $\underline{p}$ for the vector $(\partial^i p)_{i \geq 0}$.

Euclidean Algorithm for Geometric Invariants Construction of a geometric invariant of order at $\underline{\gamma(t)}$ equal to the greatest common divisor of orders at $\underline{\gamma(t)}$ of two other geometric invariants, for $\gamma(t) \in \mathbb{C}[t]^2$ a polynomial pair

---

Input two geometric invariants p and q & a polynomial pair $\gamma(t) \in \mathbb{C}[t]^2$
Output pair of geometric invariants $(\hat{\kappa}_1, \hat{\kappa}_2)$ with $\operatorname{ord}\big(\hat{\kappa}_1(\underline{\gamma(t)})\big) = \gcd(n, m)$, where $n = \operatorname{ord}\big(p(\underline{\gamma(t)})\big)$ and $m = \operatorname{ord}\big(q(\underline{\gamma(t)})\big)$, and with $\operatorname{ord}\big(\hat{\kappa}_2(\underline{\gamma(t)})\big) = r$, where r is the penultimate non-zero remainder in the Euclidean algorithm applied to the pair (n, m), or FAIL if either n and m both equal zero or $p(\underline{\gamma(t)}) \cdot q(\underline{\gamma(t)}) = 0$ or if at least one of n and m is negative
procedure $\textsc{gcd}(p, q; \gamma(t))$

1st step:
Set

$$\mathsf{n} := \operatorname{ord}\big(p(\underline{\gamma(t)})\big) \text{ and } \mathsf{m} := \operatorname{ord}\big(q(\underline{\gamma(t)})\big).$$

If $0 \leq \mathsf{n}, \mathsf{m} < \infty$ not both equal to 0, then $p(\underline{\gamma(t)}), q(\underline{\gamma(t)}) \in \mathbb{C}\{t\}$ are both power series different from zero and at least one of them has no constant coefficient. In this case we continue with the 2nd step of the algorithm.
Otherwise, **return** FAIL.

2nd step:
We proceed according to the Euclidean algorithm. Let us write the Euclidean algorithm for n and m as the sequence of following equations:

$$\begin{aligned}
\mathsf{m} &= q_0\mathsf{n} + r_0 \\
\mathsf{n} &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\vdots \\
r_{N-2} &= q_N r_{N-1} + r_N \\
r_{N-1} &= q_{N+1} r_N + r_{N+1},
\end{aligned}$$

where $0 = r_{N+1} < r_N = \gcd(\mathsf{n}, \mathsf{m}) < r_{N-1} < \cdots < r_0 < \mathsf{n}$.

Define now geometric invariants $z_i \in I_F$ for $1 \leq i \leq N$ according to the Euclidean algorithm and use Remark 3.5 to compute the orders of their evaluation at $\underline{\gamma(t)}$:

$$\begin{aligned}
z_0 &:= \kappa_{q_o-1}(\underline{p}, \underline{q}) & \operatorname{ord}\big(z_0(\underline{\gamma(t)})\big) &= r_0, \\
z_1 &:= \kappa_{q_1-1}(\underline{z_0}, \underline{p}) & \operatorname{ord}\big(z_1(\underline{\gamma(t)})\big) &= r_1, \\
z_2 &:= \kappa_{q_2-1}(\underline{z_1}, \underline{z_0}) & \operatorname{ord}\big(z_2(\underline{\gamma(t)})\big) &= r_2, \\
&\vdots & &\vdots \\
z_N &:= \kappa_{q_N-1}(\underline{z_{N-1}}, \underline{z_{N-2}}) & \operatorname{ord}\big(z_N(\underline{\gamma(t)})\big) &= r_N,
\end{aligned}$$

where in the case $q_0 = 0$ we set

$$\kappa_{-1}(\underline{p}, \underline{q}) := q.$$

We **return**

$$(z_N, z_{N-1}).$$

Example 3.6. Let us consider the geometric invariants $p = x^{(0)}$ and $q = \kappa_0$ and a Puiseux parametrization $\gamma(t) = (t^3, t^8)$. Let us for the orders

$$\begin{aligned}
5 &= \operatorname{ord}\big(q(\underline{\gamma(t)})\big) = \operatorname{ord}\left(\frac{y'(t)}{x'(t)}\right) = \operatorname{ord}\left(\frac{8t^7}{3t^2}\right) = \operatorname{ord}\left(\frac{8}{3}t^5\right), \\
3 &= \operatorname{ord}\big(p(\underline{\gamma(t)})\big) = \operatorname{ord}(x(t)) = \operatorname{ord}(t^3),
\end{aligned}$$

write the Euclidean algorithm and construct:

$$\begin{aligned} 5 &= 1\cdot 3+2 & \qquad z_0 &= \kappa_0(\underline{x^{(0)}},\underline{\kappa_0}) \\ 3 &= 1\cdot 2+1 & z_1 &= \kappa_0(\underline{z_0},\underline{x^{(0)}}) \\ 2 &= 2\cdot 1+0 & & \end{aligned}$$

Then

$$z_0 = \frac{\partial\kappa_0}{x^{(1)}} = \kappa_1,$$

and

$$z_1 = \kappa_0(\underline{\kappa_1},\underline{x^{(0)}}) = \frac{x^{(1)}}{\partial\kappa_1} = \kappa_2^{-1}.$$

With Remark 3.5 we see immediately that $\operatorname{ord}\big(z_1)(\underline{\gamma(t)})\big) = 1$.

The strategy now is the following: for a Puiseux parametrization $\eta(t)$ of X at 0, we want to go through all its characteristic exponents $\beta_1,\dots,\beta_g$ (see Appendix A for their definition) and construct inductively geometric invariants $p_1,\dots,p_g \in I_F$ whose evaluation at $\underline{\eta(t)}$ have orders $e_1 = \gcd(n,\beta_1),\dots,1 = e_g = \gcd(n,\beta_1,\dots,\beta_g)$, respectively. To do so, we will have to combine our last algorithm with triangular coordinate changes:

Triangular Coordinate Change for Plane Curves Elimination of the first terms of the evaluation of a geometric invariant at $\underline{\gamma(t)}$, for $\gamma(t)\in\mathbb{C}[t]^2$, that have degrees divisible by the order of the evaluation of another geometric invariant at $\underline{\gamma(t)}$

Input polynomial pair $\gamma(t)\in\mathbb{C}[t]^2$ & two geometric invariants p and q so that $q(\underline{\gamma(t)}) \notin \mathbb{C}[p(\underline{\gamma(t)})]$ and $0<\operatorname{ord}\big(p(\underline{\gamma(t)})\big),\operatorname{ord}\big(q(\underline{\gamma(t)})\big)<\infty$
Output geometric invariant $z = q-(c_1p+\cdots+c_kp^k)$, for some $k\in\mathbb{N}$ and $c_i\in\mathbb{C}$ satisfying $\operatorname{ord}\big(p(\underline{\gamma(t)})\big)\nmid\operatorname{ord}\big(z(\underline{\gamma(t)})\big)$
procedure $\textsc{Trian}(p,q;\gamma(t))$

1st step:

Set

$$\mathsf{y}_0 := q \;\text{ and }\; \mathsf{n} := \operatorname{ord}\big(p(\underline{\gamma(t)})\big),\; \mathsf{m}_0 := \operatorname{ord}\big(q(\underline{\gamma(t)})\big).$$

2nd step:

Consider the geometric invariant y_{i-1} and the corresponding order $\mathsf{m}_{i-1} := \operatorname{ord}\big(\mathsf{y}_{i-1}(\underline{\gamma(t)})\big)$.
If $n\nmid m_{i-1}$ then **return** y_{i-1}.

Otherwise continue with the next step of the algorithm.

3rd step:

Set

$$\mathsf{y}_i := \mathsf{y}_{i-1} - p^{q_{i-1}},$$

where

$$\mathsf{m}_{i-1} = \mathsf{n} \cdot q_{i-1},$$

and go back to the 2nd step of the algorithm.

Finally, we present our resolution algorithm for plane curves with one singularity at the origin:

Resolution Algorithm for Plane Curves - One Singularity Construction of a crucial height

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Input polynomial pair $\gamma(t) = (t^n, y(t)) \in \mathbb{C}[t]^2$
Output geometric invariant $\widetilde{\kappa}$ satisfying $\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = \gcd\big(n, i \,:\, i \in \operatorname{supp}(y(t))\big)$
procedure PLANECURVATURE($\gamma(t)$)

1st step:
If $n \leq \operatorname{ord}(y(t))$, then define

$$\mathsf{x}_0 \coloneqq x^{(0)},\ \mathsf{y}_0 \coloneqq y^{(0)} \text{ and } \mathsf{m}_0 \coloneqq \operatorname{ord}(y(t)),\ \mathsf{n}_0 \coloneqq n.$$

otherwise

$$\mathsf{x}_0 \coloneqq y^{(0)},\ \mathsf{y}_0 \coloneqq x^{(0)} \text{ and } \mathsf{m}_0 \coloneqq n,\ \mathsf{n}_0 \coloneqq \operatorname{ord}(y(t)).$$

Further set

$$\alpha \coloneqq \gcd\big(n, i \,:\, i \in \operatorname{supp}(y(t))\big).$$

2nd step:

Assume that $\mathsf{x}_{i-1}, \mathsf{y}_{i-1}$ and $\mathsf{n}_{i-1}, \mathsf{m}_{i-1}$ have been already constructed. If $\mathsf{n}_{i-1} = \alpha$, then **return** x_{i-1}.

Otherwise continue with the next step of the algorithm.

3rd step:

Define

$$Y_i \coloneqq \textsc{Trian}(\mathsf{x}_{i-1}, \mathsf{y}_{i-1}; \gamma(t)) \text{ and } M_i \coloneqq \operatorname{ord}\big(Y_i(\underline{\gamma(t)})\big).$$

4th step:

Set

$$(\mathsf{x}_i, \mathsf{y}_i) \coloneqq \textsc{gcd}(\mathsf{x}_{i-1}, Y_i; \gamma(t)) \text{ and } \mathsf{m}_i \coloneqq \operatorname{ord}\big(\mathsf{y}_i(\underline{\gamma(t)})\big).$$

Then, the geometric invariant x_i satisfies

$$\operatorname{ord}\big(\mathsf{x}_i(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_{i-1}, M_i)$$

and we set

$$\mathsf{n}_i \coloneqq \operatorname{ord}\big(\mathsf{x}_i(\underline{\gamma(t)})\big).$$

Turn back to the 2nd step of the algorithm.

Termination of the algorithm PLANECURVATURE**:**

Let us first discuss the case when $\gamma(t) = (t^n, y(t))$ is a Puiseux parametrization of a plane algebraic curve, i.e., $n \leq \operatorname{ord}(y(t))$ and $\gcd\big(n, i \,:\, i \in \operatorname{supp}(y(t))\big) = 1$. Show that in this case the algorithm constructs already a crucial height of the curve parametrized by $\gamma(t)$. The general case then follows by a suitable variable substitution and reparametrization.

1. **$\gamma(t)$ is a Puiseux parametrization:**

   Observe first that the algorithm terminates in the 1st step if and only if $n = 1$, which would mean that the curve parametrized by $\gamma(t)$ is smooth at the origin and the variable x_0 is a crucial height of the curve. Let us therefore consider a Puiseux parametrization $\gamma(t) = (t^n, y(t))$ with $n > 1$.

   The goal is to show that we have $\mathsf{n}_1 < \mathsf{n}_0$. Then, the claim follows by induction. First observe, that the Puiseux characteristic does not change under triangular coordinate changes. Further we have:

   Theorem 3.7. *Let $\eta(t) = (t^n, y(t)) \in \mathbb{C}[t]^2$ be a Puiseux parametrization of a branch at 0. Let $(n|\beta_1, \dots, \beta_g)$ be the Puiseux characteristic of the branch. Assume that $\operatorname{ord}(y(t)) = \beta_1$. Let $k = \lfloor \frac{\beta_1}{n} \rfloor$. Then the Puiseux characteristic of the branch parametrized by the pair $(\kappa_{k-1}(\underline{\eta(t)}), t^n)$ equals*

   (i) $(\beta_1 - kn|n, \beta_2 - \beta_1 + n, \dots, \beta_g - \beta_1 + n)$ *if* $(\beta_1 - kn) \nmid n$,
   (ii) $(\beta_1 - kn|\beta_2 - \beta_1 + n, \dots, \beta_g - \beta_1 + n)$ *if* $(\beta_1 - kn)|n$.

   Proof. The statement can be proven by the same argument as Theorem 3.5.5 in [Wa04]. □

   It follows directly from the definition of the individual steps of the algorithm PLANECURVATURE, that we have
   $$\mathsf{n}_1 \leq \beta_1 - kn < n = \mathsf{n}_0,$$
   which proves the termination of the algorithm applied to a Puiseux parametrization.

2. **$\gamma(t)$ is not a Puiseux parametrization:**

   In this case we have two possibilities:

   a) either $\gcd\big(n, i \,:\, i \in \operatorname{supp}(y(t))\big) = 1$ but $n > m = \operatorname{ord}(y(t))$,
   b) or $\gcd\big(n, i \,:\, i \in \operatorname{supp}(y(t))\big) = \alpha > 1$.

   In the case a), after a suitable reparametrization we obtain a Puiseux parametrization $((x \circ \varphi)(t), t^m)$ of the branch to which our above analysis applies.

   In the case b), there exists a Puiseux parametrization $\widetilde{\gamma}(t)$ of the branch parametrized by $\gamma(t)$ such that $\gamma(t) = \widetilde{\gamma}(t^\alpha)$. Moreover, using the fact that for each geometric invariant $p \in I_F$, we have $p(\underline{\gamma(t)}) = p(\underline{\widetilde{\gamma}(t)})|_{t=t^\alpha}$ (for more details see [Me20b]) we conclude
   $$\text{PLANECURVATURE}(\gamma(t)) = \text{PLANECURVATURE}(\widetilde{\gamma}(t))|_{t=t^\alpha} = t^\alpha.$$

Example 3.8 (Construction of a crucial height with the algorithm PLANECURVATURE)**.** Consider the curve

$$X = \{-x^3 + (3y^2 - 6y + 1)x^2 + (-3y^4 - 2y^3)x + y^6 = 0\}.$$

The pair

$$\gamma(t) = (t^6, t^2 + t^3)$$

defines a parametrization of X at 0. We construct now $\widetilde{\kappa} = \text{PLANECURVATURE}(\gamma(t))$ according to the algorithm.

1st step: $\mathsf{x}_0 = y^{(0)}, \mathsf{y}_0 = x^{(0)}, \mathsf{n}_0 = \operatorname{ord}(t^2 + t^3) = 2, \mathsf{m}_0 = \operatorname{ord}(t^6) = 6, \alpha = \gcd(6, 2, 3) = 1.$

2nd step: Since $\mathsf{n}_0 = 2 \neq 1 = \alpha$, we continue with the next step.

3rd step: As $\mathsf{n}_0 | \mathsf{m}_0$, a triangular coordinate change is necessary. We define

$$Y_1 = \text{TRIAN}(\mathsf{x}_0, \mathsf{y}_0; \gamma(t)) = \mathsf{y}_0 - \mathsf{x}_0^3$$

and

$$M_1 = \operatorname{ord}\big(Y_1(\underline{\gamma(t)})\big) = \operatorname{ord}(t^6 - (t^2 + t^3)^3) = \operatorname{ord}(-3t^7 - 3t^8 - t^9) = 7.$$

4th step: We compute the geometric invaraint whose evaluation at $\underline{\gamma(t)}$ is of order $\gcd(2, 7) = 1$. We follow the Euclidean Algorithm for 2 and 7:

$$\begin{aligned} &7 = 3 \cdot 2 + 1 && \mathsf{x}_1 = \kappa_2(\underline{\mathsf{x}_0}, \underline{Y_1}) && \mathsf{n}_1 = \operatorname{ord}\big(\mathsf{x}_1(\underline{\gamma(t)})\big) = 1 \\ &2 = 1 \cdot 2 + 0 \end{aligned}$$

A computation shows that

$$\mathsf{x}_1 = \kappa_2(\underline{y^{(0)}}, \underline{x^{(0)}}) - 6.$$

5th step: As $\mathsf{n}_1 = 1$ we stop here and return

$$\kappa_2(\underline{y^{(0)}}, \underline{x^{(0)}}) - 6.$$

Thus, for a plane algebraic curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ with only one singular point $0 \in X$ and only one analytic branch at the origin, we have just proven the following theorem:

Theorem 3.9. *For any Puiseux parametrization $\eta(t)$ of X at the origin, the algorithm* PLANECURVATURE *applied to the characteristic part $\eta^X(t)$ of $\eta(t)$ constructs a crucial height of X, i.e., a geometric invariant $\widetilde{\kappa}$ that satisfies*

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\eta(t)})\big) = 1.$$

Moreover, the blowup of X in the ideal $(\widetilde{\kappa}(X)_1, \widetilde{\kappa}(X)_2)$ defines a resolution of singularities of X.

In this section we established an algorithm only for resolution of plane algebraic curves with only one singular point. The resolution of plane curves with several singularities is discussed in Section 5 of this article.

4 Resolution of Analytically Irreducible Singular Space Curves with only one Singularity

We present in this section an algorithm for resolution of analytically irreducible space curves with a single singular point. Our algorithm is, as in the plane curve case, based on the existence of characteristic exponents of Puiseux parametrizations. Given an algebraic space curve $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ with only one singularity at the origin and only one analytic branch at the origin, our algorithm constructs a geometric invariant of space curves $\widetilde{\kappa}$ satisfying the property

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1, \tag{3}$$

for at least one parametrization $\gamma(t) \in \mathbb{C}[\![t]\!]^{n+1}$ of X at the origin.

Definition 4.1. We call a geometric invariant $\widetilde{\kappa}$, that satisfies the property (3), a *crucial height* of X (at the origin).

Given a Puiseux parametrization $\gamma(t)$ of X at 0, the strategy of our algorithm is to project the space curve X to the coordinate planes and using the algorithm PLANECURVATURE there to construct for each projection a geometric invariant of minimal possible order (when evaluating at $\underline{\gamma(t)}$). The orders of the evaluations of these geometric invariants at $\underline{\gamma(t)}$ contain by construction a complete information about the Puiseux characteristic of each projection. Moreover, their greatest common divisor should be, by construction, equal one.

To be more precise: Let $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ be a space algebraic curve with only one singular point $0 \in X$. Let us assume that X is analytically irreducible at 0. Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$ be an ideal defining X. As in the case of plane curves, we will construct a resolution $\widetilde{X}$ of X via a blowup in a suitable ideal $I + (\widetilde{\kappa}(X)_1, \widetilde{\kappa}(X)_2)$ defined in X by the numerator and denominator of an implicit expression of a crucial height $\widetilde{\kappa}$.

The resolution $\widetilde{X}$ again equals the Zariski closure $\widetilde{X}_{\widetilde{\kappa}}$ of the graph of the map induced by the crucial height $\widetilde{\kappa}$:

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to \mathbb{P}_{\mathbb{C}}^1 \\ a &\mapsto (\widetilde{\kappa}(X)_1(a) : \widetilde{\kappa}(X)_2(a)), \end{aligned}$$

with $Z = V\big(I + (\widetilde{\kappa}(X)_1, \widetilde{\kappa}(X)_2)\big)$. Notice that Lemma 3.3 generalizes also to the space curve case and the same holds also for Corollary 3.4 — both proofs go along the same line as for plane curves. Thus, a crucial height of X yields already a resolution of X.

Let us w.l.o.g. assume that $x \notin I$ and $y_j \notin I$ for each $j = 1, \ldots, n$ (otherwise we embed the curve in $\mathbb{A}_{\mathbb{C}}^n$). Then according to Theorem A.2, X admits a parametrization

$$\eta(t) = (x(t), y_1(t) \ldots, y_n(t)) = (t^l, y_1(t) \ldots, y_n(t)) \in \mathbb{C}\{t\}^{n+1}$$

at the origin for some $l \in \mathbb{N}$. Let us w.l.o.g. assume that $l \leq \min\{\operatorname{ord}(y_i(t)) \, : \, i = 1, \ldots, n\}$ (otherwise apply an affine coordinate change), i.e., that $\eta(t)$ is a Puiseux parametrization of X at 0. As already mentioned, the strategy of our algorithm is to project the curve X with the n

projections

$$\pi_i \colon \mathbb{A}^{n+1}_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$$
$$(x, y_1, \ldots, y_n) \mapsto (x, y_i)$$

to plane curves X_i parametrized by $\eta_i(t) = (t^l, y_i(t))$ at the origin and to apply the algorithm PLANECURVATURE to these projections in order to construct for each of the parametrizations $\eta_i(t)$ a geometric invariant of minimal possible order. More precisely:

Remark 4.2. Let $\eta^\chi(t)$ be the characteristic part of the Puiseux parametrization $\eta(t)$. Then, $\eta_i(t)$ itself is not a Puiseux parametrization, but $\eta_i(t)$ can be written as $\widetilde{\eta}_i(t^{\frac{l}{l_i}})$ for some Puiseux parametrization $\widetilde{\eta}_i(t)$ of the branch parametrized by $\eta_i(t)$ and some divisor l_i of l. Notice, that l_i is the multiplicity of $\widetilde{\eta}_i(t)$. Thus l is the product of the multiplicities l_i of Puiseux parametrizations of branches that are parametrized by $\eta_i(t) = (t^l, y_i(t))$ (for more details see Appendix A), i.e., $l = \prod_i l_i$. By construction, we have

$$\operatorname{ord}\Big(\big(\text{PLANECURVATURE}(\gamma_i(t))\big)(\underline{\gamma_i(t)})\Big) = \frac{l}{l_i}.$$

Resolution Algorithm for Space Curves - one Singularity Construction of a crucial height

Input polynomial pair $\gamma(t) = (t^l, y_1(t), \ldots, y_n(t),) \in \mathbb{C}[t]^{n+1}$ & the embedding dimension $N = n + 1$
Output geometric invariant $\widetilde{\kappa}$ satisfying
$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = \gcd\big(l, i_1, \ldots, i_n \,:\, i_j \in \operatorname{supp}(y_j(t))\big)$
procedure SPACECURVATURE$(\gamma(t); N)$

1st step:
Let $m = \min\{l, \operatorname{ord}(y_j(t)) \,:\, j = 1, \ldots, n\}$. If $m = 1$, then **return**

$$\begin{cases} x^{(0)} & \text{if} \quad l = 1 \\ y_j^{(0)} & \text{if} \quad \operatorname{ord}(y_j(t)) = 1 \end{cases}$$

Else, if $m = l$, then set

$$\mathsf{x} \coloneqq x^{(0)},\ \mathsf{y}_1 \coloneqq y_1^{(0)}, \ldots\ ,\ \mathsf{y}_{N-1} \coloneqq y_{N-1}^{(0)}.$$

Otherwise, if $y_j(t)$ is the component of $\gamma(t)$ satisfying $m = \operatorname{ord}(y_j)$, then we set

$$\mathsf{x} \coloneqq y_j^{(0)},\ \mathsf{y}_1 \coloneqq y_1^{(0)}, \ldots\ \mathsf{y}_{j-1} \coloneqq y_{j-1}^{(0)},\ \mathsf{y}_j \coloneqq x^{(0)},\ \mathsf{y}_{j+1} \coloneqq y_{j+1}^{(0)}, \ldots\ ,\ \mathsf{y}_{N-1} \coloneqq y_{N-1}^{(0)}.$$

2nd step:

For each $i = 1, \ldots, N-1$, set

$$z_i \coloneqq \text{PLANECURVATURE}\big(\mathsf{x}(\underline{\gamma(t)}), \mathsf{y}_i(\underline{\gamma(t)})\big).$$

For each $i = 1, \ldots, N-1$, we apply the substitution

$$\begin{aligned} \lambda_i \colon \mathbb{C}(x^{(j)}, y^{(j)} \,:\, j \in \mathbb{N}) &\to \mathbb{C}(x^{(j)}, y_i^{(j)} \,:\, i, j \in \mathbb{N}, 1 \leq i \leq n) \\ x^{(j)} &\mapsto \partial^j \mathsf{x} \\ y^{(j)} &\mapsto \partial^j \mathsf{y}_i \end{aligned}$$

in order to obtain geometric invariants of space curves

$$\mathsf{z}_i := \lambda_i(z_i)$$

of order at $\underline{\gamma(t)}$ equal to

$$\mathsf{n}_i := \operatorname{ord}(\mathsf{z}_i(\underline{\gamma(t)})).$$

3rd step:

Define iteratively and observe

$$\begin{aligned}
&\widetilde{\mathsf{z}}_2 := \mathrm{GCD}(\mathsf{z}_1, \mathsf{z}_2; \gamma(t)) && \operatorname{ord}\big(\widetilde{\mathsf{z}}_2(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \mathsf{n}_2), \\
&\widetilde{\mathsf{z}}_3 := \mathrm{GCD}(\widetilde{\mathsf{z}}_2, \mathsf{z}_3; \gamma(t)) && \operatorname{ord}\big(\widetilde{\mathsf{z}}_3(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \mathsf{n}_2, \mathsf{n}_3), \\
&\quad \vdots && \quad \vdots \\
&\widetilde{\mathsf{z}}_{N-1} := \mathrm{GCD}(\widetilde{\mathsf{z}}_{N-2}, \mathsf{z}_{N-1}; \gamma(t)) && \operatorname{ord}\big(\widetilde{\mathsf{z}}_{N-1}(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \ldots, \mathsf{n}_{N-1}).
\end{aligned}$$

Finally, **return** $\widetilde{\mathsf{z}}_{N-1}$.

Correctness of the algorithm SPACECURVATURE:
In fact, by construction, by the properties of the algorithm PLANECURVATURE, and by the same arguments we used to prove the correctness of the algorithm PLANECURVATURE, the order of $\widetilde{\kappa} = \text{SPACECURVATURE}(\gamma(t); N)$ at $\underline{\gamma(t)}$ satisfies

$$\operatorname{ord}(\widetilde{\kappa}(\underline{\gamma(t)})) = \gcd\big(l, i_1, \ldots, i_n \,:\, i_j \in \operatorname{supp}(y_j(t))\big) = 1.$$

Thus, for an algebraic space curve $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ with only one singularity $0 \in X$ and only one analytic branch at the origin, we have just proven the following theorem:

Theorem 4.3. *For the characteristic part $\eta^{\chi}(t)$ of a Puiseux parametrization $\eta(t) = (x(t), y_1(t), \ldots, y_n(t))$ of X at the origin, the algorithm* SPACECURVATURE *constructs a crucial height of X, i.e., a geometric invariant $\widetilde{\kappa}$ that satisfies*

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\eta(t)})\big) = 1.$$

Moreover, for $I \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$ a defining ideal of X, the blowup of X in the ideal

$$\big(I + (\widetilde{\kappa}(X)_1, \widetilde{\kappa}(X)_2)\big)$$

defines a resolution of singularities of X.

However, the algorithm SPACECURVATURE constructs a resolution only for analytically irreducible space curves with only one singular point. If X has more than one singular point, iterations of the algorithm SPACECURVATURE are needed as we will see in the next section.

5 Resolution of Analytically Irreducible Singular Plane and Space Curves with more than one Singularity

Let us fix a plane or space curve $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$, $n \geq 1$, with m singular points

$$\operatorname{Sing}(X) = \{a_1, \dots, a_m\}.$$

Let $I \subseteq \mathbb{C}[x, y_1 \dots, y_n]$ be a defining ideal of X. Let us assume that X is analytically irreducible at each point. The goal of this section is to present an algorithm for construction of a resolution of X based on the algorithms PLANECURVATURE and SPACECURVATURE presented in Sections 3 and 4, respectively.

Let us fix for each $i = 1, \dots, m$, a Puiseux parametrization $\eta_i(t) \in \mathbb{C}\{t\}^{n+1}$ of X at the singular point a_i. Further, consider for each i the following coordinate change:

$$\begin{aligned} \lambda_{a_i} \colon \mathbb{A}_{\mathbb{C}}^{n+1} &\to \mathbb{A}_{\mathbb{C}}^{n+1} \\ (x, y_1, \dots, y_n) &\mapsto (x, y_1, \dots, y_n) - a_i, \end{aligned}$$

under which a_i moves to the origin. Let us further denote by X_{a_i} the image of the curve X under λ_{a_i}. Then, $\eta_i(t) - a_i = (x_i(t), y_{i,1}(t), \dots, y_{i,n}(t))$ is a Puiseux parametrization of X_{a_i} at 0. Let us w.l.o.g. assume that $x_i(t) \neq 0$ and $y_{i,j}(t) \neq 0$ is fulfilled for all i, j (otherwise we could embed X in $\mathbb{A}_{\mathbb{C}}^{n}$).

We now present an algorithm that constructs with SPACECURVATURE for each singular point a_i on X a crucial height $\widetilde{\kappa}_i = \frac{\widetilde{\kappa}_{i,1}}{\widetilde{\kappa}_{i,2}}$ of X_{a_i} at 0 which is at the same time also a crucial height of X at a_i. The claim is that for $\gamma(t)$ a parametrization of X, the curve in $\mathbb{A}_{\mathbb{C}}^{n+1} \times (\mathbb{P}_{\mathbb{C}}^1)^m$ parametrized by the vector

$$\widetilde{\gamma}(t) = \gamma(t) \times \big(\widetilde{\kappa}_{i,1}(\underline{\gamma(t)}) : \widetilde{\kappa}_{i,2}(\underline{\gamma(t)})\big) \times \cdots \times \big(\widetilde{\kappa}_{m,1}(\underline{\gamma(t)}) : \widetilde{\kappa}_{m,2}(\underline{\gamma(t)})\big)$$

defines a resolution of singularities of X.

Resolution Algorithm - Several Singularities Construction of a crucial curvature at each singular point

---

Input number of singularities m & for each $i = 1, \dots, m$ the characteristic part $\eta_i^{\chi}(t) = a_i + (x_i(t), y_{i,1}(t), \dots, y_{i,n}(t)) \in \mathbb{C}[t]^{n+1}$ of a Puiseux parametrization of X at a_i & the embedding dimension $N = n + 1$

Output vector $\widetilde{\kappa} = (\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_m)$ of geometric invariants satisfying $\operatorname{ord}\big(\widetilde{\kappa}_i(\underline{\eta_i(t)})\big) = 1$

procedure CURVATURES$(\eta_1^{\chi}(t), \dots, \eta_m^{\chi}(t)\,; m, N)$

For each $i = 1, \dots, m$ compute

$$\widetilde{\kappa}_i := \text{SPACECURVATURE}(\eta_i(t) - \eta_i(0)\,; N).$$

Finally **return** the list

$$(\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_k).$$

Correctness of the algorithm CURVATURES**:**
We have

$$\widetilde{\kappa}_i(\underline{\eta_i(t)}) = \widetilde{\kappa}_i(\underline{\eta_i(t) - \eta_i(0)}),$$

for each $i = 1, \dots, n$, and so $\operatorname{ord}\big(\widetilde{\kappa}_i(\underline{\eta_i(t)})\big) = 1$.

Let $\widetilde{\kappa} = (\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_m) = \text{CURVATURES}(\eta_1^X(t), \dots, \eta_k^X(t); m, n+1)$ be the list of crucial curvatures produced by the algorithm CURVATURES. Consider the map

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to (\mathbb{P}^1_{\mathbb{C}})^m \\ a &\mapsto \big(\widetilde{\kappa}_1\big(X\big)_1(a) : \widetilde{\kappa}_1\big(X\big)_2(a)\big) \times \cdots \times \big(\widetilde{\kappa}_m\big(X\big)_1(a) : \widetilde{\kappa}_m\big(X\big)_2(a)\big), \end{aligned}$$

where $Z = V\big(I + (\widetilde{\kappa}_i\big(X\big)_j \,:\, 1 \leq i \leq m, j = 1, 2)\big)$ and where $\widetilde{\kappa}_i\big(X\big)_j$ denote the numerator for $j = 1$ and the denominator for $j = 2$ of an implicit expression of $\widetilde{\kappa}_i$, respectively. Let $\widetilde{X}_{\widetilde{\kappa}}$ denote the Zariski closure of the graph of the map $\phi_{\widetilde{\kappa}}$.

Proposition 5.1. *The projection morphism*

$$\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$$

induced by the projection $\pi : \mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^m \to \mathbb{A}^{n+1}_{\mathbb{C}}$ *is a birational and projective morphism which is an isomorphism*

$$\pi : \widetilde{X}_{\widetilde{\kappa}} \backslash E \to X \backslash Z$$

outside $E = \pi^{-1}(Z)$*, where* $Z = V\big(I + (\widetilde{\kappa}_i\big(X\big)_j \,:\, 1 \leq i \leq m, j = 1, 2)\big)$.

Proof. For each $1 \leq l \leq m$, we denote by $\widetilde{X}^l_{\widetilde{\kappa}}$ the Zariski closure of the graph of the map

$$\begin{aligned} \phi^l_{\widetilde{\kappa}} \colon X \backslash Z_l &\to (\mathbb{P}^1_{\mathbb{C}})^l \\ a &\mapsto (\widetilde{\kappa}_1\big(X\big)_1(a) : \widetilde{\kappa}_1\big(X\big)_2(a)) \times \dots (\widetilde{\kappa}_l\big(X\big)_1(a) : \widetilde{\kappa}_l\big(X\big)_2(a)), \end{aligned}$$

with $Z_l = V\big(I + (\widetilde{\kappa}_i\big(X\big)_j \,:\, 1 \leq i \leq l, j = 1, 2)\big)$. Notice that each map $\pi_l : \widetilde{X}^l_{\widetilde{\kappa}} \to X$ induced by the projection $\mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^l \to \mathbb{A}^{n+1}_{\mathbb{C}}$ defines a birational morphism and moreover an isomorphism $\pi_l : \widetilde{X}^l_{\widetilde{\kappa}} \backslash E_l \to X \backslash Z_l$ outside $E_l = \pi_l^{-1}(Z_l)$ with the inverse map given by $a \mapsto a \times \phi^l_{\widetilde{\kappa}}(a)$. We proceed now by induction on l to show that π_l is projective for each $l = 1, \dots, m$. For $l = 1$, $X^1_{\widetilde{\kappa}}$ is the blowup of X in the ideal $(I + (\widetilde{\kappa}_1\big(X\big)_1, \widetilde{\kappa}_1\big(X\big)_2))$ and the claim follows directly. For $l + 1$, we observe that $\widetilde{X}^{l+1}_{\widetilde{\kappa}}$ is the Zariski closure of the image of the map

$$\widetilde{X}^l_{\widetilde{\kappa}} \backslash V\big(I + (\widetilde{\kappa}_{l+1}\big(X\big)_1, \widetilde{\kappa}_{l+1}\big(X\big)_2)\big) \to \mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^l \times \mathbb{P}^1_{\mathbb{C}}$$

induced by the geometric invariant $\widetilde{\kappa}_{l+1}$. The claim follows now from the induction hypothesis on $\widetilde{X}^l_{\widetilde{\kappa}}$. □

Moreover, using [Ha77, Chapter II, Theorem 7.17.] the following corollary follows:

Corollary 5.2. *There exists an ideal* $J \subseteq \mathbb{C}[x, y_1, \ldots, y_n]$ *such that the curve* $\widetilde{X}_{\widetilde{\kappa}}$ *together with the projection* $\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$ *is a blowup of* X *in the ideal* J*.*

Thus, we have proven resolution of singularities of X via the map $\phi_{\widetilde{\kappa}}$:

Theorem 5.3. *The Zariski closure* $\widetilde{X}_{\widetilde{\kappa}}$ *of the graph of* $\phi_{\widetilde{\kappa}}$ *defines together with the projection morphism* $\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$ *a resolution of singularities of* X*.*

6 Analytically Reducible Singular Curves

The algorithms presented in Sections 3, 4 and 5 were constructed in order to resolve analytically irreducible curves and unfortunately do not give a resolution of curves with several analytic branches at their singular points. In fact, it is not clear for the moment how to use geometric invariants for resolution of analytically reducible curves. Whereas it is very easy to use the slope of the tangent vector to separate two transversal branches, it is in general not clear how to use the (higher) algebraic curvatures to separate two analytic branches that meet tangentially, as the illustrated in the following example:

Example 6.1. Let us consider the (even algebraically, not only analytically) reducible plane curve X defined as the union of two parabolas X_1 and X_2 given by their respective equations $x = y^2$ and $y = -x^2$ and parametrized by $\gamma_1(t) = (t^2, t)$ and $\gamma_2(t) = (-t^2, t)$, respectively.

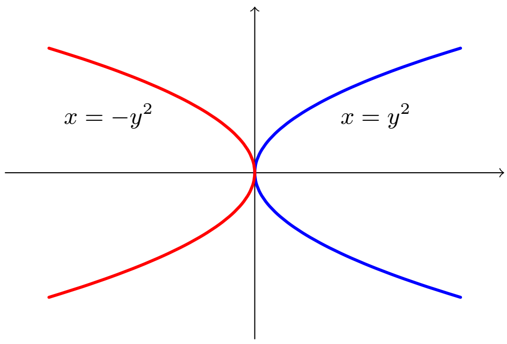

Figure 3: Two symmetric horizontal parabolas $x = y^2$ (blue) and $x = -y^2$ (red) meeting at the origin.

Then the evaluation $\kappa_i(\underline{\gamma_j(t)})$, with $j = 1, 2$, of each algebraic curvature at $\underline{\gamma_j(t)}$ is a Laurent series of order $-2i - 1$ and, moreover, we have the equality

$$\kappa_i(\gamma_1(t)) = (-1)^{i+1}\kappa_i(\gamma_2(t)).$$

The (higher) algebraic curvatures are therefore not able to distinguish between both curves at the origin and hence are also not able to tear them apart.

Therefore, to establish a resolution of analytically reducible curves by means of geometric invariants, a more refined method would be necessary. This problem remains for the moment on a list with open questions for the moment.

A Puiseux Parametrizations

In this section we introduce some terminology concerning Puiseux parametrizations of algebraic curves (plane curves in $\mathbb{A}^2_{\mathbb{C}}$ and also space curves in $\mathbb{A}^{n+1}_{\mathbb{C}}$) and collect some basic facts about them which play a crucial role in our resolution algorithms presented in Sections 3, 4 and 5. All facts about Puiseux parametrizations of plane algebraic curves and their analytic branches listed here are standard and can be found for example in E. Casas-Alvero's book [Ca00, Chapter 1] or in C. T. C. Wall's book [Wa04, Chapter 3]. For the construction of Puiseux parametrizations of space curves we refer to J. Maurer's work [Ma80].

Let us start by recalling some facts about y-roots of bivariate power series and the Puiseux parametrizations of plane curves. Let $f \in \mathbb{C}\{x, y\}$ be a bivariate holomorphic power series with $f(0, 0) = 0$ and $f(0, y) \neq 0$. And assume that f is irreducible. (In case of a reducible power series f, one has to look at its irreducible components.) Then Puiseux' Theorem states that if we want to solve the equation $f(x, y) = 0$ for y in terms of x, then there always exists a solution which is fractionary power series in x, i.e., a power series in x with fractional exponents with bounded denominator, called a *Puiseux series*. In other words:

Newton-Puiseux theorem. *For any power series $f \in \mathbb{C}\{x, y\}$ with zero constant term and with $f(0, x) \neq 0$, there exists a pair convergent of power series*

$$(x(t), y(t)) = (t^n, \sum_{i \geq 1} a_i t^i) \in \mathbb{C}\{t\}^2,$$

for some $n \in \mathbb{N}$, such that $f(x(t), y(t)) = 0$.

Each such pair $(x(t), y(t)) = (t^n, y(t)) = (t^n, \sum_{i \geq 1} a_i t^i)$ of power series with minimal integer n satisfying $f(t^n, y(t)) = 0$ can be constructed using Newton-Puiseux algorithm. By the minimality here we mean that the values of i with $a_i \neq 0$, together with n, have the greatest common divisor equal to 1. This allows us to define the following set:

$$\begin{aligned} \beta_1 &:= \min\{i : a_i \neq 0 \text{ and } n \nmid i\}, \\ e_1 &:= \gcd(n, \beta_1) \\ \beta_{j+1} &:= \min\{i : a_i \neq 0 \text{ and } e_j \nmid i\}, \\ e_{j+1} &:= \gcd(n, \beta_1, \ldots, \beta_{j+1}). \end{aligned} \tag{4}$$

Let g be the minimal integer satisfying $e_g = 1$. The numbers $\beta_1, \ldots, \beta_g$ are then called the *characteristic exponents* of $(x(t), y(t))$. In the case that n additionally satisfies $n \leq m = \operatorname{ord}(y(t))$, we call the pair $(x(t), y(t))$ a *Puiseux parametrization* of the the analytic curve defined by f at 0 and the sequence of integers

$$(n|\beta_1, \ldots, \beta_g)$$

its *Puiseux characteristic*. In this case, n equals the order of f and is called the *multiplicity* of the curve. Further, we define

$$(x(t), y(t))^{\chi}(t) := (t^n, \sum_{i=1}^{\beta_g} a_i t^i)$$

and call it the *characteristic part* of the Puiseux parametrization $(x(t), y(t))$.

If $n > m = \operatorname{ord}(y(t))$, we parametrize the curve by the pair $(x(t), t^m)$ and take the characteristic exponents of this parametrization and m to define the Puiseux characteristic of the curve. It follows directly form the Newton-Puiseux algorithm and also from [Wa04, Lemma 3.5.4 and the proof of Theorem 3.5.5], that in the case that $n = m$, the Puiseux characteristic does not depend on the choice of a Puiseux parametrization. Hence, it is a local invariant of the curve.

Example A.1. Let ξ be a primitive 6-th root of unity. Then it can be shown that the polynomial

$$\begin{aligned}f(x,y) =& \prod_{i=1}^{6}(y - \xi^{3i}x^{1/2} + 2\xi^{4i}x^{2/3} + 3\xi^{5i}x^{5/6}) \\ =& x^6 - 3x^4y - 56x^3y^2 - 162x^2y^3 + 972xy^4 - 729y^5 + 3x^2y^2 + 120xy^3 + 730y^4 - y^3\end{aligned}$$

is analytically irreducible, i.e., irreducible as a power series, and that the pair

$$\gamma(t) = (t^6, t^3 - 2t^4 - 3t^5)$$

parametrizes the curve X defined by f. Notice that $\gcd(6, 3, 4) = 1$, hence, the characteristic exponents of $\gamma(t)$ are 3 and 4. However, $\gamma(t)$ is not a Puiseux parametrization of the curve. After reparametrizing $\gamma(t)$ suitably, we obtain the Puiseux parametrization

$$\eta(t) = (t^6 + 4t^7 + \frac{62}{3}t^8 + \ldots, t^3)$$

of X. $\eta(t)$ has only one characteristic exponent, namely 7. Thus, the Puiseux characteristic of X equals

$$(3|7).$$

The classical Newton-Puiseux algorithm constructs parametrizations only for plane algebraic curves and their analytic branches. However, since Puiseux' study of fractional power series, several generalizations of the algorithm for solving more general systems of polynomial equations were established. In 1980, J. Maurer gave in his paper [Ma80] a constructive proof for the existence of parametrizations of space curves, i.e., he solved the problem of finding y_i-roots of a system of convergent power series

$$f_j(x, y_1, \ldots, y_n) = 0 \tag{5}$$

defining an analytic space curve and gave an algorithm constructing all its parametrizations of the form $\gamma(t) = (t^l, y_1(t), \ldots, y_n(t)) \in \mathbb{C}\{t\}^{n+1}$. Another proof of the existence of y_i-roots of (5) can for instance be found in the paper [JMM08] by A. N. Jensen, H. Markwig and T. Markwig.

Theorem A.2. *Let $I \subseteq \mathbb{C}\{x, y_1, \ldots, y_n\}$ be an ideal defining a branch of an algebraic space curve at 0. Suppose that $x \notin I$ and $y_j \notin I$ for any $1 \leq j \leq n$. Then this branch can be parametrized by $\gamma(t) = (t^l, y_1(t), \ldots, y_n(t))$ for some $l \in \mathbb{N}$ and convergent power series $y_j(t) \in \mathbb{C}\{t\}^{n+1}$ for $j = 1, \ldots, n$.*

Moreover, similarly as in the plane curve case, for a an analytic space curve Y in $\mathbb{A}^{n+1}_{\mathbb{C}}$, with $0 \in Y$, one can always construct a parametrization $\gamma(t) = (t^l, y_1(t), \ldots, y_n(t))$ with l minimal, i.e., l has no non-trivial divisor with the set of all exponents appearing in the power series expansions of $y_1(t), \ldots, y_n(t)$. We call a parametrization $\gamma(t)$ of Y, which is (up to permutation of the components of the parametrization) of the form $\gamma(t) = (t^l, y_1(t), \ldots, y_n(t))$, with $l \leq \min\{\mathrm{ord}(y_i(t)) \,:\, i = 1, \ldots, n\}$, a *Puiseux parametrization* of Y. Let us consider the analytic plane curves Y_i parametrized by the tuples $\gamma_i(t) = (t^l, y_i(t))$ for $i = 1, \ldots, n$. Then for each $i = 1, \ldots, n$, there exists a Puiseux parametrization $\eta_i(t) = (t^{l_i}, \widetilde{y}_i(t))$ of the curve Y_i so that $\gamma_i(t) = \eta_i(t^{\frac{l}{l_i}})$. Thus, l is the product of the multiplicities l_i of the Puiseux parametrizations of the curves Y_i. Let, further, $(n_i|\beta_{i,1}, \ldots, \beta_{i,g})$ be the Puiseux characteristic of Y_i. Notice, that by the minimality of l we have

$$\gcd\left(l, \frac{l}{l_i} \cdot \beta_{i,1}, \ldots, \frac{l}{l_i} \cdot \beta_{i,g} \,:\, i = 1, \ldots, n\right) = 1.$$

Let $M := \max\{\frac{l}{l_i} \cdot \beta_{i,1}, \ldots, \frac{l}{l_i} \cdot \beta_{i,g} \,:\, i = 1, \ldots, n\}$. We then define for the Puiseux parametrization $\gamma(t)$ its *characteristic part* by

$$\gamma^{\chi}(t) := \left(t^l, \sum_{i=1}^{M} a_{1,i} t^i, \ldots, \sum_{i=1}^{M} a_{n,i} t^i\right),$$

where each $y_j(t), j = 1, \ldots, n$ has its power series expansion equal to $\sum_{i \geq 1} a_{j,i} t^i$.

Acknowledgements. The author is grateful to Mariemi Alonso, Ana Bravo, Eduardo Casas-Alvero, Hiraku Kawanoue, Stefan Perlega and Orlando Villamayor for their valuable discussions during the research phase. Further, I would like to thank Christopher Chiu, Herwig Hauser, Mohab Safey El Din and Josef Schicho for sharing their ideas with me. I would like to express my sincere gratitude also to Alin Bostan and Matteo Gallet for their time and patience during the corrections of the version of this text. Finally, I am very thankful to Harald Grobner whose (not only financial) support helped a lot during the final work on this text.

The author was supported partially by Austrian Academy of Sciences, ÖAW, Doc Stipendium FA506081, and partially by START-Prize Y-966 of the Austrian Science Fund.

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