MUSTAFIN DEGENERATIONS: BETWEEN APPLIED AND ARITHMETIC GEOMETRY

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Degenerations have been at the forefront of algebraic geometry for the past decades. They provide a conceptual framework for the idea of transforming a geometric object of interest X into a new and simpler geometric object X_0 in a limiting process. Therefore degenerations enable to study X in terms of X_0 and among others open up the possibility to investigate X from a discrete and combinatorial view point.
A particularly interesting class of degenerations for which this combinatorial view point plays a central role are so-called *Mustafin varieties*. Mustafin varieties are degenerations of projective spaces induced by a finite set of points in a *Bruhat–Tits building*. Mustafin varieties were first introduced by Mustafin in [Mus78] in order to generalise Mumford's seminal work on the uniformisation of curves [Mum72]. Since then, they have attracted a lot of interest. In particular, they play an important role in Faltings' work on Shimura varieties [Fal05] and the study of Keel and Tevelev of Chow quotients of Grassmannians in [KT06]. In recent years, especially the interesting feature that their reductions may be described in terms of the combinatorics of *convex hulls* in Bruhat–Tits buildings has been studied. A conceptual combinatorial framework for this perspective was developed by Cartwright, Häbich, Sturmfels and Werner in [CHSW11].
Mustafin varieties have since given rise to a wide array of applications, among others in Brill–Noether theory [HZ19]. In this talk, we will give an overview of these applications with a particular focus on the following two topics:

(1) Recent advances towards a p-adic Narasimhan–Seshadri theorem
(2) Families of multi-view varieties in computer vision

This talk is based on the author's joint work with Binglin Li in [HL20], with Annette Werner in [HW19] and the author's work in [Hah20a, Hah20b].

1. Basic notions and algebraic Set-up

In this section, we outline some of the basics of the theory of Mustafin varieties and fix our algebraic set-up.
Let K be a discretely valued field, $\mathcal{O}_K$ its ring of integers and k its residue field. We fix a uniformiser π. As an example take $K = \mathbb{C}((\pi))$ as the ring of formal Laurent series over $\mathbb{C}$ with discrete valuation $v(\sum_{n\geq l} a_n\pi^n) = l$ for $l \in \mathbb{Z}$ and $a_n \in \mathbb{C}$ with $a_l \neq 0$. Then $\mathcal{O}_K = \{\sum_{n\geq l} a_n\pi^n : l \in \mathbb{Z}_{\geq 0}\}$ and $k = \mathbb{C}$. Moreover, let V be vector space of dimension d over K. We define $\mathbb{P}(V) = \mathrm{ProjSym}(V^*)$ as paramatrising lines through V. We call free $\mathcal{O}_K-$modules $L \subset V$ of rank d *lattices* and define $\mathbb{P}(L) = \mathrm{ProjSym}(L^*)$, where $L^* = \mathrm{Hom}_{\mathcal{O}_K}(L, \mathcal{O}_K)$. Note, that we will mostly consider lattices up to homothety, i.e. $L \backsim L'$ if $L = c \cdot L'$ for some $c \in K^\times$. We denote the homothety class of L by $[L]$.

Definition 1.1 (Mustafin varieties)**.** Let $\Gamma = \{[L_1], \ldots, [L_n]\}$ be a set of lattice classes in V. Then $\mathbb{P}(L_1), \ldots, \mathbb{P}(L_n)$ are projective spaces over $\mathcal{O}_K$ whose generic fibers are canonically isomorphic to $\mathbb{P}(V) \simeq \mathbb{P}^{d-1}_K$. The open immersions $\mathbb{P}(V) \hookrightarrow \mathbb{P}(L_i)$ give rise to a map

$$f_\Gamma \colon \mathbb{P}(V) \longrightarrow \mathbb{P}(L_1) \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \mathbb{P}(L_n).$$

We denote the closure of the image endowed with the reduced scheme structure by $\mathcal{M}(\Gamma)$. We call $\mathcal{M}(\Gamma)$ the *associated Mustafin variety*. Its special fiber $\mathcal{M}(\Gamma)_k$ is a scheme over k.

While the generic fiber of such a scheme is isomorphic to $\mathbb{P}(V)$ the special fiber has many interesting properties.

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Mustafin varieties may be studied by means of computer algebra since one may choose global coordinates on them. We now outline how this is achieved. Consider the diagonal map

$$\Delta : \mathbb{P}(V) \longrightarrow \mathbb{P}(V)^n = \mathbb{P}(V) \times_K \cdots \times_K \mathbb{P}(V).$$

The image of Δ is the subvariety of $\mathbb{P}(V)^n$ cut out by the ideal generated by the 2×2 minors of a matrix $X = (x_{ij})_{\substack{i=1,\dots,d \\ j=1,\dots,n}}$ of unknowns, where the jth column corresponds to coordinates in the jth factor.
Start with an element $g \in \mathrm{GL}(V)$, it is represented by an invertible $n \times n$ matrix over K. It induces a dual map $g^t : V^* \to V^*$ and thus a morphism $g : \mathbb{P}(V) \to \mathbb{P}(V)$. For n elements $g_1, \dots, g_n \in \mathrm{GL}(V)$, the image of

$$\mathbb{P}(V) \xrightarrow{\Delta} \mathbb{P}(V)^n \xrightarrow{g_1 \times \cdots \times g_n} \mathbb{P}(V)^n$$

is the subvariety of $\mathbb{P}(V)^n$ cut out by the multihomogeneous prime ideal

$$I_2((g_1^{-1}, \dots, g_n^{-1})(X)) \subset K[X],$$

where $(g_1^{-1}, \dots, g_n^{-1})(X)$ is the matrix whose jth column is given by

$$g_j^{-1} \begin{pmatrix} x_{1j} \\ \vdots \\ x_{dj} \end{pmatrix}$$

and where I_2 is the ideal generated by the $2 \times 2-$minors

Consider a reference lattice $L = \mathcal{O}_K e_1 + \cdots + \mathcal{O}_K e_d$. For any set $\Gamma = \{[L_1], \dots, [L_n]\}$ of lattice classes in $\mathfrak{B}_d^0$, we choose g_i, such that $g_i L_i = L$ for all i. The following diagram commutes:

$$\begin{tikzcd}
\mathbb{P}(V) \arrow[r, "(g_1 \times \cdots \times g_n) \circ \Delta"] \arrow[d] & \mathbb{P}(V)^n \arrow[d] \\
\prod_{\mathcal{O}_K} \mathbb{P}(L_i) \arrow[r, "(g_1 \times \cdots \times g_n)"] & \mathbb{P}(L)^n
\end{tikzcd}$$

It follows immediately that the Mustafin Variety $\mathcal{M}(\Gamma)$ is isomorphic to the subscheme of $\mathbb{P}(L)^n \cong (\mathbb{P}_R^{d-1})^n$ cut out by the multihomogeneous ideal $I_2\left((g_1^{-1}, \dots, g_n^{-1})(X)\right) \cap R[X]$ in $R[X]$.

As noted above, Mustafin varieties are related to Bruhat–Tits buildings and may studied in terms of convex hulls. We now make this precise.
Bruhat–Tits buildings are geometric realisations of simplicial complexes induced by a reductive algebraic group. Mustafin varieties are intimately connected to the Bruhat–Tits building $\mathfrak{B}_d$ associated to $\mathrm{GL}(V)$, where V is a $d-$dimensional vector space over K. Furthermore, we denote by $\mathfrak{B}_d^0$ the set of homothety classes of lattices in V. The set $\mathfrak{B}_d^0$ may be viewed as the set of integral points of $\mathfrak{B}_d$. We call two lattice classes $[L], [M]$ in $\mathfrak{B}_d^0$ adjacent if there exist representatives $L' \in [L]$ and $M' \in [M]$, such that $\pi M' \subset L' \subset M'$. Furthermore, we call a subset $\Gamma \subset \mathfrak{B}_d^0$ a simplex if its elements are pairwise adjacent. This defines a simplicial complex structure on $\mathfrak{B}_d$.

Definition 1.2. Let $\Gamma = \{[L_1], \dots, [L_n]\} \subset \mathfrak{B}_d^0$ a finite set of homothety classes of lattices. Then, we define its convex hull by

$$\mathrm{conv}(\Gamma) := \{[\pi^{m_1} L_1 \cap \cdots \cap \pi^{m_n} L_n] \mid m_1, \dots, m_n \in \mathbb{Z}\}.$$

The adjacency relation on $\mathfrak{B}_d^0$ naturally induces a simplicial complex structure on $\mathrm{conv}(\Gamma)$ by restriction.

As proved by Faltings in [Falo5], the convex hull of a finite subset $\Gamma \subset \mathfrak{B}_d^0$ is always a *finite* simplicial complex. In order to state the relation between Mustafin varieties and convex hulls in the Bruhat–Tits building $\mathfrak{B}_d$, we need the notion of a *dual reduction comlex* of a Mustafin variety.

Definition 1.3. Let $\Gamma = \{[L_1], \ldots, [L_n]\} \subset \mathfrak{B}_d^0$ a finite set of homothety classes of lattices and let $\mathcal{M}(\Gamma)$ be the associated Mustafin variety. The dual reduction complex of $\mathcal{M}(\Gamma)$ is the simplicial complex with one vertex for each irreducible component of $\mathcal{M}(\Gamma)_k$ and where a set of vertices forms a simplex if the respective irreducible components have non-empty intersection.

Then, we have following result, which should be viewed as a combinatorial classification of the special fibres of Mustafin varieties.

Theorem 1.4. *Let $\Gamma = \mathrm{conv}(\Gamma)$. Then, the convex hull $\mathrm{conv}(\Gamma)$ and the dual reduction complex of $\mathcal{M}(\Gamma)$ are isomorphic as simplicial complexes.*

2. Classification of Mustafin varieties

In this section, we give a brief description of the author's joint work with Binglin Li in [HL20]. The main result of this work is a complete classification of the irreducible components of $\mathcal{M}(\Gamma)_k$. We obtain this classification in the language of images of rational maps.
More precisely, let $g_1 \ldots, g_n \in \mathrm{Mat}(k, d \times d)$ and consider the associated rational map

$$\underline{g}\colon \mathbb{P}(W) \xrightarrow{(g_1,\ldots,g_n)} \mathbb{P}(W)^n.$$

We then denote $X(W, g_1, \ldots, g_n) := \overline{\mathrm{Im}(\underline{g})}$. Varieties of this form were studied extensively by Li in [Li18]. In particular, the Hilbert function and the multidegrees were computed.
In [HL20], we observed that for a finite subset $\Gamma \subset \mathfrak{B}_d^0$, one may associate to each $[L] \in \mathrm{conv}(\Gamma)$ a variety $X_{[L]}$ over k of the form $X_(W, g_1, \ldots, g_n)$. By a careful analysis of the multidegrees of $\mathcal{M}(\Gamma)_k$, the multidegrees of $X_{[L]}$ and the combinatorial classification in theorem 1.4, we prove the following theorem.

Theorem 2.1 ([HL20])**.** *Let $\Gamma = \{[L_1], \ldots, [L_n]\} \subset \mathfrak{B}_d^0$. For any irreducible component X of $\mathcal{M}(\Gamma)_k$, there exist $g_1, \ldots, g_n \in \mathrm{Mat}(k, d \times d)$, such that $X \cong X(W, g_1, \ldots, g_n)$.*
Each such component corresponds to a unique lattice class in $\mathrm{conv}(\Gamma)$.

Based on our proof of this result, we give a linear algebra algorithm to compute the special fibres of $\mathcal{M}(\Gamma)_k$.

3. A computer algebra programme towards a p−adic Narasimhan-Seshadri theorem

The classical Simpson correspondence in dimension one establishes a correspondence between semistable degree zero Higgs bundles on a Riemann surface X and representations of its topological fundamental group [Sim90, Sim92]. In recent years, the question whether a similar result holds in the p−adic setting has developed to a major research theme in arithmetic geometry [DW05, DW10, Fal05, AGT16, LZ17]. In [Fal05], Faltings proved an equivalence of categories between Higgs bundles on a p−adic curve and so-called *generalised representations* of its étale fundamental group. These generalised representations contain the continuous representations of the étale fundamental group as a full subcategory. The remaining challenge is to identify the subcategory of Higgs bundles, which is equivalent to the category of continuous representations. This is still an open problem. An approach for Higgs bundles with trivial Higgs field was introduced in [DW05, DW10] by Deninger and Werner and shown to be compatible with Faltings' functor in [Xu17] (see also [Wür19]). More precisely, it is shown that if a semistable vector bundle on a proper, smooth p-adic curve X has strongly semistable reduction of degree zero after pullback to a finite covering of the curve, it admits p-adic parallel transport and hence gives rise to a continuous representation of the étale fundamental group. One is of course tempted to speculate that the desired category is that of semistable Higgs bundles of degree zero. By the results of Deninger and Werner a positive answer for semistable degree zero bundles with trivial Higgs fields can be given if we prove a potentially strongly semistable reduction theorem for all such bundles. This involves – possibly after pull-back to a finite covering – finding for any semistable vector bundle E of degree zero on a smooth projective p−adic curve X, a model $\mathcal{E}$ on a model of the curve $\mathcal{X}$, such that the special fibre of $\mathcal{E}$ is strongly semistable on each irreducible component. This is expected to be a very difficult task.

In [HW19] and [Hah20a], Annette Werner and the author propose a constructive computer algebra programme towards $p-$adic Narasimhan–Seshadri theorem whose core objects are *Mustafin degenerations.* The idea is to first construct suitable models $\mathcal{X}$ of the underlying curve X. For this, let $\Gamma = \{[L_1], \ldots, [L_n]\} \subset \mathfrak{B}_d^0$ be a finite set of homothety classes. Moreover, we consider the map

$$f_\Gamma \colon \mathbb{P}(V) \longrightarrow \mathbb{P}(L_1) \times \cdots \times \mathbb{P}(L_n)$$

as in definition 1.1. Let $X \subset \mathbb{P}(V)$ be a subvariety. Then, we call $\overline{f_\Gamma(X)}$ endowed with the reduced scheme structure the associated *Mustafin degeneration of* X and denote it by $X(\Gamma)$.
In [HW19], we prove the following classification result on the structure of the special fibres $X(\Gamma)_k$.

Theorem 3.1. *Let $d = 3$ and $\mathbb{Q}_p$. Moreover, let $\Gamma = \{[L_1], \ldots, [L_n]\} \subset \mathfrak{B}_3^0$ be a "generic" subset of homothety classes of lattices and $X \subset \mathbb{P}(\mathbb{Q}_p^3)$ a smooth $p-$adic curve. Then, we have*

(1) the special fibre $X(\Gamma)_k$ consists of n irreducible components, each isomorphic to $\mathbb{P}_k^1$
(2) all irreducible components intersect in a single point.

Moreover, we determine the irreducible components of $\mathcal{M}(\Gamma)_k$, in which each component of $X(\Gamma)_k$ is contained.

Our proof proceeds by gluing all possible special fibres of Mustafin degenerations $X(\Gamma)_k$ to a scheme of finite type over a large polynomial ring. The classification is then obtained from a careful analysis of the generic fibre of this scheme. In [Hah20a], we observed that this technique is essentially an arithmo–geometric formulation of a parametrisation of Groebner bases. Motivated by this observation, we develop a novel theory of parametrised Groebner bases over unique factorisation rings in [Hah20a], which allows us to generalise theorem 3.1 to the case of $d = 4$ and $p \gg 0$. Since any smooth projective curve may be embedded into $\mathbb{P}(\mathbb{Q}_p^3)$, we in particular obtain a classification of a large family of Mustafin degenerations for all smooth and projective $p-$adic curves for $p \gg 0$. This classification allows us to prove the following theorem.

Theorem 3.2. *Let X be a smooth and projective $p-$adic. Then, for any $r \geq 0$, there exists an infinite family of semistable vector bundles of degree 0 on X that satisfy the criterion of Deninger an Werner. In particular, we construct an appropriate model for every element in these families.*

As a first application of this result, we were able to solve an open problem posed by Brenner in 2005 in [Bre05]. In [Bre05], Brenner proposed a counter example to the claim that any semistable degree zero vector bundle on a $p-$adic curve satisfies the criterion in Deninger and Werner. More precisely, Brenner considered the syzygy bundle $\mathrm{Syz}_m(x^2, y^2, z^2)(3)$ on the Fermat curve $X_m = V(x^m + y^m + z^m) \subset \mathbb{P}(\mathbb{Q}_p^3)$, defined by the following exact sequence

$$0 \to \mathrm{Syz}^{(m)}(x^2, y^2, z^2)(3) \to \mathcal{O}_{X_m}(1)^3 \xrightarrow{(x^2, y^2, z^2)} \mathcal{O}_{X_m}(3) \to 0$$

and proved that for infinitely many m, the obvious model over $\mathbb{Z}_p$ does not yield a semistable special fibre. Brenner posed the question, whether there exists another suitable model, which remained open until our work in [HW19], in which we proved the following result by employing explicit computations in Singular.

Theorem 3.3. *For any m, there exists a smooth finite covering of X_m, such that the pullback of $\mathrm{Syz}^{(m)}(x^2, y^2, z^2)(3)$ along this covering admits strongly semistable reduction and therefore satisfies the criterion of Deninger and Werner.*

We consider theorems 3.1 to 3.3 to be strong evidence for the existence of a $p-$adic Narasimhan–Seshadri theorem and hope to extend our approach to all semistable degree zero vector bundles on smooth projective $p-$adic curves.

4. Families of multi-view varieties in computer vision

In this final section, we outline how Mustafin varieties may be used to study families of multi-view varieties in computer vision.
In recent years, the application of algebraic geometry in the field of computer vision has attracted a

lot of interest, particularly in the context of so-called *multi-view geometry* [AST13, JKSW16, DKLP19, LM20, APT19, PST17]. The field of multi-view geometry is concerned with the situation of several cameras taking pictures of the same object simultaneously. For so-called *pinhole cameras* taking a picture is modelled as a projection, i.e. each camera is represented by a 3×4 matrix A_i of rank 3 over a field K. For a tuple $\underline{A} = (A_1, \ldots, A_n)$ of such matrices – where each matrix represents a camera – one therefore considers the rational map

$$f_{\underline{A}} \colon \mathbb{P}^3 \overset{(A_1,\ldots,A_n)}{\dashrightarrow} \left(\mathbb{P}^2\right)^n$$

which we call the *vision map*. Furthermore, we denote $M_{\underline{A}} = \overline{\mathrm{Im}(f_{\underline{A}})}$ and call $M_{\underline{A}}$ the *multi-view variety* associated to $\underline{A}$. Multi-view varieties are central objects in reconstruction problems in computer vision [HZ03].
In recent years, the study of families of multi-view varieties has gained a lot of interest, in particular due to surprising connections to Hilbert schemes [AST13, LM20]. In [AST13], the following beautiful theorem was proved.

Theorem 4.1 ([AST13]). *Let k be an algebraically closed field. The set $\{M_{\underline{A}} \mid \underline{A}\,\text{generic}\}$ lies dense in a distinguished irreducible component C_n of the Hilbert scheme $\mathcal{H}_n$ associated to the Hilbert function of the generic multi-view variety.*

In [Hah20b], we ask the following question:

Question. *What varieties do the points in $C_n \backslash \{M_{\underline{A}} \mid \underline{A}\,\text{generic}\}$ correspond to?*

We aim to approach this question with the techniques of Mustafin degenerations. The starting point of our work is the following theorem of Werner in [Wer01, Wer04].

Theorem 4.2 ([Wer01, Wer04]). *The Bruhat–Tits building $\mathfrak{B}_d$ may be compactified by homothety classes of quotient lattices, i.e. the integral points of the boundary are naturally identified with classes of free $\mathcal{O}_K-$modules of the form*

$$\left[{}^{L}\!/\!_{M}\right] \coloneqq \{{}^{c \cdot L}\!/\!_{c \cdot M} \mid c \in K^*, L \text{ a lattice in } V,\ M \text{ a free } \mathcal{O}_K - \text{submodule of } L\}.$$

In [Hah20b], we consider this compactification of $\mathfrak{B}_d$ by quotient lattices – that we denote by $\overline{\mathfrak{B}}_d$ – and generalise Mustafin varieties to subsets $\Gamma \subset \overline{\mathfrak{B}}_d^0$ of quotient lattices.

Definition 4.3. Let $\Gamma = \{\left[{}^{L_1}\!/\!_{M_1}\right], \ldots, \left[{}^{L_n}\!/\!_{M_n}\right]\}$ be a set of homothety classes of quotient lattices in V. Then, the generic fibre of $\mathbb{P}\left({}^{L_i}\!/\!_{M_i}\right)$ is canonically isomorphic to ${}^{L_i \otimes K}\!/\!_{M_i \otimes K}$. The natural rational maps to the generic fibers $\mathbb{P}(V) \dashrightarrow \mathbb{P}\left({}^{L_i}\!/\!_{M_i}\right)$ give rise to a map

$$f_\Gamma \colon \mathbb{P}(V) \longrightarrow \mathbb{P}\left({}^{L_1}\!/\!_{M_1}\right) \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \mathbb{P}\left({}^{L_n}\!/\!_{M_n}\right).$$

We denote the closure of the image endowed with the reduced scheme structure by $\mathcal{M}(\Gamma)$. We call $\mathcal{M}(\Gamma)$ the associated *generalised Mustafin variety*. Its special fiber $\mathcal{M}(\Gamma)_k$ is a scheme over k.

Finally, we show the following theorem.

Theorem 4.4. *Let k be an algebraically closed field and $\Gamma = \left\{\left[{}^{L_1}\!/\!_{M_1}\right], \ldots, \left[{}^{L_n}\!/\!_{M_n}\right]\right\}$ be a set of homothety classes of quotient lattices in V. Then, the special fibre of the associated generalised Mustafin variety $\mathcal{M}(\Gamma)_k$ corresponds to a point in C_n.*
It is equidimensional and connected. We obtain a partial classification of their irreducible components and derive a sharp upper bound on the number of irreducible components.

Our proof proceeds by a careful analysis of the multidegrees of the generic fibre of $\mathcal{M}(\Gamma)$ and by a choice of coordinates on the generalised Mustafin variety similar to the choice of coordinates on Mustafin varieties outlined above.
Finally, we note that in [Hah20b], we also show that with generalised Mustafin varieties one may reach a very singular point of C_n. Therefore, we hope to show in the future that all points in $C_n \backslash \{M_{\underline{A}} \mid \underline{A}\,\text{generic}\}$ arise as special fibres of Mustafin degenerations.

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