

# KP Solitons from Tropical Limits

Daniele Agostini, Claudia Fevola, Yelena Mandelshtam and Bernd Sturmfels

## Abstract

We study solutions to the Kadomtsev-Petviashvili equation whose underlying algebraic curves undergo tropical degenerations. Riemann's theta function becomes a finite exponential sum that is supported on a Delaunay polytope. We introduce the Hirota variety which parametrizes all tau functions arising from such a sum. We compute tau functions from points on the Sato Grassmannian that represent Riemann-Roch spaces and we present an algorithm that finds a soliton solution from a rational nodal curve.

## 1 Introduction

In the interplay between integrable systems and algebraic geometry [1, 3, 7, 15, 16, 18, 19], the study of complex algebraic curves is connected to the *Kadomtsev-Petviashvili (KP) equation*

$$\frac{\partial}{\partial x} (4p_t - 6pp_x - p_{xxx}) = 3p_{yy}. \quad (1)$$

Given a smooth curve with Riemann matrix  $B$ , one considers the Riemann theta function

$$\theta = \theta(\mathbf{z} | B) = \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp \left[ \frac{1}{2} \mathbf{c}^T B \mathbf{c} + \mathbf{c}^T \mathbf{z} \right]. \quad (2)$$

Krichever [16] constructed  $g$ -phase solutions to the KP equation as follows. Let  $\tau(x, y, t)$  be obtained from (2) by setting  $\mathbf{z} = \mathbf{u}x + \mathbf{v}y + \mathbf{w}t$ . Here,  $\mathbf{u} = (u_1, \dots, u_g)$ ,  $\mathbf{v} = (v_1, \dots, v_g)$ ,  $\mathbf{w} = (w_1, \dots, w_g)$  are coordinates on the weighted projective space  $\mathbb{WP}^{3g-1}$  that is defined by

$$\deg(u_i) = 1, \quad \deg(v_i) = 2, \quad \deg(w_i) = 3 \quad \text{for } i = 1, 2, \dots, g. \quad (3)$$

We require that the trivariate tau function  $\tau(x, y, t)$  satisfies Hirota's differential equation

$$\tau\tau_{xxxx} - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 + 4\tau_x\tau_t - 4\tau\tau_{xt} + 3\tau\tau_{yy} - 3\tau_y^2 = 0. \quad (4)$$

Under this hypothesis, the following function satisfies (1), and we call it the *KP solution*:

$$p(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \tau(x, y, t). \quad (5)$$

The *Dubrovin threefold* of [3] comprises all points  $(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in  $\mathbb{WP}^{3g-1}$  for which (4) holds.

In the present paper we study these objects when the smooth curve is defined over a non-archimedean field  $\mathbb{K}$  such as  $\mathbb{Q}(\epsilon)$  or the Puiseux series  $\mathbb{C}\{\{\epsilon\}\}$ . The Riemann matrix  $B_\epsilon$  depends analytically on the parameter  $\epsilon$ , and hence so do the tau function and KP solution. For  $\epsilon \rightarrow 0$ , the function  $p(x, y, t)$  becomes a soliton solution of (1). Our aim is to compute these degenerations and resulting KP solitons [1, 14] explicitly using computer algebra.

In the tropical limit, the infinite sum over  $\mathbb{Z}^g$  in the theta function becomes a finite sum

$$\theta_{\mathcal{C}}(\mathbf{z}) = a_1 \exp[\mathbf{c}_1^T \mathbf{z}] + a_2 \exp[\mathbf{c}_2^T \mathbf{z}] + \cdots + a_m \exp[\mathbf{c}_m^T \mathbf{z}], \quad (6)$$

where  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m\}$  is a certain subset of the integer lattice  $\mathbb{Z}^g$ . Each lattice point  $\mathbf{c}_i = (c_{i1}, \dots, c_{ig})$  specifies a linear form  $\mathbf{c}_i^T \mathbf{z} = \sum_{j=1}^g c_{ij} z_j$ , just like in (2). The coefficients  $\mathbf{a} = (a_1, a_2, \dots, a_m)$  are unknowns that serve as coordinates on the algebraic torus  $(\mathbb{K}^*)^m$ .

**Example 1** ( $g = 2$ ). Consider a genus two curve  $y^2 = f_i(x)$  where  $f_i(x)$  is a polynomial of degree six with coefficients in  $\mathbb{Q}(\epsilon)$ . Here are two instances corresponding to Figures 1 and 2:

$$\begin{aligned} f_1(x) &= (x-1)(x-2\epsilon)(x-3\epsilon^2)(x-4\epsilon^3)(x-5\epsilon^4)(x-6\epsilon^5), \\ f_2(x) &= (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon). \end{aligned} \quad (7)$$

Note that  $f_2$  is an example for [19, §7]. For any fixed  $\epsilon \in \mathbb{C}^*$ , we can compute the Dubrovin threefold in  $\mathbb{WP}^5$ , using [3, Theorem 3.7], and derive KP solutions from its points. The difficulty is to maintain  $\epsilon$  as a parameter and to understand what happens for  $\epsilon \rightarrow 0$ .

One configuration in  $\mathbb{Z}^2$  that arises here is the square  $\mathcal{C} = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ . In order for the associated theta function  $\theta_{\mathcal{C}} = a_{00} + a_{10} \exp[z_1] + a_{01} \exp[z_2] + a_{11} \exp[z_1 + z_2]$  to yield a KP solution, the following three polynomial identities are necessary and sufficient:

$$\begin{aligned} u_1^4 - 4u_1 w_1 + 3v_1^2 &= 0, & ((u_1+u_2)^4 - 4(u_1+u_2)(w_1+w_2) + 3(v_1+v_2)^2) a_{00} a_{11} \\ u_2^4 - 4u_2 w_2 + 3v_2^2 &= 0, & + ((u_1-u_2)^4 - 4(u_1-u_2)(w_1-w_2) + 3(v_1-v_2)^2) a_{01} a_{10} = 0. \end{aligned}$$

If these conditions hold then  $p(x, y, t)$  can be written as a (2, 4)-soliton by [14, §2.5].

This article is organized as follows. In Section 2 we review the derivation of tropical Riemann matrices. Theorem 3 characterizes degenerations of theta functions from algebraic curves over  $\mathbb{K}$ . Proposition 4 shows that  $\mathcal{C}$  is the vertex set of a Delaunay polytope in  $\mathbb{Z}^g$ . In Section 3 we study the Hirota variety  $\mathcal{H}_{\mathcal{C}}$ , which lives in  $(\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}$ . A point  $(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w}))$  lies on the Hirota variety if and only if (4) holds for (6). We saw  $\mathcal{H}_{\mathcal{C}}$  for  $g = 2$  and  $\mathcal{C} = \{0, 1\}^2$  in Example 1. Theorem 10 characterizes the Hirota variety of the  $g$ -simplex.

A key idea in this paper is to never compute a Riemann matrix or the theta function (2). Instead we follow the approach in [15, 18, 19] that rests on the Sato Grassmannian (Theorem 13). This setting is entirely algebraic and hence amenable to symbolic computation over  $\mathbb{K}$ . Section 4 explains the computation of tau functions from points on the Sato Grassmannian.

In Section 5 we start from an algebraic curve  $X$  over  $\mathbb{K}$ . Certain Riemann-Roch spaces on  $X$  are encoded as points on the Sato Grassmannian. Following [19], we present an algorithm and its `Maple` implementation for computing these points and the resulting tau functions, for  $X$  hyperelliptic. Proposition 23 addresses the case when  $X$  is reducible. This is followed up in Section 6, where we present Algorithm 27 for KP solitons from nodal rational curves.

## 2 Tropical Curves and Delaunay Polytopes

We work over a field  $\mathbb{K}$  of characteristic zero with a non-archimedean valuation. Let  $X$  be a *Mumford curve* of genus  $g$ , that is,  $X$  is a smooth curve over  $\mathbb{K}$  whose Berkovich analytification is a graph with  $g$  cycles. This metric graph is the tropicalization  $\text{Trop}(X)$  of a faithful embedding of  $X$ . In spite of the recent advances in [12], computing  $\text{Trop}(X)$  from  $X$  remains a nontrivial task. All our examples were derived with methods described in [4].

If the curve  $X$  is hyperelliptic, given by an equation  $y^2 = f(x)$ , then  $\text{Trop}(X)$  is a metric graph with a harmonic two-to-one map onto the phylogenetic tree encoding the roots of  $f(x)$ .

**Example 2** ( $g = 2$ ). Let  $f(x)$  be of degree six. The six roots determine a subtree with six leaves in the Berkovich line. The edge lengths are invariants of the semistable model [4] over the valuation ring of  $\mathbb{K}$ . There are two combinatorial types of trivalent trees with six leaves, the *caterpillar* and the *snowflake*. These are realized by the two polynomials in Example 1.

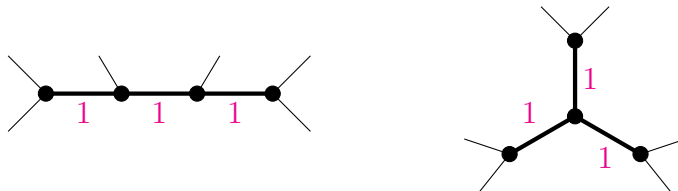


Figure 1: The metric trees defined by the polynomials  $f_1$  (left) and  $f_2$  (right) in (7)

Each metric tree with  $2g + 2$  leaves has a unique hyperelliptic covering by a metric graph of genus  $g$ . Its edge lengths are obtained from those of the tree by stretching or shrinking by a factor of 2. Figure 2 shows the graphs that give a two-to-one map to the trees in Figure 1.

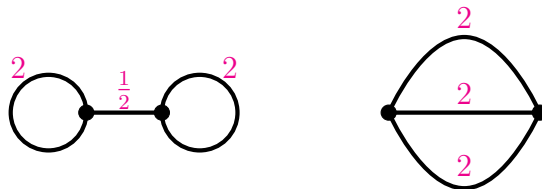


Figure 2: The metric graphs  $\text{Trop}(X)$  for the curves  $X$  in Example 1 and Figure 1

From the tropical curve  $\text{Trop}(X)$  we can read off the tropical Riemann matrix  $Q$ . This is a positive definite real symmetric  $g \times g$  matrix. Fix a basis of cycles in  $\text{Trop}(X)$  and write them as the  $g$  rows of a matrix  $\Lambda$  whose columns are labeled by the edges. Let  $\Delta$  be the diagonal matrix whose entries are the edge lengths of  $\text{Trop}(X)$ . Then we have  $Q = \Lambda \Delta \Lambda^T$ .

The genus two graphs in Figure 2 have three edges. Their Riemann matrices are

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

We now consider the degeneration of our curve  $X$  over  $\mathbb{K} = \mathbb{C}\{\{\epsilon\}\}$  to its tropical limit. The Riemann matrix can be written in the form  $B_\epsilon = \frac{1}{\epsilon}Q + R(\epsilon)$ , where  $R(\epsilon)$  is a symmetric  $g \times g$  matrix whose entries are complex analytic functions in  $\epsilon$  that converge as  $\epsilon \rightarrow 0$ .

Fix a point  $\mathbf{a} \in \mathbb{R}^g$ . Replacing  $\mathbf{z}$  by  $\mathbf{z} - \frac{1}{\epsilon}Q\mathbf{a}$  in the Riemann theta function (2), we obtain

$$\theta(\mathbf{z} - \frac{1}{\epsilon}Q\mathbf{a} | B_\epsilon) = \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp \left[ \frac{1}{2\epsilon} \mathbf{c}^T Q \mathbf{c} - \frac{1}{\epsilon} \mathbf{c}^T Q \mathbf{a} \right] \cdot \exp \left[ \frac{1}{2} \mathbf{c}^T R(\epsilon) \mathbf{c} + \mathbf{c}^T \mathbf{z} \right]. \quad (8)$$

This expression converges for  $\epsilon \rightarrow 0$  provided  $\mathbf{c}^T Q \mathbf{c} - 2\mathbf{c}^T Q \mathbf{a} \geq 0$  for all  $\mathbf{c} \in \mathbb{Z}^g$ . Equivalently,

$$\mathbf{a}^T Q \mathbf{a} \leq (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g. \quad (9)$$

This means that the distance from  $\mathbf{a}$  to the origin, in the metric given by  $Q$ , is at most the distance to any other lattice point  $\mathbf{c} \in \mathbb{Z}^g$ . In other words, (9) means that  $\mathbf{a}$  belongs to the *Voronoi cell* for  $Q$ . Under this hypothesis, we now consider the associated *Delaunay set*

$$\mathcal{D}_{\mathbf{a}, Q} = \left\{ \mathbf{c} \in \mathbb{Z}^g : \mathbf{a}^T Q \mathbf{a} = (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c}) \right\}. \quad (10)$$

This is the set of vertices of a polytope in the Delaunay subdivision of  $\mathbb{Z}^g$  induced by  $Q$ . If  $\mathbf{a}$  is a vertex of the Voronoi cell then the Delaunay polytope  $\text{conv}(\mathcal{D}_{\mathbf{a}, Q})$  is  $g$ -dimensional.

As in [2, §4], we observe that  $\exp \left[ \frac{1}{2\epsilon} \mathbf{c}^T Q \mathbf{c} - \frac{1}{\epsilon} \mathbf{c}^T Q \mathbf{a} \right]$  converges to 1 for  $\mathbf{c} \in \mathcal{D}_{\mathbf{a}, Q}$  and to 0 for  $\mathbf{c} \in \mathbb{Z}^g \setminus \mathcal{D}_{\mathbf{a}, Q}$ . We have thus derived the following generalization of [2, Theorem 4]:

**Theorem 3.** *Fix  $\mathbf{a}$  in the Voronoi cell of the tropical Riemann matrix  $Q$ . For  $\epsilon \rightarrow 0$ , the series (8) converges to a theta function (6) supported on the Delaunay set  $\mathcal{C} = \mathcal{D}_{\mathbf{a}, Q}$ , namely*

$$\theta_{\mathcal{C}}(\mathbf{x}) = \sum_{\mathbf{c} \in \mathcal{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T \mathbf{z}], \quad \text{where } a_{\mathbf{c}} = \exp \left[ \frac{1}{2} \mathbf{c}^T R(0) \mathbf{c} \right]. \quad (11)$$

The Delaunay polytope  $\text{conv}(\mathcal{C})$  can have any dimension between 0 and  $g$ , depending on the location of  $\mathbf{a}$  in the Voronoi cell (9). If  $\mathbf{a}$  lies in the interior then  $\mathcal{C} = \{\mathbf{0}\}$  is just the origin. We are most interested in the case when  $\mathbf{a}$  is a vertex of the Voronoi cell, and we now assume this to be the case. This ensures that  $\mathcal{C}$  is the vertex set of a  $g$ -dimensional Delaunay polytope. For fixed  $g$ , there is only a finite list of Delaunay polytopes, up to lattice isomorphism. Thanks to [8] and its references, that list is known for  $g \leq 6$ . However, not every Delaunay polytope arises from a curve  $X$  and its tropical Riemann matrix  $Q = \Lambda \Delta \Lambda^T$ . To illustrate these points, we present the list of all relevant Delaunay polytopes for  $g \leq 4$ .

**Proposition 4.** *The complete list of Delaunay polytopes  $\mathcal{C}$  arising from metric graphs for  $g \leq 4$  is as follows. For  $g = 2$  there are two types: triangle and square. For  $g = 3$  there are five types: tetrahedron, square-based pyramid, octahedron, triangular prism and cube. For  $g = 4$  there are 17 types. These  $\mathcal{C}$  have between 5 and 16 vertices. They are listed in Table 1.*

*Proof.* For any edge  $e$  of the graph, let  $\lambda_e \in \mathbb{Z}^g$  be the associated column of  $\Lambda$ . The Voronoi cell is a zonotope, obtained by summing line segments parallel to  $\lambda_e$  for all  $e$ . It has  $\mathbf{a}$  as a

vertex. After reorienting edges, in the corresponding expression of  $\mathbf{a}$  as a linear combination of the vectors  $\lambda_e$ , all coefficients are positive. This means that the Delaunay polytope equals

$$\text{conv}(\mathcal{C}) = \{ \mathbf{c} \in \mathbb{R}^g : 0 \leq \lambda_e^T \mathbf{c} \leq 1 \text{ for all edges } e \}. \quad (12)$$

Our task is to classify the polytopes (12) for all graphs of genus  $g$  and all their orientations. For  $g = 2$  this is easy, and for  $g = 3$  it was done in [2, Theorem 4]. We see from [2, Figure 2] that every Delaunay polytope can be realized by a curve over  $\mathbb{K}$ . For  $g = 4$  we started from the classification of 19 Delaunay polytopes in [9, Theorem 6.2], labeled  $1, 2, \dots, 16$  in [9, Table V] and labeled A,B,C in [9, Table VI]. Two types do not arise from graphs, namely the pyramid over the octahedron (#B) and the cross polytope (#C). The other 17 Delaunay polytopes all arise from graphs. They are listed in Table 1. The second row gives the number of vertices. The third row gives the number of facets. These two numbers uniquely identify the isomorphism type of  $\mathcal{C}$ . The last row indicates which graphs give rise to that Delaunay polytope. We refer to the 16 graphs of genus 4 by the labeling used in [5, Table 1]. Table 1 was constructed by a direct computation. It establishes the  $g = 4$  case in Proposition 4.  $\square$

polytopes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	A
vertices	5	6	7	7	8	8	8	9	9	9	10	10	10	12	12	16	6
facets	5	6	6	8	7	9	6	7	9	6	7	12	10	7	10	8	9
graphs	$\begin{smallmatrix} 1,2,3,4 \\ 5,7,10,13 \end{smallmatrix}$	$\begin{smallmatrix} 1,3,4 \\ 5,6,9 \end{smallmatrix}$	3,7,10	4	7	5	8,11,15	6	10	12	11	9	13	12	15	16	2

Table 1: The 17 Delaunay polytopes that arise from the 16 graphs of genus 4. Polytopes are labeled as in [9, Tables V and VI] and graphs are labeled as in [5, Table 1]. For instance, the complete bipartite graph  $K_{3,3}$  is #2, and it has two Delaunay polytopes, namely the simplex (#1) and the cyclic 4-polytope with 6 vertices (#A). The polytope #3 has 7 vertices and 6 facets. It is the pyramid over the triangular prism, and it arises from three graphs (#3,7,10).

### 3 Hirota Varieties

Starting from the theta function of the configuration  $\mathcal{C}$  in (6), we consider the tau function

$$\tau(x, y, t) = \theta_{\mathcal{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t) = \sum_{i=1}^m a_i \exp \left[ \left( \sum_{j=1}^g c_{ij} u_j \right) x + \left( \sum_{j=1}^g c_{ij} v_j \right) y + \left( \sum_{j=1}^g c_{ij} w_j \right) t \right].$$

The *Hirota variety*  $\mathcal{H}_{\mathcal{C}}$  consists of all points  $(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w}))$  in the parameter space  $(\mathbb{K}^*)^m \times \mathbb{W}\mathbb{P}^{3g-1}$  such that  $\tau(x, y, t)$  satisfies Hirota's differential equation (4). Thus  $\mathcal{H}_{\mathcal{C}}$  is an analogue to the Dubrovin threefold [3] for the classical Riemann theta function of a smooth curve.

We recall from [14, equation (2.25)] that (4) can be written via the Hirota differential operators as  $P(D_x, D_y, D_t)\tau \bullet \tau = 0$ , for the special polynomial  $P(x, y, t) = x^4 - 4xt + 3y^2$ . For any fixed index  $j$ , the equation  $P(u_j, v_j, w_j) = 0$  defines a curve in the weighted

projective plane  $\mathbb{WP}^2$ . More generally, for any two indices  $k, \ell$  in  $\{1, \dots, m\}$ , we consider the hypersurface in  $\mathbb{WP}^{3g-1}$  defined by

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := P((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w}).$$

This expression is unchanged if we switch  $k$  and  $\ell$ . The defining equations of the Hirota variety  $\mathcal{H}_\mathcal{C}$  can be obtained from the following lemma, which is proved by direct computation.

**Lemma 5.** *The result of applying the differential operator (4) to the function  $\tau(x, y, t)$  equals*

$$\sum_{1 \leq k < \ell \leq m} a_k a_\ell P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \exp\left[\left((\mathbf{c}_k + \mathbf{c}_\ell) \cdot \mathbf{u}\right)x + \left((\mathbf{c}_k + \mathbf{c}_\ell) \cdot \mathbf{v}\right)y + \left((\mathbf{c}_k + \mathbf{c}_\ell) \cdot \mathbf{w}\right)t\right]. \quad (13)$$

The polynomials defining the Hirota variety of  $\mathcal{C}$  are the coefficients we obtain by writing (13) as a linear combination of distinct exponentials. These correspond to points in the set

$$\mathcal{C}^{[2]} = \left\{ \mathbf{c}_k + \mathbf{c}_\ell : 1 \leq k < \ell \leq m \right\} \subset \mathbb{Z}^g.$$

A point  $\mathbf{d}$  in  $\mathcal{C}^{[2]}$  is *uniquely attained* if there exists precisely one index pair  $(k, \ell)$  such that  $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{d}$ . In that case,  $(k, \ell)$  is a *unique pair*, and this contributes the quartic  $P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w})$  to our defining equations. If  $\mathbf{d} \in \mathcal{C}^{[2]}$  is not uniquely attained, then the coefficient we seek is

$$\sum_{\substack{1 \leq k < \ell \leq m \\ \mathbf{c}_k + \mathbf{c}_\ell = \mathbf{d}}} P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) a_k a_\ell. \quad (14)$$

**Corollary 6.** *The Hirota variety  $\mathcal{H}_\mathcal{C}$  is defined by the quartics  $P_{k\ell}$  for all unique pairs  $(k, \ell)$  and by the polynomials (14) for all non-uniquely attained points  $\mathbf{d} \in \mathcal{C}^{[2]}$ . If all points in  $\mathcal{C}^{[2]}$  are uniquely attained then  $\mathcal{H}_\mathcal{C}$  is defined by the  $\binom{m}{2}$  quartics  $P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ , so its equations do not involve the coefficients  $a_1, \dots, a_m$ . This is the case when  $\mathcal{C}$  is the vertex set of a simplex.*

**Example 7** (The Square). Let  $g = 2$  and  $\mathcal{C} = \{0, 1\}^2$  as in Example 1. Here,  $\mathcal{C}^{[2]} = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)\}$ . The Hirota variety  $\mathcal{H}_\mathcal{C}$  is a complete intersection of codimension three in  $(\mathbb{K}^*)^4 \times \mathbb{WP}^5$ . There are four unique pairs  $(k, \ell)$  and these contribute the two quartics  $P_{13} = P_{24} = u_1^4 - 4u_1w_1 + 3v_1^2$  and  $P_{12} = P_{34} = u_2^4 - 4u_2w_2 + 3v_2^2$ . The point  $\mathbf{d} = (1, 1)$  is not uniquely attained in  $\mathcal{C}^{[2]}$ . The polynomial (14) contributed by this  $\mathbf{d}$  equals

$$P(u_1 + u_2, v_1 + v_2, w_1 + w_2) a_{00} a_{11} + P(u_1 - u_2, v_1 - v_2, w_1 - w_2) a_{01} a_{10}. \quad (15)$$

For any point in  $\mathcal{H}_\mathcal{C}$ , we can write  $\tau(x, y, t)$  as a  $(2, 4)$ -soliton, as shown in [14, § 2.5].

**Example 8** (The Cube). Let  $g = 3$  and consider the tropical degeneration of a smooth plane quartic to a rational quartic. By [2, Example 6], the associated theta function equals

$$\begin{aligned} \theta_\mathcal{C} = & a_{000} + a_{100} \exp[z_1] + a_{010} \exp[z_2] + a_{001} \exp[z_3] + a_{110} \exp[z_1 + z_2] \\ & + a_{101} \exp[z_1 + z_3] + a_{011} \exp[z_2 + z_3] + a_{111} \exp[z_1 + z_2 + z_3]. \end{aligned} \quad (16)$$

We compute the Hirota variety in  $(\mathbb{K}^*)^8 \times \mathbb{WP}^8$ . The set  $\mathcal{C}^{[2]}$  consists of 19 points. Twelve are uniquely attained, one for each edge of the cube. These give rise to the three familiar

quartics  $u_j^4 - 4u_j w_j + 3v_j^2$ , one for each edge direction  $\mathbf{c}_k - \mathbf{c}_\ell$ . Six points in  $\mathcal{C}^{[2]}$  are attained twice. They contribute equations like (15), one for each of the six facets of the cube. Finally, the point  $\mathbf{d} = (1, 1, 1)$  is attained four times. The polynomial (14) contributed by  $\mathbf{d}$  equals

$$\begin{aligned} & P(u_1 + u_2 + u_3, v_1 + v_2 + v_3, w_1 + w_2 + w_3) a_{000} a_{111} \\ & + P(u_1 + u_2 - u_3, v_1 + v_2 - v_3, w_1 + w_2 - w_3) a_{001} a_{110} \\ & + P(u_1 - u_2 + u_3, v_1 - v_2 + v_3, w_1 - w_2 + w_3) a_{010} a_{101} \\ & + P(-u_1 + u_2 + u_3, -v_1 + v_2 + v_3, -w_1 + w_2 + w_3) a_{100} a_{011}. \end{aligned} \quad (17)$$

We now restrict to the 9-dimensional component of  $\mathcal{H}_C$  that lies in  $\{a_{000} a_{110} a_{101} a_{011} = a_{001} a_{010} a_{100} a_{111}\}$ . Its image in  $\mathbb{W}\mathbb{P}^8$  has dimension 5, with fibers that are cones over  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . They are defined by seven equations arising from non-unique  $(k, \ell)$ . Six of these are binomials (15). Extending [14, §2.5], we identify  $\tau(x, y, t)$  with (3, 6)-solitons for

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (18)$$

By definition, a (3, 6)-soliton for the matrix  $A$  has the form

$$\tilde{\tau}(x, y, t) = \sum_I \prod_{\substack{i, j \in I \\ i < j}} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right], \quad (19)$$

where  $I$  runs over the eight bases 135, 136, 145, 146, 235, 236, 245, 246. To get from (16) to this form, we use the following parametric representation of the main component in  $\mathcal{H}_C$ :

$$\begin{aligned} u_1 &= \kappa_1 - \kappa_2, & v_1 &= \kappa_1^2 - \kappa_2^2, & w_1 &= \kappa_1^3 - \kappa_2^3, \\ u_2 &= \kappa_3 - \kappa_4, & v_2 &= \kappa_3^2 - \kappa_4^2, & w_2 &= \kappa_3^3 - \kappa_4^3, \\ u_3 &= \kappa_5 - \kappa_6, & v_3 &= \kappa_5^2 - \kappa_6^2, & w_3 &= \kappa_5^3 - \kappa_6^3, \\ a_{111} &= (\kappa_3 - \kappa_5)(\kappa_1 - \kappa_5)(\kappa_1 - \kappa_3) \lambda_0 \lambda_1 \lambda_2 \lambda_3, & a_{011} &= (\kappa_3 - \kappa_5)(\kappa_2 - \kappa_5)(\kappa_2 - \kappa_3) \lambda_0 \lambda_2 \lambda_3, \\ a_{101} &= (\kappa_4 - \kappa_5)(\kappa_1 - \kappa_5)(\kappa_1 - \kappa_4) \lambda_0 \lambda_1 \lambda_3, & a_{001} &= (\kappa_4 - \kappa_5)(\kappa_2 - \kappa_5)(\kappa_2 - \kappa_4) \lambda_0 \lambda_3, \\ a_{110} &= (\kappa_3 - \kappa_6)(\kappa_1 - \kappa_6)(\kappa_1 - \kappa_3) \lambda_0 \lambda_1 \lambda_2, & a_{010} &= (\kappa_3 - \kappa_6)(\kappa_2 - \kappa_6)(\kappa_2 - \kappa_3) \lambda_0 \lambda_2, \\ a_{100} &= (\kappa_4 - \kappa_6)(\kappa_1 - \kappa_6)(\kappa_1 - \kappa_4) \lambda_0 \lambda_1, & a_{000} &= (\kappa_4 - \kappa_6)(\kappa_2 - \kappa_6)(\kappa_2 - \kappa_4) \lambda_0. \end{aligned}$$

If we multiply (19) by  $\exp[-(\kappa_2 + \kappa_4 + \kappa_6)x - (\kappa_2^2 + \kappa_4^2 + \kappa_6^2)y - (\kappa_2^3 + \kappa_4^3 + \kappa_6^3)t]$  then we obtain the desired function  $\theta_C(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t)$  for the above generic point on the Hirota variety. The extraneous exponential factor disappears after we pass from  $\tilde{\tau}(x, y, t)$  to  $\partial_x^2 \log(\tilde{\tau}(x, y, t))$ . Both versions of the (3, 6)-soliton satisfy (4) and they represent the same solution to the KP equation (1). An analogous construction works for the cube  $\mathcal{C} = \{0, 1\}^g$  in any dimension  $g$ .

We now consider the simplex  $\mathcal{C} = \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_g\}$ . This arises from plane quartics ( $g = 3$ ) that degenerate to four lines or to a conic plus two lines [2, Example 5]. The tau function is

$$\tau(x, y, t) = a_0 + a_1 \exp[u_1 x + v_1 y + w_1 t] + a_2 \exp[u_2 x + v_2 y + w_2 t] + \dots + a_g \exp[u_g x + v_g y + w_g t].$$

We know from Corollary 6 that the conditions imposed by Hirota's differential equation (4) do not depend on  $\mathbf{a}$  but only on  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ . We thus consider the Hirota variety  $\mathcal{H}_C$  in  $\mathbb{W}\mathbb{P}^{3g-1}$ .

**Lemma 9.** *The Hirota variety  $\mathcal{H}_C$  is the union of two irreducible components of dimension  $g$  in  $\mathbb{W}\mathbb{P}^{3g-1}$ . One of the two components has the following parametric representation:*

$$u_j \mapsto \kappa_j - \kappa_0, \quad v_j \mapsto \kappa_j^2 - \kappa_0^2, \quad w_j \mapsto \kappa_j^3 - \kappa_0^3 \quad \text{for } j = 1, 2, \dots, g. \quad (20)$$

*The other component is obtained from (20) by the sign change  $v_j \mapsto -v_j$  for  $j = 1, \dots, g$ .*

*Proof.* By Corollary 6, the variety  $\mathcal{H}_C$  is defined by the quartics  $P(u_i, v_i, w_i)$  and  $P(u_i - u_j, v_i - v_j, w_i - w_j)$ . The first  $g$  quartics imply  $u_j = \kappa_j - \kappa_{j+g}$ ,  $v_j = \kappa_j^2 - \kappa_{j+g}^2$ ,  $w_j = \kappa_j^3 - \kappa_{j+g}^3$  for  $j = 1, \dots, g$ . Under these substitutions, the remaining  $\binom{g}{2}$  quartics factor into products of expressions  $\kappa_i - \kappa_j$ . Analyzing all cases up to symmetry reveals the two components.  $\square$

Setting  $t = \kappa_0$  and  $\kappa_j = u_j + t$ , the parameterization (20) of  $\mathcal{H}_C$  can be written as follows:

$$u_j \mapsto u_j, \quad v_j \mapsto 2u_j t + u_j^2, \quad w_j \mapsto 3u_j t^2 + 3u_j^2 t + u_j^3 \quad \text{for } j = 1, 2, \dots, g. \quad (21)$$

**Theorem 10.** *The prime ideal of the Hirota variety in (20) is minimally generated by*

- (a) *the  $\binom{g}{2}$  cubics  $\underline{v_i u_j} - v_j u_i - u_i u_j (u_i - u_j)$  for  $1 \leq i < j \leq g$ ,*
- (b) *the  $g$  quartics  $\underline{4w_i u_i} - 3v_i^2 - u_i^4$  for  $i = 1, \dots, g$ ,*
- (c) *the  $g(g-1)$  quartics  $\underline{4w_j u_i} - 3v_i v_j + 3u_i(u_i - u_j)v_j - u_i u_j^3$  for  $i \neq j$ , and*
- (d) *the  $\binom{g}{2}$  quintics  $\underline{4w_i v_j} - 4w_j v_i + 3u_i v_j (v_j - v_i) + u_i v_j (u_j - u_i)(u_i - 2u_j) + u_i u_j^3 (u_i - u_j)$ .*

*These  $2g^2 - g$  ideal generators are a minimal Gröbner basis with the underlined leading terms.*

*Proof.* Consider the subalgebra of  $\mathbb{K}[t, u_1, \dots, u_g]$  generated by the  $3g$  polynomials in the parametrization (21). We sort terms by  $t$ -degree. We claim that this is a *Khovanskii basis*, or *canonical basis*, as defined in [13] or [22, Chapter 11]. The parametrization given by the leading monomials  $u_j \mapsto u_j$ ,  $v_j \mapsto 2u_j t$ ,  $w_j \mapsto 3u_j t^2$  defines a toric variety, namely the embedding of  $\mathbb{P}^1 \times \mathbb{P}^{g-1}$ , into  $\mathbb{P}^{3g-1}$  by the very ample line bundle  $\mathcal{O}(2, 1)$ . Its toric ideal is generated by the leading binomials  $v_i u_j - v_j u_i$ ,  $\underline{4w_i u_i} - 3v_i^2$ ,  $\underline{4w_j u_i} - 3v_i v_j$ ,  $w_i v_j - w_j v_i$  seen in (a)-(d). In fact, by [22, §14.A], these  $2g^2 - g$  quadrics form a square-free Gröbner basis with underlined leading monomials. Under the correspondence in [22, Theorem 8.3], this initial ideal corresponds to a unimodular triangulation of the associated polytope  $(2\Delta_1) \times \Delta_{g-1}$ .

One checks directly that the polynomials (a), (b), (c), (d) vanish for (21). Since only two indices  $i$  and  $j$  appear, by symmetry, it suffices to do this check for  $g = 2$ . Hence the generators of the toric ideal are the leading binomials of certain polynomials that vanish on the Hirota variety. By [13, Theorem 2.17] or [22, Corollary 11.5], this proves the Khovanskii basis property. Geometrically speaking, we have constructed a toric degeneration from the Hirota variety to a toric variety in  $\mathbb{W}\mathbb{P}^{3g-1}$ . Furthermore, using [13, Proposition 5.2] or [22, Corollary 11.6 (1)] we conclude that the polynomials in (a)-(d) are a Gröbner basis for the prime ideal of (21), where the term order is chosen to select the underlined leading terms.  $\square$

Using the methods described above, we can compute the Hirota variety  $\mathcal{H}_C$  for each of the known Delaunay polytopes  $\mathcal{C}$ , starting with those in Proposition 4. We did this above for the triangle, the square, the tetrahedron and the cube. Here is one more example.



**Example 11** (Triangular prism). Let  $g = 3$  and take  $\theta_{\mathcal{C}}$  to be the six-term theta function

$$a_{000} + a_{100} \exp[z_1] + a_{001} \exp[z_3] + a_{101} \exp[z_1 + z_3] + a_{011} \exp[z_2 + z_3] + a_{111} \exp[z_1 + z_2 + z_3].$$

The prism  $\mathcal{C}$  arises in the degeneration as in Theorem 3 from a smooth quartic to a cubic plus a line. This is the second diagram in Figures 1 and 2 in [3, page 11]. The Hirota variety is cut out by four quartics in  $u_i, v_i, w_i$ , one for each edge direction, plus three relations involving the  $a_{ijk}$ , one for each of the three quadrangle facets. The edges from the two triangle facets define a reducible variety of codimension 3. One irreducible component is given by

$$\langle u_1^4 + 3v_1^2 - 4u_1w_1, u_2^4 + 3v_2^2 - 4u_2w_2, u_1^2u_2 + u_1u_2^2 - u_2v_1 + u_1v_2 \rangle.$$

Together with the four other relations, this defines an irreducible variety of codimension 4 inside  $(\mathbb{K}^*)^6 \times \mathbb{W}\mathbb{P}^8$ . That irreducible Hirota variety has the parametric representation

$$\begin{aligned} u_1 &= \kappa_1 - \kappa_2, & v_1 &= \kappa_1^2 - \kappa_2^2, & w_1 &= \kappa_1^3 - \kappa_2^3, \\ u_2 &= \kappa_2 - \kappa_3, & v_2 &= \kappa_2^2 - \kappa_3^2, & w_2 &= \kappa_2^3 - \kappa_3^3, \\ u_3 &= \kappa_4 - \kappa_5, & v_3 &= \kappa_4^2 - \kappa_5^2, & w_3 &= \kappa_4^3 - \kappa_5^3, \\ a_{000} &= (\kappa_1 - \kappa_4)\lambda_0, & a_{100} &= (\kappa_2 - \kappa_4)\lambda_0\lambda_1, & a_{110} &= (\kappa_3 - \kappa_4)\lambda_0\lambda_1\lambda_2, \\ a_{001} &= (\kappa_1 - \kappa_5)\lambda_0\lambda_3, & a_{101} &= (\kappa_2 - \kappa_5)\lambda_0\lambda_1\lambda_3, & a_{111} &= (\kappa_3 - \kappa_5)\lambda_0\lambda_1\lambda_2\lambda_3. \end{aligned}$$

This allows us to write the  $\tau$ -function as a  $(2, 5)$ -soliton, with  $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ . The six bases of the matrix  $A$  correspond to the six terms in  $\theta_{\mathcal{C}}$ , in analogy to the cube (16).

## 4 The Sato Grassmannian

The Sato Grassmannian is a device for encoding all solutions to the KP equation. Recall that the classical Grassmannian  $\text{Gr}(k, n)$  parametrizes  $k$ -dimensional subspaces of  $\mathbb{K}^n$ . It is a projective variety in  $\mathbb{P}^{\binom{n}{k}-1}$ , cut out by quadratic relations known as *Plücker relations*. Following [17, Chapter 5], the Plücker coordinates  $p_I$  are indexed by  $k$ -element subsets  $I$  of  $\{1, 2, \dots, n\}$ . As is customary in Schubert calculus [17, §5.3], we identify these  $\binom{n}{k}$  subsets with partitions  $\lambda$  that fit into a  $k \times (n - k)$  rectangle. Such a partition  $\lambda$  is a sequence  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  of integers that satisfy  $n - k \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$ . The corresponding Plücker coordinate  $c_\lambda = p_I$  is the maximal minor of a  $k \times n$  matrix  $M$  of unknowns, as in [17, §5.1], where the columns are indexed by  $I = \{\lambda_k + 1, \lambda_{k-1} + 2, \dots, \lambda_2 + k - 1, \lambda_1 + k\}$ . With this notation, the Plücker relations for  $\text{Gr}(k, n)$  are quadrics in the unknowns  $c_\lambda$ .

**Example 12.** We revisit [17, Example 5.9] with Plücker coordinates indexed by partitions. The Grassmannian  $\text{Gr}(3, 6)$  is a 9-dimensional subvariety in  $\mathbb{P}^{19}$ . Its prime ideal is generated by 35 Plücker quadrics. These are found easily by the following two lines in Macaulay2 [10]:

$$\begin{aligned} \mathbb{R} &= \mathbb{Q}\mathbb{Q}[c, c1, c11, c111, c2, c21, c211, c22, c221, c222, c3, c31, c311, \\ &\quad c32, c321, c322, c33, c331, c332, c333]; \quad \mathbb{I} = \text{Grassmannian}(2, 5, \mathbb{R}) \end{aligned}$$

The output consists of 30 three-term relations, like  $\underline{c_{211}c_{22}} - c_{21}c_{221} + c_2c_{222}$  and five four-term relations, like  $\underline{c_{221}c_{31}} - c_{21}c_{321} + c_{11}c_{331} + c_{333}$ . These quadrics form a minimal Gröbner basis.

The *Sato Grassmannian* SGM is the zero set of the Plücker relations in the unknowns  $c_\lambda$ , where we now drop the constraint that  $\lambda$  fits into a  $k \times (n - k)$ -rectangle. Instead, we allow arbitrary partitions  $\lambda$ . What follows is the description of a minimal Gröbner basis for SGM.

Partitions are order ideals in the poset  $\mathbb{N}^2$ . The set of all order ideals, ordered by inclusion, is a distributive lattice, known as *Young's lattice*. Consider any two partitions  $\lambda$  and  $\mu$  that are incomparable in Young's lattice. They fit into a common  $k \times (n - k)$ -rectangle, for some  $k$  and  $n$ . There is a canonical Plücker relation for  $\text{Gr}(k, n)$  that has leading monomial  $c_\lambda c_\mu$ . It is known that these *straightening relations* form a minimal Gröbner basis for fixed  $k$  and  $n$ . This property persists as  $k$  and  $n - k$  increase, hence yielding a Gröbner basis for SGM.

The previous paragraph paraphrases the definition in [6, 20] of the Sato Grassmannian as an inverse limit of projective varieties. This comes from the diagram of maps  $\text{Gr}(k, n+1) \dashrightarrow \text{Gr}(k, n)$  and  $\text{Gr}(k+1, n+1) \dashrightarrow \text{Gr}(k, n)$ , where these rational maps are given by dropping the last index. This corresponds to *deletion* and *contraction* in matroid theory [17, Chapter 13]. One checks that the simultaneous inverse limit for  $k \rightarrow \infty$  and  $n - k \rightarrow \infty$  is well-defined. The straightening relations in our equational description above are those in [6, Example 4.1]. That they form a Gröbner basis is best seen using Khovanskii bases [13, Example 7.3].

We next present the parametric representation of SGM that is commonly used in KP theory. Let  $V = \mathbb{K}((z))$  be the field of formal Laurent series with coefficients in our ground field  $\mathbb{K}$ . Consider the natural projection map  $\pi: V \rightarrow \mathbb{K}[z^{-1}]$  onto the polynomial ring in  $z^{-1}$ . We regard  $V$  and  $\mathbb{K}[z^{-1}]$  as  $\mathbb{K}$ -vector spaces, with Laurent monomials  $z^i$  serving as bases. Points in the Sato Grassmannian SGM correspond to  $\mathbb{K}$ -subspaces  $U \subset V$  such that

$$\dim \text{Ker } \pi|_U = \dim \text{Coker } \pi|_U, \quad (22)$$

and this common dimension is finite. We can represent  $U \in \text{SGM}$  via a doubly infinite matrix as follows. For any basis  $(f_1, f_2, f_3, \dots)$  of  $U$ , the  $j$ th basis vector is a Laurent series,

$$f_j(z) = \sum_{i=-\infty}^{+\infty} \xi_{i,j} z^{i+1}.$$

Then  $U$  is the column span of the infinite matrix  $\xi = (\xi_{i,j})$  whose rows are indexed from top to bottom by  $\mathbb{Z}$  and whose columns are indexed from right to left by  $\mathbb{N}$ . The  $i$ -th row of  $\xi$  corresponds to the coefficients of  $z^{i+1}$ . Sato proved that a subspace  $U$  of  $V$  satisfies (22) if and only if there is a basis, called a *frame* of  $U$ , whose corresponding matrix has the shape

$$\xi = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \mathbf{1} & 0 & 0 & 0 & \cdots & 0 \\ \cdots & * & \mathbf{1} & 0 & 0 & \cdots & 0 \\ \cdots & * & * & \xi_{-l,l} & \xi_{-l,l-1} & \cdots & \xi_{-l,1} \\ \cdots & * & * & \xi_{-l+1,l} & \xi_{-l+1,l-1} & \cdots & \xi_{-l+1,1} \\ & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & * & * & \xi_{-1,l} & \xi_{-1,l-1} & \cdots & \xi_{-1,1} \\ \cdots & * & * & \xi_{0,l} & \xi_{0,l-1} & \cdots & \xi_{0,1} \\ \cdots & * & * & \xi_{1,l} & \xi_{1,l-1} & \cdots & \xi_{1,1} \\ & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix}. \quad (23)$$

This matrix is infinite vertically, infinite on the left and, most importantly, it is eventually lower triangular with 1 on the diagonal, at the  $(-n, n)$  positions. The space  $U$  is described by the positive integer  $\ell$  and the submatrix with  $\ell$  linearly independent columns whose upper left entry is  $\xi_{-\ell, \ell}$ . This description implies that a subspace  $U$  of  $V$  satisfies (22) if and only if

$$\text{there exists } \ell \in \mathbb{N} \text{ such that } \dim U \cap V_n = n + 1 \quad \text{for all } n \geq \ell, \quad (24)$$

where  $V_n = z^{-n} \mathbb{K}[[z]]$  denotes the space of Laurent series with a pole of order at most  $n$ .

The Plücker coordinates on SGM are computed as minors  $\xi_\lambda$  of the matrix  $\xi$ . Think of a partition  $\lambda$  as a weakly decreasing sequence of nonnegative integers that are eventually zero. Setting  $m_i = \lambda_i - i$  for  $i \in \mathbb{N}$ , we obtain the associated *Maya diagram*  $(m_1, m_2, m_3, \dots)$ . This is a vector of strictly decreasing integers  $m_1 > m_2 > \dots$  such that  $m_i = -i$  for large enough  $i$ . Partitions and Maya diagrams are in natural bijection. Given any partition  $\lambda$ , we consider the matrix  $(\xi_{m_i, j})_{i, j \geq 1}$  whose row indices  $m_1, m_2, m_3, \dots$  are the entries in the Maya diagram of  $\lambda$ . Thanks to the shape of the matrix  $\xi$ , it makes sense to take the determinant

$$\xi_\lambda := \det(\xi_{m_i, j}). \quad (25)$$

This Plücker coordinate is a scalar in  $\mathbb{K}$  that can be computed as a maximal minor of the finite matrix to the lower right of  $\xi_{-\ell, \ell}$  in (23). We summarize our discussion as a theorem.

**Theorem 13.** *The Sato Grassmannian SGM is the inverse limit of the classical Grassmannians  $\text{Gr}(k, n) \subset \mathbb{P}^{\binom{n}{k}-1}$  as both  $k$  and  $n - k$  tend to infinity. A parametrization of SGM is given by the matrix minors  $c_\lambda = \xi_\lambda$  in (25), where  $\lambda$  runs over all partitions. The equations of SGM are the quadratic Plücker relations, shown in [6, Example 4.1] and in Example 12.*

We now connect the Grassmannians above to our study of solutions to the KP equation. Fix positive integers  $k < n$  and a vector of parameters  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$ . For each  $k$ -element index set  $I \in \binom{[n]}{k}$  we introduce an unknown  $p_I$  that serves as a Plücker coordinate. Our ansatz for solving (4) is now the following linear combination of exponential functions:

$$\tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i, j \in I \\ i < j}} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right]. \quad (26)$$

**Proposition 14.** *The function  $\tau$  is a solution to Hirota's equation (4) if and only if the point  $p = (p_I)_{I \in \binom{[n]}{k}}$  lies in the Grassmannian  $\text{Gr}(k, n)$ , i.e. there is a  $k \times n$  matrix  $A = (a_{ij})$  such that, for all  $I \in \binom{[n]}{k}$ , the coefficient  $p_I$  is the  $k \times k$ -minor of  $A$  with column indices  $I$ .*

*Proof.* This follows from [14, Theorem 1.3]. □

We define a  $(k, n)$ -soliton to be any function  $\tau(x, y, t)$  where  $\kappa \in \mathbb{R}^n$  and  $p \in \text{Gr}(k, n)$ . Even the case  $k = 1$  is interesting. Writing  $A = (a_1 \ a_2 \ \dots \ a_n)$ , the  $(1, n)$ -soliton equals

$$\tau(x, y, t) = \sum_{i=1}^n a_i \exp \left[ x \cdot \kappa_i + y \cdot \kappa_i^2 + t \cdot \kappa_i^3 \right].$$

If we now set  $n = g + 1$  and we divide the sum above by its first exponential term then we obtain the tau function that was associated with the  $g$ -simplex in Lemma 9. Hence the KP solutions that arise when the Delaunay polytope is a simplex are precisely the  $(1, n)$ -solitons.

We next derive the Sato representation in [14, Definition 1.3], that is, we express  $\tau(x, y, t)$  as a linear combination of Schur polynomials. Let  $\lambda$  be a partition with at most three parts, written  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ . Following [14, §1.2.2], the associated Schur polynomial  $\sigma_\lambda(x, y, t)$  can be defined as follows. We first introduce the *elementary Schur polynomial*  $\varphi_j(x, y, t)$  by the series  $\exp[x\lambda + y\lambda^2 + t\lambda^3] = \sum_{j=0}^{\infty} \varphi_j(x, y, t)\lambda^j$ . The *Schur polynomial*  $\sigma_\lambda$  for the partition  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is the determinant of the Jacobi-Trudi matrix of size  $3 \times 3$ :

$$\sigma_\lambda(x, y, t) = \det(\varphi_{\lambda_i - i + j}(x, y, t))_{1 \leq i, j \leq 3}.$$

To be completely explicit, we list Schur polynomials for partitions  $\lambda$  with  $\lambda_1 + \lambda_2 + \lambda_3 \leq 4$ :

$$\begin{aligned} \sigma_\emptyset &= 1, & \sigma_1 &= x, & \sigma_{11} &= \frac{1}{2}x^2 - y, & \sigma_2 &= \frac{1}{2}x^2 + y, & \sigma_{111} &= \frac{1}{6}x^3 - xy + t, & \sigma_3 &= \frac{1}{6}x^3 + xy + t, \\ \sigma_{21} &= \frac{1}{3}x^3 - t, & \sigma_{211} &= \frac{1}{8}x^4 - \frac{1}{2}x^2y - \frac{1}{2}y^2, & \sigma_{22} &= \frac{1}{12}x^4 - tx + y^2, & \sigma_{31} &= \frac{1}{8}x^4 + \frac{1}{2}x^2y - \frac{1}{2}y^2, \dots \end{aligned}$$

For a partition  $\lambda$  as above, we set  $\lambda_4 = \dots = \lambda_k = 0$ . For  $I = \{i_1 < i_2 < \dots < i_k\}$  we set

$$\Delta_\lambda(\kappa_i, i \in I) := \det \begin{pmatrix} \kappa_{i_1}^{\lambda_1 + k - 1} & \kappa_{i_2}^{\lambda_1 + k - 1} & \dots & \kappa_{i_k}^{\lambda_1 + k - 1} \\ \kappa_{i_1}^{\lambda_2 + k - 2} & \kappa_{i_2}^{\lambda_2 + k - 2} & \dots & \kappa_{i_k}^{\lambda_2 + k - 2} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{i_1}^{\lambda_k} & \kappa_{i_2}^{\lambda_k} & \dots & \kappa_{i_k}^{\lambda_k} \end{pmatrix}.$$

The empty partition gives the Vandermonde determinant  $\Delta_\emptyset(\kappa_i, i \in I) = \prod_{\substack{i, j \in I \\ i < j}} (\kappa_j - \kappa_i)$ .

**Lemma 15.** *The exponential function indexed by  $I$  in the formula (26) has the expansion*

$$\exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right] = \Delta_\emptyset(\kappa_i, i \in I)^{-1} \cdot \sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} \Delta_\lambda(\kappa_i, i \in I) \cdot \sigma_\lambda(x, y, t).$$

*Proof.* The unknowns  $x, y, t$  play the role of power sum symmetric functions in  $r_1, r_2, \dots$ :

$$x = r_1 + r_2 + r_3 = p_1(r), \quad y = \frac{1}{2}(r_1^2 + r_2^2 + r_3^2) = \frac{1}{2}p_2(r), \quad t = \frac{1}{3}(r_1^3 + r_2^3 + r_3^3) = \frac{1}{3}p_3(r).$$

It suffices to prove the statement after this substitution. By [14, Remark 1.5], we have  $\sigma_\lambda(x, y, t) = s_\lambda(r_1, r_2, r_3)$ , where  $s_\lambda$  is the usual Schur function as a symmetric polynomial, which satisfies  $\Delta_\lambda(\kappa_i, i \in I) = s_\lambda(\kappa_i, i \in I) \cdot \Delta_\emptyset(\kappa_i, i \in I)$ . Our identity can be rewritten as

$$\exp \left[ p_1(w) \cdot p_1(\kappa) + \frac{1}{2}p_2(w) \cdot p_2(\kappa) + \frac{1}{3}p_3(w) \cdot p_3(\kappa) \right] = \sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} s_\lambda(\kappa_i, i \in I) \cdot s_\lambda(r_1, r_2, r_3).$$

This is precisely the classical Cauchy identity, as stated in [21, page 386].  $\square$

By substituting the formula in Lemma 15 into the right hand side of (26), we obtain :

**Proposition 16.** *The  $(k, n)$ -soliton has the following expansion into Schur polynomials*

$$\tau(x, y, t) = \sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} c_\lambda \cdot \sigma_\lambda(x, y, t), \quad \text{where } c_\lambda = \sum_{I \in \binom{[n]}{k}} p_I \cdot \Delta_\lambda(\kappa_i, i \in I). \quad (27)$$

For any point  $\xi$  in the Sato Grassmannian SGM we now define a tau function as follows:

$$\tau_\xi(x, y, t) = \sum_{\lambda} \xi_\lambda \sigma_\lambda(x, y, t). \quad (28)$$

The sum is over all possible partitions. We can now state the main result of Sato's theory.

**Theorem 17** (Sato). *For any  $\xi \in \text{SGM}$ , the tau function  $\tau_\xi$  satisfies Hirota's equation (4).*

Actually, Sato's theorem is much more general. From a frame  $\xi$  as in (23), we can define

$$\tau(t_1, t_2, t_3, t_4, \dots) = \sum_{\lambda} \xi_\lambda \sigma_\lambda(t_1, t_2, t_3, t_4, \dots)$$

The sum is over all partitions. This function in infinitely many variables is a solution to the *KP hierarchy*, which is an infinite set of differential equations which generalize the KP equation. Moreover, every solution to the KP hierarchy arises from the Sato Grassmannian in this way. The tau functions that we consider here arise from the general case by setting

$$t_1 = x, \quad t_2 = y, \quad t_3 = t, \quad t_4 = t_5 = \dots = 0.$$

We refer to [14, Theorem 1.3] for a first introduction and numerous references. We may also start with an ansatz  $\tau(x, y, t) = \sum_{\lambda} c_\lambda \sigma_\lambda(x, y, t)$ , and examine the quadratic equations in the unknowns  $c_\lambda$  that are imposed by (4). This leads to polynomials that vanish on SGM.

**Remark 18.** We can view Proposition 14 as a special case of Theorem 17, given that the Sato Grassmannian contains all classical Grassmannians  $\text{Gr}(k, n)$ . Here is an explicit description. We fix distinct scalars  $\kappa_1, \dots, \kappa_n$  in  $\mathbb{K}^*$ . Points in  $\text{Gr}(k, n)$  are represented by matrices  $A$  in  $\mathbb{K}^{k \times n}$ . Following [15, §3.1] and [19, §2.2], we turn  $A$  into an infinite matrix  $\xi$  as in (23). Let  $\Lambda(\kappa)$  denote the  $\infty \times n$  matrix whose rows are  $(\kappa_1^\ell, \kappa_2^\ell, \dots, \kappa_n^\ell)$  for  $\ell = 0, 1, 2, \dots$ . We define  $A(\kappa) := \Lambda(\kappa) \cdot A^T$ . This is the  $\infty \times k$  matrix whose  $j$ th column is given by the coefficients of

$$\sum_{i=1}^n \frac{a_{ji}}{1 - \kappa_i z} = \sum_{\ell=0}^{\infty} \sum_{i=1}^n \kappa_i^\ell a_{ji} \cdot z^\ell.$$

This verifies [19, Theorem 3.2]. Indeed, the double infinite matrix representing  $\tau$  equals

$$\xi = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & A(\kappa) \end{bmatrix},$$

where  $\mathbf{0}$  and  $\mathbf{1}$  are infinite zero and identity matrices. In particular, the first nonzero row of  $A(\kappa)$  is at the row  $-k$  of  $\xi$ . The corresponding basis  $(f_1, f_2, f_3, \dots)$  of the space  $U$  is given by

$$f_j = \frac{1}{z^{k-1}} \sum_{i=1}^n \frac{a_{ji}}{1 - \kappa_i z}, \quad \text{for } j = 1, \dots, k, \quad f_j = \frac{1}{z^{j-1}}, \quad \text{for } j \geq k + 1. \quad (29)$$

The Plücker coordinates  $c_\lambda$  indexed by partitions with at most three parts are certain minors of  $A(\kappa)$ , and these are expressed in terms of maximal minors of  $A$  by the formula in (27).

## 5 Tau Functions from Algebraic Curves

Let  $X$  be a smooth projective curve of genus  $g$  defined over a field  $\mathbb{K}$  of characteristic zero. In this section we show how certain Riemann-Roch spaces on  $X$  define points in the Sato Grassmannian SGM. Using Theorem 17, we obtain KP solutions by choosing appropriate bases of these spaces. The relevant theory is known since the 1980s; see [14, 15, 20]. We begin with the exposition in [19, §4]. Our aim is to develop tools to carry this out in practice.

Fix a divisor  $D$  of degree  $g - 1$  on  $X$  and a distinguished point  $p \in X$ , both defined over  $\mathbb{K}$ . For any integer  $n \in \mathbb{N}$ , we consider the Riemann-Roch space  $H^0(X, D + np)$ . For  $m < n$  there is an inclusion  $H^0(X, D + mp) \subseteq H^0(X, D + np)$ . As  $n$  increases, we obtain a space  $H^0(X, D + \infty p)$  of rational functions on the curve  $X$  whose pole order at  $p$  is unconstrained.

Let  $z$  denote a local coordinate on  $X$  at  $p$ . Each element in  $H^0(X, D + \infty p)$  has a unique Laurent series expansion in  $z$  and hence determines an element in  $V = \mathbb{K}((z))$ . Let  $m = \text{ord}_p(D)$  be the multiplicity of  $p$  in  $D$ . Multiplication by  $z^{m+1}$  defines the  $\mathbb{K}$ -linear map:

$$\iota : H^0(X, D + \infty p) \rightarrow V, \quad s = \sum_{n \in \mathbb{Z}} s_n z^n \mapsto \sum_{n \in \mathbb{Z}} s_n z^{n+m+1}.$$

**Proposition 19** ([19, Theorem 4.1]). *The space  $U = \iota(H^0(X, D + \infty p)) \subset V$  lies in SGM.*

*Proof.* The map  $\iota$  is injective because a rational function on an irreducible curve  $X$  is uniquely determined by its Laurent series. Setting  $V_n = z^{-n} \mathbb{K}[[z]] \subset V$  as in Section 4, we have

$$\dim U \cap V_n = h^0(X, D + (n+1)p) = n + 1 + h^1(X, D + (n+1)p). \quad (30)$$

The second equality is the Riemann-Roch Theorem, with  $\deg(D) = g - 1$ . Hence (24) holds provided  $h^1(X, D + (n+1)p) = 0$ . This happens for  $n \geq g - 1$ , by degree considerations.  $\square$

Following [19], we examine the case  $g = 2$ . A smooth curve of genus two is hyperelliptic:

$$X = \{y^2 = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_6)\}.$$

Here  $\lambda_1, \lambda_2, \dots, \lambda_6 \in \mathbb{K}$  are pairwise distinct. Let  $p$  be one of the two preimages of the point at infinity under the double cover  $X \rightarrow \mathbb{P}^1$ . Using the local coordinate  $z = \frac{1}{x}$  at  $p$ , we write

$$y = \pm \sqrt{(x - \lambda_1) \cdots (x - \lambda_6)} = \pm \frac{1}{z^3} \cdot \sum_{n=0}^{+\infty} \alpha_n z^n,$$

where  $\alpha_0 = 1$  and the  $\alpha_i$  are polynomials in  $\lambda_1, \dots, \lambda_6$ . We consider three kinds of divisors:

$$D_0 = p, \quad D_1 = p_1 \quad \text{and} \quad D_2 = p_1 + p_2 - p,$$

where  $p_1 = (c_1, y_1)$ ,  $p_2 = (c_2, y_2)$  are general points on  $X$ . For  $m \geq 3$ , consider the functions

$$\begin{aligned} g_m(x) &= \sum_{j=0}^m \alpha_j x^{m-j}, \\ f_m(x, y) &= \frac{1}{2} (x^{m-3} y + g_m(x)), \\ h_j(x, y) &= \frac{f_3(x, y) - f_3(c_j, -y_j)}{x - c_j} = \frac{y + g_3(x) - (-y_j + g_3(c_j))}{2(x - c_j)} \quad \text{for } j = 1, 2. \end{aligned}$$

These rational functions are series in  $z$  with coefficients that are polynomials in  $\lambda_1, \dots, \lambda_6$ . We write  $U_i$  for the image of the Riemann-Roch space  $H^0(X, D_i + \infty p)$  under the inclusion  $\iota$ .

**Lemma 20** ([19, Lemma 5.2]). *The set  $\{1, f_3, f_4, f_5, \dots\}$  is a basis of  $U_0$ , the set  $\{1, f_3, f_4, f_5, \dots\} \cup \{h_1\}$  is a basis of  $U_1$ , and  $\{1, f_3, f_4, f_5, \dots\} \cup \{h_1, h_2\}$  is a basis of  $U_2$ .*

This lemma furnishes us with an explicit basis for the  $\mathbb{K}$ -vector space  $U$  in Proposition 19. This basis is a frame in the sense of Sato theory. It gives us the matrix  $\xi$  in (23), from which we compute the Plücker coordinates (25) and the tau function (28). This process is a symbolic computation over the ground field  $\mathbb{K}$ . No numerics are needed. For general curves of genus  $g \geq 3$ , the same is possible, but it requires computing a basis for  $U$ , e.g. using [11].

Our approach differs greatly from the computation of KP solutions from the curve  $X$  via theta functions as in [3, 7, 16]. That would require the computation of the Riemann matrix of  $X$ , which cannot be done over  $\mathbb{K}$ . This is why we adopted the SGM approach in [15, 19].

We implemented the method in `Maple` for  $D_0 = p$  on hyperelliptic curves over  $\mathbb{K} = \mathbb{Q}(\epsilon)$ . If  $\lambda$  is a partition with  $n$  parts, then the Plücker coordinate  $\xi_\lambda$  is the minor given by the  $n$  right-most columns of  $\xi$  and the rows given by the first  $n$  parts in the Maya diagram of  $\lambda$ . Since the tau function (28) is an infinite sum over all partitions, our code does not provide an exact solution to the Hirota equation (4). Instead, it computes the truncated tau function

$$\tau[n] := \sum_{i=1}^n \sum_{\lambda \vdash i} \xi_\lambda \sigma_\lambda(x, y, t), \quad (31)$$

where  $n$  is the order of precision. In our experiments we evaluated (31) up to  $n = 12$  on a range of hyperelliptic curves of genus  $g = 2, 3, 4$ . The first non-zero  $\tau[n]$  is  $\tau[g] = \sigma_{(g)}(x, y, t)$  [19, Proposition 6.3]. When plugging (31) into the left hand side of (4), we get an expression in  $x, y, t$  whose terms of low order vanish. The following facts were observed for this expression. For  $n > g + 2$ , the term of lowest degree has degree  $n + g - 3$ , and the monomial that appears in that lowest degree  $n + g - 3 = 1, 2, 3, \dots$  is  $x, y, t, xt, yt, t^2, xt^2, yt^2, t^3, xt^3, yt^3, t^4, \dots$ .

We use our `Maple` code to study  $(k, n)$ -solitons arising from the degenerations in [19]. Namely, we explore the limit for  $\epsilon \rightarrow 0$  for hyperelliptic curves of genus  $g = n - 1$  given by

$$y^2 = (x - \kappa_1)(x - \kappa_1 - \epsilon) \cdots (x - \kappa_n)(x - \kappa_n - \epsilon). \quad (32)$$

Set  $h(z) = (1 - \kappa_1 z) \cdots (1 - \kappa_n z)$ . For  $\epsilon \rightarrow 0$  the frame found in Lemma 20 degenerates to

$$U = \{1, z^{-n}h(z), z^{-(n+1)}h(z), z^{-(n+2)}h(z), z^{-(n+3)}h(z), \dots\}. \quad (33)$$

Observe that  $z^{-n}h(z)$  gets expanded to

$$z^{-n} + z^{-(n-1)} \left( - \sum_{i=1}^n \kappa_i \right) + z^{-(n-2)} \left( \sum_{1 \leq i < j \leq n} \kappa_i \kappa_j \right) + z^{-(n-3)} \left( \sum_{1 \leq i < j < l \leq n} \kappa_i \kappa_j \kappa_l \right) + O(z^{-(n-4)}).$$

Following [19, §7], one multiplies all elements in  $U$  by  $h(z)^{-1}$  in order to obtain a soliton solution. By [19, Theorem 3.2], we obtain a  $(1, n)$ -soliton solution given by the matrix

$$A = \left( \left( \prod_{i \neq 1} (\kappa_1 - \kappa_i) \right)^{-1} \quad \left( \prod_{i \neq 2} (\kappa_2 - \kappa_i) \right)^{-1} \quad \cdots \quad \left( \prod_{i \neq n} (\kappa_n - \kappa_i) \right)^{-1} \right).$$





separates the nodes of  $X$ . The divisor  $E$  lifts to  $\tilde{X}$ , and  $H^0(X, E)$  is a subspace of  $H^0(\tilde{X}, E)$ . It consists of rational functions which coincide on the points of  $\tilde{X}$  that map to the nodes of  $X$ . If  $X = X_0 \cup \dots \cup X_r$  is reducible, then  $H^0(X, E)$  is a subspace of  $\bigoplus_{i=0}^r H^0(X_i, E|_{X_i})$ . Its elements are tuples  $(f_0, f_1, \dots, f_r)$  where  $f_i$  and  $f_j$  coincide on  $X_i \cap X_j$ .

Fix a divisor  $D$  of degree  $g - 1$  and a point  $p$ , where all points are smooth on  $X$  and defined over  $\mathbb{K}$ . We wish to compute  $H^0(X, D + \infty p)$ . Riemann-Roch holds for  $X$  and hence so does (30). In order for the proof of Proposition 19 to go through, we need two conditions:

(\*) A rational function in  $H^0(X, D + np)$  is uniquely determined by its Laurent series at  $p$ .

(\*\*) We have  $h^1(X, D + np) = 0$  for  $n \gg 0$ .

Our next result characterizes when these two conditions hold. Let  $X_0$  be the irreducible component of  $X$  that contains  $p$ , and let  $X'_0 = \overline{X \setminus X_0}$  be the curve obtained removing  $X_0$ . Set  $Z = X_0 \cap X'_0$  and denote the restrictions of the divisor  $D$  to  $X_0, X'_0$  by  $D_0, D'_0$  respectively.

**Proposition 23.** *Condition (\*) holds if and only if  $H^0(X'_0, D'_0 - Z) = 0$ . Condition (\*\*) holds if and only if  $H^1(X'_0, D'_0) = 0$ . These are vanishing conditions on the curve  $X'_0$ .*

*Proof.* If  $X$  is irreducible then (\*) holds since rational functions are determined by their series on the normalization  $\tilde{X}$ . If  $X$  is reducible then  $X'_0$  is nonempty. We need that the restriction

$$H^0(X, D + np) \longrightarrow H^0(X_0, D_0 + np)$$

is injective. The kernel of this map consists exactly of those rational functions in  $H^0(X'_0, D'_0)$  which vanish on  $Z$ . In other words, the kernel is the space  $H^0(X'_0, D'_0 - Z)$ , as desired.

Also for the second statement, we can take  $X$  to be reducible. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D + np) \longrightarrow \mathcal{O}_{X_0}(D_0 + np) \oplus \mathcal{O}_{X'_0}(D'_0) \longrightarrow \mathcal{O}_Z(D_0 + np) \longrightarrow 0.$$

Taking global sections we see that  $H^1(X, D + np)$  surjects onto  $H^1(X'_0, D'_0)$ , since  $\dim(Z) = 0$ . Hence  $H^1(Z, D_0 + np) = 0$ . In particular, if  $H^1(X, D + np) = 0$  then  $H^1(X'_0, D'_0) = 0$ .

Conversely, suppose  $H^1(X'_0, D'_0) = 0$ . Since  $X_0$  is irreducible, we have  $H^1(X_0, D_0 + np) = 0$  for  $n \gg 0$ . The long exact sequence tells us that  $H^1(X, D + np) = 0$  as soon as the map

$$H^0(X_0, D_0 + np) \oplus H^0(X'_0, D'_0) \longrightarrow H^0(Z, D + np), \quad (f_0, f'_0) \mapsto f_0|_Z - f'_0|_Z$$

is surjective. Actually, the map  $H^0(X_0, D + np) \rightarrow H^0(Z, D + np)$  is surjective for  $n \gg 0$ . Indeed,  $H^1(X_0, D - Z + np) = 0$  for  $n \gg 0$  since  $p$  is an ample divisor on the curve  $X_0$ .  $\square$

**Remark 24.** Here we presented the case of nodal curves for simplicity, but the same discussion holds true, with essentially the same proofs, for an arbitrary singular curve.

**Remark 25.** The two conditions in Proposition 23 are automatically satisfied when the curve  $X$  is irreducible. In that case we always get a point  $U$  in the Sato Grassmannian.

## 6 Nodal Rational Curves

Our long-term goal is to fully understand the points  $U(\epsilon)$  in the Sato Grassmannian that represent Riemann-Roch spaces of a smooth curve over a valued field, such as  $\mathbb{K} = \mathbb{Q}(\epsilon)$ . We explained how these points are computed, and we implemented this in `Maple` for the case of hyperelliptic curves. Our approach is similar to [15, 18, 19]. For a given Mumford curve, it remains a challenge to lift the computation to the valuation ring (such as  $\mathbb{Q}[\epsilon]$ ) and correctly encode the limiting process as  $\epsilon \rightarrow 0$ . In this section we focus on what happens in the limit.

Consider a nodal reducible curve  $X = X_0 \cup \cdots \cup X_r$ , where each irreducible component  $X_i$  is rational. The arithmetic genus  $g$  is the genus of the dual graph. We present an algorithm whose input is a divisor  $D$  of degree  $g - 1$  and a point  $p$ , supported in the smooth locus of  $X$ . The algorithm checks the conditions in Proposition 23, and, if these are satisfied, it outputs a soliton solution that corresponds to  $U = \iota(H^0(X, D + \infty p))$ .

We start with some remarks on interpolation of rational functions on  $\mathbb{P}^1$ . Consider distinct points  $\kappa_1, \dots, \kappa_a$  and  $\kappa_{1,1}, \kappa_{1,2}, \dots, \kappa_{b,1}, \kappa_{b,2}$  on  $\mathbb{P}^1$ . We also choose a divisor  $D_0 = m_1 p_1 + \cdots + m_s p_s + m p$ , which is supported away from the previous points. Choose also scalars  $\lambda_1, \dots, \lambda_a, \mu_1, \dots, \mu_b \in \mathbb{K}$ . We wish to compute all functions  $f$  in  $H^0(\mathbb{P}^1, D_0 + \infty p)$  satisfying

$$f(\kappa_j) = \lambda_j \text{ for } j = 1, \dots, a \quad \text{and} \quad f(\kappa_{j,1}) = f(\kappa_{j,2}) = \mu_j \text{ for } j = 1, \dots, b. \quad (34)$$

To do so, we choose an affine coordinate  $x$  on  $\mathbb{P}^1$  such that  $p = \infty$ . Then we define

$$P(x) := \prod_{j=1}^s (x - p_j)^{m_j} \quad \text{and} \quad K(x) := \prod_{j=1}^a (x - \kappa_j) \cdot \prod_{j=1}^b (x - \kappa_{j,1})(x - \kappa_{j,2}).$$

Write  $K'(x)$  for the derivative of the polynomial  $K(x)$ . An interpolation argument shows:

**Lemma 26.** *A rational function  $f$  in  $H^0(\mathbb{P}^1, D_0 + \infty p)$  satisfies condition (34) if and only if*

$$f(x) = \frac{K(x)}{P(x)} \left[ \sum_{j=1}^a \lambda_j \frac{P(\kappa_j)}{K'(\kappa_j)} \frac{1}{x - \kappa_j} + \sum_{j=1}^b \mu_j \left( \frac{P(\kappa_{j,1})}{K'(\kappa_{j,1})} \frac{1}{x - \kappa_{j,1}} + \frac{P(\kappa_{j,2})}{K'(\kappa_{j,2})} \frac{1}{x - \kappa_{j,2}} \right) + H(x) \right],$$

where  $\mu_1, \dots, \mu_b \in \mathbb{K}$  and  $H(x)$  is a polynomial in  $\mathbb{K}[x]$ .

Lemma 26 gives a way to compute the Riemann-Roch space  $H^0(X, E)$  when  $E$  is a divisor on a nodal rational curve  $X$  as above. The normalization of such a curve is an union of projective lines. On each line we need to compute rational functions with prescribed values at certain points (corresponding to the intersection of two components of  $X$ ) and at certain pairs of points (corresponding to the nodes in the components of  $X$ ).

**Algorithm 27.** The following steps compute the soliton (26) associated to the curve data.

**Input:** A reducible curve  $X = X_0 \cup \cdots \cup X_r$  as above, with a smooth point  $p$  and a divisor  $D$  of degree  $g - 1$  supported also on smooth points. Everything is defined over  $\mathbb{K}$ .

- (1) Let  $X_0, X'_0, D_0, D'_0, Z$  be as in Section 5. Write  $Z = \{q_1, \dots, q_a\}$  and let  $n_1, \dots, n_b$  be the nodes in  $X_0$ . If  $\nu: \mathbb{P}^1 \rightarrow X_0$  is the normalization of  $X_0$  we set  $\kappa_j := \nu^{-1}(q_j)$  and  $\{\kappa_{j,1}, \kappa_{j,2}\} = \nu^{-1}(n_j)$ . We also write  $D_0 = m_1 p_1 + \dots + m_s p_s + m p$ , we fix an affine coordinate  $x$  on  $\mathbb{P}^1$  such that  $p = \infty$ , and we compute  $P(x)$  and  $K(x)$  as in (34).
- (2) Compute a basis  $Q_1, Q_2, \dots, Q_\ell$  of  $H^0(X'_0, D'_0)$ . If  $\ell = \deg D'_0 + 1 - p_a(X'_0)$  then proceed. Otherwise return “Condition (\*\*) in Lemma 23 fails” and terminate.
- (3) Compute the Riemann-Roch space  $H^0(X'_0, D'_0 - Z)$ . If this is zero then proceed. Otherwise return “Condition (\*) in Lemma 23 fails” and terminate.
- (4) Define the  $\ell \times a$  matrix  $A$  and the  $b \times 2b$  matrix  $B$  by

$$A_{i,j} := \frac{Q_i(p_j)P(\kappa_j)}{K'(\kappa_j)}, \quad B_{j,2j-1} := \frac{P(\kappa_{j,1})}{K'(\kappa_{j,i})}, \quad B_{j,2j} := \frac{P(\kappa_{j,2})}{K'(\kappa_{j,2})}, \quad B_{i,j} := 0 \text{ otherwise.}$$

**Output:** The  $(\ell+b) \times (a+2b)$  matrix  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . This represents the soliton solution for the point  $\iota(H^0(X, D + \infty p))$  in the Sato Grassmannian SGM, after a gauge transformation.

**Proposition 28.** *Algorithm 27 is correct.*

*Proof.* By Riemann-Roch, we have  $h^0(X'_0, D'_0) = h^1(X'_0, D'_0) + \deg D'_0 + 1 - p_a(X'_0)$ . Hence condition (\*\*) in Proposition 23 is satisfied if and only if the condition in step (2) of Algorithm 27 is satisfied. Moreover, condition (\*) in Proposition 23 is precisely the condition in step (3). Hence, we need to show that the output of the algorithm corresponds to  $\iota(H^0(X, D + \infty p))$ , after a gauge transformation. However, we know that any element of  $H^0(X, D + \infty p)$  can be written as  $(f, \sum_j \lambda_j Q_j)$  such that  $f \in H^0(X_0, D_0 + \infty p)$  and

$$f(\kappa_j) = \sum_i \lambda_i Q_i(\kappa_j), \text{ for } j = 1, \dots, a \quad \text{and} \quad f(\kappa_{j,1}) = f(\kappa_{j,2}) \text{ for } j = 1, \dots, b.$$

At this point, Lemma 26 gives us a basis of  $\iota(H^0(X, D + \infty p))$ . Remark 18 shows that this corresponds exactly to the matrix given by the algorithm, after a gauge transformation.  $\square$

We illustrate the algorithm in the following examples.

**Example 29.** Let  $X$  be an irreducible rational curve with  $g$  nodes. Algorithm 27 returns a matrix  $B$  for a  $(g, 2g)$ -soliton. This is consistent with (18). Note that  $X$  is a tropical limit where the graph is one node with  $g$  loops and the Delaunay polytope is the  $g$ -cube.

**Example 30** ( $g = 2$ ). Let  $X$  be the union of two smooth rational curves  $X_0, X_1$  meeting at three points  $Z = \{q_1, q_2, q_3\}$ . This curve is the special fiber of the genus 2 curve  $\{y^2 = f_2(x)\}$  in Example 1. It corresponds to the graph on the right in Figure 2. We choose a smooth point  $p \in X_0 \setminus Z$ , and we consider three different divisors of degree one:  $p$ ,  $-2q + 3p$  and  $3q - 2p$ , where  $q$  is a smooth point in  $X_1$ . We apply Algorithm 27 to these three instances.

- Take  $D = p$ . Then  $H^0(X'_0, D'_0) = H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{K}$  has the constant function 1 as basis. The conditions in steps (2) and (3) are both satisfied, and the algorithm gives us the soliton solution corresponding to the matrix  $A = \begin{pmatrix} \frac{1}{K'(\kappa_1)} & \frac{1}{K'(\kappa_2)} & \frac{1}{K'(\kappa_3)} \end{pmatrix}$ . Note that the Delaunay polytope  $\mathcal{C}$  is the triangle, and the approach in Section 3 leads to the gauge-equivalent matrix  $A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ . This also arises for  $z_1 = 0$  in Example 11.
- Take  $D_2 = -2q + 3p$ . Then  $H^0(X'_0, D'_0) \cong H^0(\mathbb{P}^1, -2q) = 0$  and the condition in step (2) is not satisfied. Hence we do not get a point in the Sato Grassmannian.
- Take  $D_3 = 3q - 2p$ . Then  $H^0(X'_0, D'_0) \cong H^0(\mathbb{P}^1, 3q)$  has dimension 4 and the condition in step (2) is satisfied. However  $H^0(X'_0, D'_0 - Z) \cong H^0(\mathbb{P}^1, \mathcal{O}) \neq 0$  so the condition in step (3) is not satisfied, and we do not get a point in the Sato Grassmannian.

**Example 31** ( $g = 3$ ). Consider four general lines  $X = X_0 \cup X_1 \cup X_2 \cup X_3$  in  $\mathbb{P}^2$ . Set  $X_0 \cap X_i = \kappa_i$  and  $X_i \cap X_j = q_{ij}$  for  $i, j \in \{1, 2, 3\}$ . We fix the divisor  $D = p_1 + p_2 + p_3 - p$ , for general points  $p \in X_0$  and  $p_i \in X_i$  for  $i = 1, 2, 3$ . After the preparatory set-up in step (1), we compute  $H^0(X'_0, D'_0)$  in step (2). This is the space of functions  $(g_1, g_2, g_3)$  in  $\bigoplus_{i=1}^3 H^0(X_i, p_i)$  such that  $g_i(q_{ij}) = g_j(q_{ij})$  for  $i, j \in \{1, 2, 3\}$ . Choose affine coordinates  $x_i$  on  $X_i$  for  $i = 1, 2, 3$  such that  $p_i = \infty$ . We compute the following basis for  $H^0(X'_0, D'_0)$ :

$$Q_1 = \left(0, \frac{x_2 - q_{12}}{q_{23} - q_{12}}, \frac{x_3 - q_{13}}{q_{23} - q_{12}}\right), \quad Q_2 = \left(\frac{x_1 - q_{12}}{q_{13} - q_{12}}, 0, \frac{x_3 - q_{23}}{q_{13} - q_{23}}\right), \quad Q_3 = \left(\frac{x_1 - q_{13}}{q_{12} - q_{13}}, \frac{x_2 - q_{23}}{q_{12} - q_{13}}, 0\right).$$

Hence  $\ell = 3$  and the condition in step (2) holds. We also find that  $H^0(X'_0, D'_0 - Z) = 0$ , so that the condition in step (3) is satisfied as well. Algorithm 27 outputs the soliton matrix

$$A = \begin{pmatrix} 0 & \frac{\kappa_1 - q_{12}}{q_{13} - q_{12}} \frac{1}{K'(\kappa_1)} & \frac{\kappa_1 - q_{13}}{q_{12} - q_{13}} \frac{1}{K'(\kappa_1)} \\ \frac{\kappa_2 - q_{12}}{q_{23} - q_{12}} \frac{1}{K'(\kappa_2)} & 0 & \frac{\kappa_2 - q_{23}}{q_{12} - q_{13}} \frac{1}{K'(\kappa_2)} \\ \frac{\kappa_3 - q_{13}}{q_{23} - q_{12}} \frac{1}{K'(\kappa_3)} & \frac{\kappa_3 - q_{23}}{q_{13} - q_{23}} \frac{1}{K'(\kappa_3)} & 0 \end{pmatrix}. \quad (35)$$

The curve  $X$  is the last one in [2, Figure 2]. The Delaunay polytope  $\mathcal{C}$  is a tetrahedron, so Theorem 10 applies. It would be desirable to better understand the relationship between the soliton solution (35), the Hirota variety  $\mathcal{H}_{\mathcal{C}}$ , and the Dubrovin variety in [3, Example 6.2].

We end with a few words of conclusion. There are two ways to obtain a tau function from a smooth curve: via the theta function and the Dubrovin threefold as in [3], or via the Sato Grassmannian as in Section 5. In this paper we presented a parallel for tropical limits of smooth curves: namely, the Delaunay polytopes and Hirota varieties of Sections 1–3, or the Sato Grassmannian as in Section 6. Our next goal is to better understand this process in families. An essential step is to clarify the relation between the degeneration of theta functions via  $\mathbf{a} \in \mathcal{C}$  as in (8) and the choice of the divisor  $D$  and point  $p$  in Algorithm 27.

## References

- [1] A. Abenda and P. Grinevich: *Rational degenerations of M-curves, totally positive Grassmannians and KP2-solitons*, Communications in Math. Physics **361** (2018) 1029–1081.
- [2] D. Agostini, T. Çelik Özlüm, J. Struwe and B. Sturmfels: *Theta surfaces*, Vietnam J. Math. (2021).
- [3] D. Agostini, T. Çelik Özlüm and B. Sturmfels: *The Dubrovin threefold of an algebraic curve*, arXiv:2005.08244.
- [4] M. Brandt: *Tropical Geometry of Curves*, PhD dissertation, UC Berkeley, 2020.
- [5] L. Chua, M. Kummer and B. Sturmfels: *Schottky algorithms: classical meets tropical*, Mathematics of Computation **88** (2019) 2541–2558.
- [6] J. Draisma and R. Eggermont: *Plücker varieties and higher secants of Sato’s Grassmannian*, J. Reine Angew. Math. **737** (2018), 189–215.
- [7] B. Dubrovin: *Theta functions and non-linear equations*, Russian Math. Surveys **36** (1981) 11–92.
- [8] M. Dutour: *The six-dimensional Delaunay polytopes*, European J. Comb. **25** (2004) 535–548.
- [9] R.M. Erdahl and S.S. Ryshkov: *The empty sphere*, Canadian J. Math. **39** (1987) 794–824.
- [10] D. Grayson and M. Stillman: *Macaulay2, a software system for research in algebraic geometry*, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [11] F. Hess: *Computing Riemann-Roch spaces in algebraic function fields and related topics*, J. Symbolic Computation **33** (2002) 425–445.
- [12] P. Jell: *Constructing smooth and fully faithful tropicalizations for Mumford curves*, Selecta Mathematica **26** (2021), Paper No 60, 23 pp.
- [13] K. Kaveh and C. Manon: *Khovanskii bases, higher rank valuations, and tropical geometry*, SIAM J. Appl. Algebra Geom. **3** (2019) 292–336.
- [14] Y. Kodama: *KP Solitons and the Grassmannians. Combinatorics and Geometry and Two-dimensional Wave Patterns*, Briefs in Mathematical Physics **22**, Springer, 2017.
- [15] Y. Kodama and Y. Xie: *Space curves and solitons of the KP hierarchy: I. The l-th generalized KdV hierarchy*, arXiv:1912.06768.
- [16] I. M. Krichever: *Methods of algebraic geometry in the theory of nonlinear equations*, Russian Mathematical Surveys **32** (1977) 185–214.
- [17] M. Michałek and B. Sturmfels: *Invitation to Nonlinear Algebra*, Graduate Studies in Mathematics, Vol 211, American Mathematical Society, Providence, 2021.
- [18] A. Nakayashiki: *Degeneration of trigonal curves and solutions of the KP-hierarchy*, Nonlinearity **31** (2018) 3567–3590.
- [19] A. Nakayashiki: *On reducible degeneration of hyperelliptic curves and soliton solutions*, SIGMA Symmetry Integrability Geom. Methods Appl. **15** (2019), Paper No. 009, 18 pp.
- [20] M. Sato: *Soliton equations as dynamical systems on infinite dimensional Grassmann manifold*, Nonlinear Partial Differential Equations in Applied Science, North-Holland, 1982, pp. 259–271.
- [21] R. Stanley: *Enumerative Combinatorics: Volume 2*, Cambridge University Press, 2001
- [22] B. Sturmfels: *Gröbner Bases and Convex Polytopes*, American Mathematical Society, University Lectures Series, No 8, Providence, Rhode Island, 1996.

Daniele Agostini, MPI-MiS Leipzig,  
Claudia Fevola, MPI-MiS Leipzig,  
Yelena Mandelshtam, UC Berkeley,  
Bernd Sturmfels, MPI-MiS Leipzig and UC Berkeley,

daniele.agostini@mis.mpg.de  
claudia.fevola@mis.mpg.de  
yelenam@berkeley.edu  
bernd@mis.mpg.de