Combinatorial Differential Algebra of x^p

Anna-Laura Sattelberger (MPI-MiS Leipzig)

based on joint work with Rida Ait El Manssour (MPI–MiS Leipzig)

arXiv:2102.03182

MEGA 2021
UiT – The Arctic University of Tromsø
June 7–11, 2021

MAX-PLANCK-GESELLSCHAFT

Combinatorial Differential Algebra of x^p

- at the interface of geometric combinatorics and differential algebra

Linking...

1. differential ideals
2. lattice polytopes arising from graphs
3. regular triangulations

Differential algebra

- study of polynomial O/PDEs with methods from Commutative Algebra

Differential rings and ideals

- $\mathbb{C}[x^{(\infty)}]$ the **ring of differential polynomials in** x **over** $\mathbb{C}$, i.e., $(\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots], \partial)$, $\partial(x^{(k)}) = x^{(k+1)}$, $\partial|_{\mathbb{C}} \equiv 0$, Leibniz' rule
- $I \triangleleft \mathbb{C}[x^{(\infty)}]$ is a **differential ideal** if $\partial(I) \subseteq I$
- For $S \subseteq \mathbb{C}[x^{(\infty)}]$, $\langle S \rangle^{\infty}$ denotes the differential ideal generated by S.

Bivariate case

$\mathbb{C}[x^{(\infty,\infty)}] := \mathbb{C}[\{x^{(k,\ell)}\}, \{\partial_s, \partial_t\}]$ with

$$\partial_s(x^{(k,\ell)}) = x^{(k+1,\ell)},\ \partial_t(x^{(k,\ell)}) = x^{(k,\ell+1)},\ \partial_s|_{\mathbb{C}} \equiv 0,\ \partial_t|_{\mathbb{C}} \equiv 0$$

the **ring of partial differential polynomials in** x **over** $\mathbb{C}$ in the two independent variables s and t.

Differential Gröbner bases

Definition

$G \subseteq \mathbb{C}[x^{(\infty)}]$ is a **differential Gröbner basis** of $\langle G \rangle^{(\infty)}$ if $\{\partial^k(g) \mid k \in \mathbb{N},\, g \in G\}$ is an algebraic Gröbner basis of $\langle G \rangle^{(\infty)}$ w.r.t. $\prec$.

Theorem (Zobnin, 2009)

The singleton $\{x^p\}$ is a differential Gröbner basis of $\langle x^p \rangle^{(\infty)}$ with respect to the reverse lexicographical ordering.

Jets of the fat point x^p on the affine line

R_n	the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$
$f_{p,n} \in R_n[t]$	the polynomial $(x_0 + x_1 t + \cdots + x_n t^n)^p$ in t
$C_{p,n} \triangleleft R_n$	the ideal generated by the coefficients of $f_{p,n}$
$I_{p,n} \triangleleft \mathbb{C}[x^{(\infty)}]$	the differential ideal generated by x^p and $x^{(n)}$

Truncating Taylor series

$C_{p,n}$ encodes certain n-jets of the fat point x^p on the affine line

Linking $C_{p,n}$ and $I_{p,n}$

$$R_n / C_{p,n} \xrightarrow{\cong} \mathbb{C}[x^{(\infty)}]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}.$$

Question

For fixed n, is $\dim_{\mathbb{C}}(R_n / C_{p,n})$ a polynomial in p of degree $n+1$?

Example: $\dim_{\mathbb{C}}(R_6/C_{p,6})_{p\in\mathbb{N}}$

The first 13 entries of the sequence $\dim_{\mathbb{C}}(R_6/C_{p,6})_{p\in\mathbb{N}}$ are[1]

$$0, 1, 34, 353, 2037, 8272, 26585, 72302, 173502, 377739, 760804, 1437799, 2576795,$$

coinciding with the sequence `https://oeis.org/A244881`.

Interpolating polynomial (computed on the values for $p = 1, \ldots, 20$):

$$\frac{17}{315}p^7 + \frac{17}{90}p^6 + \frac{53}{180}p^5 + \frac{19}{72}p^4 + \frac{13}{90}p^3 + \frac{17}{360}p^2 + \frac{1}{140}p,$$

of degree $7 = 6 + 1$.

[1] computed with `Singular`

Counting lattice points of polytopes

C	an integral d-dimensional polytope
tC	the polytope dilated by $t \in \mathbb{N}$

Then: $|tC \cap \mathbb{Z}^n|$ is a polynomial in t of degree d, the **Ehrhart polynomial** of C.

Theorem (Ait El Manssour–S., 2021)

The number $\dim_{\mathbb{C}}(R_n/C_{p,n})$ is the Ehrhart polynomial of the polytope

$$P_n := \left\{(w_0, \dots, w_n) \in (\mathbb{R}_{\geq 0})^{n+1} \middle| w_i + w_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n-1\right\}$$

evaluated at $p - 1$.

Proof: Results from graph theory +

Proposition (Bruschek–Mourtada–Schepers, 2013)

$\mathrm{in}_{\prec_{\mathrm{revlex}}}(C_{p,n})$ is generated by $\left\{x_i^{u_i} x_{i+1}^{u_{i+1}} \mid u_i + u_{i+1} = p,\, 0 \leq i \leq n-1\right\}$.

- graph G with $V = \{0, 1, \dots, n\}$ and $E = \{[i, i+1]\}_{i=0,\dots,n-1}$

Fractional stable set polytope of a graph

G	an undirected graph with vertices V and edges E
$C(G)$	the cliques of G

Two polytopes

- $\mathrm{Stab}(G) := \mathrm{conv}\left\{\chi^S \in \mathbb{R}^V \mid S \subseteq V \text{ stable}\right\}$ the **stable set polytope** of G, with $\chi^S = (\chi^S_v)_{v\in V} \in \mathbb{R}^V$ incidence vectors
- $\mathrm{QStab}(G) := \left\{x \in \mathbb{R}^V \mid 0 \leq x(v)\,\forall v \in V,\ \sum_{v\in Q} x(v) \leq 1\ \forall Q \in C(G)\right\}$ the **fractional stable set polytope** of G

Then: $\mathrm{Stab}(G) = \mathrm{conv}\{\{0,1\}^V \cap \mathrm{QStab}(G)\}$.

Theorem (Chvátal, 1975)

A graph G is perfect iff $\mathrm{Stab}(G) = \mathrm{QStab}(G)$

Bivariate case

$R_{m,n}$	the polynomial ring $\mathbb{C}[\{x_{k,\ell}\}_{0\le k\le m, 0\le \ell\le n}]$
$f_{p,(m,n)} \in R_{m,n}[s,t]$	the polynomial $(x_{00} + x_{10}s + \cdots + x_{mn}s^m t^n)^p$ in s and t
$C_{p,(m,n)} \lhd R_{m,n}$	the ideal generated by the coefficients of $f_{p,(m,n)}$
$I_{p,(m,n)} \lhd \mathbb{C}[x^{(\infty,\infty)}]$	the differential ideal generated by x^p, $x^{(m,0)}$, and $x^{(0,n)}$

Linking $C_{p,(m,n)}$ and $I_{p,(m,n)}$

$$R_{m,n}/C_{p,(m,n)} \xrightarrow{\cong} \mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m+1,n+1)}, \quad x_{k,\ell} \mapsto \frac{1}{k!\ell!}x^{(k,\ell)}.$$

Looking for monomial orderings...

...for which the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of $C_{p,(m,n)}$.

The regular triangulation $T_{m,2}$

$T_{m,2}$ the placing triangulation of the $m \times 2$-rectangle for the point configuration $[(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),\ldots,(m,0),(m,1),(m,2)]$ induced by the vector $(1,2,\ldots,2^{3m+2})$ (*lower* hull convention)

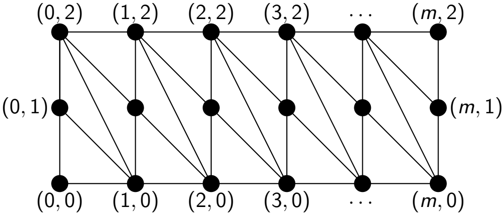


Figure: The regular triangulation $T_{m,2}$

The regular triangulation $T_{m,n}$

$T_{m,n}$ the placing triangulation of the $m \times n$-rectangle for the point configuration $[(0,0),(0,1),\ldots,(0,n),(1,0),\ldots,(1,n),\ldots,(m,0),\ldots,(m,n)]$

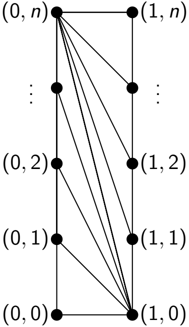


Figure: The regular triangulation $T_{1,n}$

T-orderings

T a triangulation of the $m \times n$-rectangle

Definition

A monomial ordering $\prec$ on $\mathbb{C}[\{x^{(\leq k, \leq \ell)}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}]$ is called **T-ordering** if each of the leading monomials of $(x^p)^{(k,\ell)}$ is supported on a triangle of T.

Proposition (Ait El Manssour–S., 2021)

For all k, ℓ, $(x^p)^{(k,\ell)} \in \mathbb{C}[x^{(\leq m, \leq n)}]$ has a unique monomial supported on a triangle of $T_{m,n}$. The reverse lexicographical ordering $\prec$ on $\mathbb{C}[x^{(\leq m, \leq n)}]$ is a $T_{m,n}$-ordering for all p.

A higher-dimensional analog of Zobnin's result

$\prec$ a $T_{m,2}$-ordering

Theorem (Ait El Manssour–S., 2021)

For all $m, p \in \mathbb{N}$, $\{(x^p)^{(k,\ell)}\}_{0 \leq k \leq mp,\, 0 \leq \ell \leq 2p}$ is a Gröbner basis of $\langle x^p \rangle^{(\infty,\infty)}$ in $\mathbb{C}[x^{(\leq m, \leq 2)}]$ with respect to any $T_{m,2}$-ordering.

Theorem (Ait El Manssour–S., 2021)

For all $m \in \mathbb{N}$, $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$ is the Ehrhart polynomial of the $3(m+1)$-dimensional lattice polytope

$$P_{(m,2)} := \Big\{(u_{00}, u_{01}, u_{02}, \ldots, u_{m0}, u_{m1}, u_{m2}) \in (\mathbb{R}_{\geq 0})^{3(m+1)} \Big| u_{k_1,\ell_1} + u_{k_2,\ell_2} + u_{k_3,\ell_3} \leq 1 \text{ for all indices s.t. } \{(k_1,\ell_1),(k_2,\ell_2),(k_3,\ell_3)\} \text{ is a triangle of } T_{m,2}\Big\}$$

evaluated at $p - 1$.

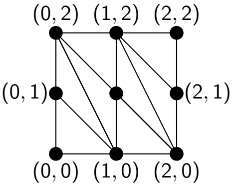
Example: $C_{3,(2,2)}$

$C_{p,(2,2)}$	the ideal in $R_{2,2} = \mathbb{C}[x_{00}, x_{10}, x_{01}, x_{02}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}]$ generated by the $(2p+1)^2$ many coefficients of $f_{p,(2,2)} \in R_{2,2}[s,t]$
$\prec$	the **weighted reverse lexicographical ordering** on $R_{2,2}$ for $w_{2,2} := \left(2^8+1, \ldots, 2^8+1\right) - \left(2^0, 2^1, \ldots 2^8\right) \in \mathbb{N}^9$

In the leading monomials of the coefficients of $f_{3,(2,2)}$, the following triples of variables show up:

$$\{x_{00}, x_{01}, x_{10}\}, \{x_{01}, x_{02}, x_{10}\}, \{x_{02}, x_{10}, x_{11}\}, \{x_{02}, x_{11}, x_{12}\},$$
$$\{x_{10}, x_{11}, x_{20}\}, \{x_{11}, x_{12}, x_{20}\}, \{x_{12}, x_{20}, x_{21}\}, \{x_{12}, x_{21}, x_{22}\}.$$

The indices of those define the triangles of the regular triangulation $T_{2,2}$:



$m = 3, n = 2$

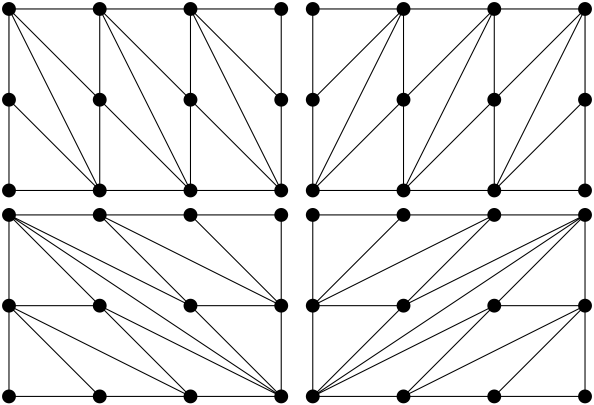


Figure: For height vectors inducing those four regular unimodular triangulations of the 3×2-rectangle, the weighted reverse lexicographical ordering turns the coefficients of $f_{p,(3,2)}$ into a Gröbner basis of $C_{p,(3,2)}$.

Open problems

Question 1

For which $m, n, p \in \mathbb{N}$ does there exist a regular unimodular triangulation T of the $m \times n$-rectangle such that the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of $C_{p,(m,n)}$ with respect to the weighted reverse lexicographical ordering for a vector inducing that triangulation in the upper hull convention?

Question 2

Are the four triangulations depicted on slide 15, continued to the $m \times 2$-rectangle, *all* regular unimodular triangulations that give rise to a Gröbner basis?

Question 3

As p varies, is $\dim_{\mathbb{C}}(R_{m,n}/C_{p,(m,n)})$ the Ehrhart polynomial of the (fractional) stable set polytope of the edge graph of T and is this graph perfect?

Inc($\mathbb{N}$)-stable ideals [KLS16, HS09, NR17]

Parallels to be worked out

Supplementary material

A "geometrical provocation" inspired by $T_{2,2}$.
Find more of them on www.alsattelberger.de!

Thank you very much for your attention!

References I

[Afs20] Pooneh Afsharijoo.
Looking for a new member of Gordon's identitites.
arXiv:2006.0921, 2020.

[Ber62] Claude Berge.
Sur une conjecture relative au problème des codes optimaux.
Comm. 13e assemblée générale de l'URSI, Tokyo, 1962.

[BJMS15] Sarah Brodsky, Michael Joswig, Ralph Morrison, and Bernd Sturmfels.
Moduli of tropical plane curves.
Research in the Mathematical Sciences, 2(4), 2015.

[BMS13] Clemens Bruschek, Hussein Mourtada, and Jan Schepers.
Arc spaces and Rogers–Ramanujan identities.
Ramanujan J., 30:9–38, 2013.

[Chv75] Václav Chvátal.
On certain polytopes associated with graphs.
J. Combin. Theory Ser. B, 18:138–154, 1975.

[DGPS20] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann.
SINGULAR 4-1-3—A computer algebra system for polynomial computations.
`http://www.singular.uni-kl.de`, 2020.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos.
Triangulations, volume 25 of *Algorithms and computations in Mathematics*.
Springer-Verlag Berlin Heidelberg, 2010.

[Ful72] Delbert R. Fulkerson.
Anti-blocking polyhedra.
J. Combin. Theory Ser. B, 12:50–71, 1972.

[GJ00] Ewgenij Gawrilow and Michael Joswig.
`polymake`: A framework for analyzing convex polytopes.
In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

References II

[GJS06] Russell A. Goward Jr. and Karen E. Smith.
The jet scheme of a monomial scheme.
Comm. Algebra, 34:1591–1598, 2006.

[Grö99] Martin Grötschel.
My Favorite Theorem.
OPTIMA, 62:2–5, 1999.

[HS09] Christopher Hillar and Seth Sullivant.
Finite Gröbner bases in infinite dimensional polynomial rings and applications.
Adv. Math., 229, 2009.

[KLS16] Robert Krone, Anton Leykin, and Andrew Snowden.
Hilbert series of symmetric ideals in infinite polynomial rings via formal languages.
J. Algebra, 485, 2016.

[Kol73] Ellis R. Kolchin.
Differential Algebra and Algebraic Groups.
Pure and applied mathematics. Academic Press, 1973.

[Lev42] Howard Levi.
On the structure of differential polynomials and on their theory of ideals.
Trans. Amer. Math. Soc., 51(3):532–568, 1942.

[Lov71] Lászlo Lovász.
Normal hypergraphs and the perfect graph conjecture.
Discrete Math., 2, 1971.

[MS10] Rahim Moosa and Thomas Scanlon.
Jet and prolongation spaces.
J. Inst. Math. Jussieu, 9:391–430, 2010.

[NR17] Uwe Nagel and Tim Römer.
Equivariant Hilbert Series in non-Noetherian Polynomial Rings.
J. Algebra, 486, 2017.

References III

[Oll90] François Ollivier.
Standard Bases of Differential Ideals.
Lecture Notes in Comput. Sci., 508:304–321, 1990.

[OSC] OSCAR Computer Algebra System.
`https://oscar.computeralgebra.de`.

[Rit50] Joseph F. Ritt.
Differential Algebra, volume 14 of *American Mathematical Society: Colloquium publications*.
American Mathematical Society, 1950.

[Stu96] Bernd Sturmfels.
Gröbner bases and convex polytopes, volume 8 of *University lecture series*.
American Mathematical Society, Providence, R.I., 1996.

[Yue06] Cornelia Yuen.
One higher dimensional analog of jet schemes.
arXiv:math/0608633, 2006.