Schur apolarity

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Introduction

Introdution and basic definitions

The aim of this talk is to show the main project of my Ph.D. studies.

Inspired by the classic apolarity theory for symmetric tensors, the purpose of my work is to develop an analogue theory for tensors associated to SL_{n+1}-rational homogeneous variety.

Additive decompositions

We will work always over the field of complex numbers $\mathbb{C}$.

We are interested in the problem of finding additive decompositions of structured tensors. A notion which we will use

Definition
Let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible variety. The X-*rank* of a point $p \in \mathbb{P}^N$ is the integer

$$r_X(p) := \min\{r : p \in \langle p_1, \dots, p_r \rangle,\ p_i \in X\}.$$

Classic apolarity theory - I

Let $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ be a Veronese variety. It can be obtained as image of the embedding

$$\begin{aligned} \nu_d : \mathbb{P}(V) &\longrightarrow \mathbb{P}(\mathrm{Sym}^d(V)) \\ [l] &\longmapsto [l^d] \end{aligned}$$

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Given $d \geq e \geq 0$, the *apolarity action* is the map

$$\varphi : \mathrm{Sym}^d V \otimes \mathrm{Sym}^e V^* \longrightarrow \mathrm{Sym}^{d-e} V$$

which acts as a derivation.

Lemma (of Apolarity, [IK99])

Let $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ be a Veronese variety. Let $p_1, \dots, p_r \in X$ and $[F] \in \mathbb{P}^{\binom{n+d}{d}-1}$. The following are equivalent:

(1) there exists $c_1, \dots, c_r \in \mathbb{C}$ such that $[F] = c_1 p_1 + \cdots + c_r p_r$,

Lemma (of Apolarity, [IK99])

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(1) there exists $c_1, \ldots, c_r \in \mathbb{C}$ such that $[F] = c_1 p_1 + \cdots + c_r p_r$,

(2) there is the inclusion of ideals $I(p_1, \ldots, p_r) \subset F^\perp$, where

- $I(p_1, \ldots, p_r)$ is the ideal of the union of the points $p_1, \ldots, p_r$,
- $F^\perp$ is the set of all derivations which kill F.

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Yes via *Non-abelian apolarity* using vector bundles techniques, [LO13].

Yes for any representation $\mathbb{S}_\lambda V$ of SL_{n+1} with apolarity action

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

The apolarity action

Definition
Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_h)$ be two partitions. We say that $\mu \subset \lambda$ if $h \leq k$ and $\mu_i \leq \lambda_i$ for all i.

Skew diagrams

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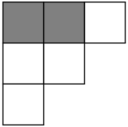
In this case the *skew Young diagram* λ/μ is the diagram of λ without the diagram of μ in the left upper corner. For instance if $\lambda = (3, 2, 1)$ and $\mu = (2)$, then

$\lambda/\mu =$ 

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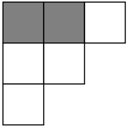
$\lambda/\mu =$ 

A *skew Schur module* $\mathbb{S}_{\lambda/\mu} V$ is obtained as a Schur module using the skew diagram λ/μ.

Let λ be a partition and V vector space of dimension $n+1$. The minimal orbit X via the SL_{n+1} action inside $\mathbb{P}(\mathbb{S}_\lambda V)$ is the Flag variety

$$\mathbb{F}(k_1, \ldots, k_s; V) := \{(V_1, \ldots, V_s) : V_1 \subset \cdots \subset V_s \subset V,\ \dim(V_i) = k_i\}$$

embedded with $\mathcal{O}(d_1, \ldots, d_s)$.

Geometry

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embedded with $\mathcal{O}(d_1, \ldots, d_s)$. The points of X-rank 1 are of the form

$$(v_1 \wedge \cdots \wedge v_{k_s})^{\otimes d_s} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{k_1})^{\otimes d_1}$$

representing the flag

$$\langle v_1, \ldots, v_{k_1} \rangle \subset \cdots \subset \langle v_1, \ldots, v_{k_s} \rangle.$$

The setting

Some particular features of the apolarity theory are the *apolarity action*, the *ideal of a point of X-rank* 1 and a *ring*.

Since we want tho build a global apolarity theory, we have lost some of this properties. For instance via the Littlewood-Richardson rule we may have several multiplication maps

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$$

with different ν (or not!). Hence in our theory the concepts of ring and ideal are replaced with suitable vector spaces and subspaces.

The ambient space - I

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For example in $V^{\otimes 3}$ there are 2 copies of $\mathbb{S}_{(2,1)}V$. The h.w.v. in both of them is

$$e_1 \wedge e_2 \otimes e_1 = e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1,$$

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Since we are not interested on how this tensors embeds in $V^{\otimes d}$, we reduce to work in the vector space

$$\begin{aligned}\mathbb{S}^\bullet V &:= \mathrm{Sym}^\bullet(V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^{n+1} V)/I^\bullet \\ &\simeq \left(\bigoplus_{(a_1,\ldots,a_{n+1})\in\mathbb{N}^{n+1}} \mathrm{Sym}^{a_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n+1}}(\wedge^{n+1} V) \right) / I^\bullet\end{aligned}$$

The ambient space - II

where for any $p \geq q \geq 0$, the ideal $I^\bullet$ is generated by

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q)$$

known as *Plücker relations*. Note that in $\mathbb{S}^\bullet(V)$ every Schur module appears once.

The apolarity action - I

The apolarity action is defined using the skew-symmetric apolarity action which is given for any $0 \leq h \leq k < \dim(V)$ by the composition

$$\wedge^h V^* \otimes \wedge^k V \longrightarrow \wedge^h V^* \otimes \wedge^h V \otimes \wedge^{k-h} V \longrightarrow \wedge^{k-h} V.$$

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Recall then that via definiton with Young symmetrizers we have the inclusions

$$\mathbb{S}_\lambda V \subset \wedge^{\lambda'_1} V \otimes \cdots \otimes \wedge^{\lambda'_k} V =: \wedge_{\lambda'} V$$

The apolarity action - II

Definition The *Schur apolarity action* is the map

$$\varphi : \mathbb{S}^{\bullet} V \otimes \mathbb{S}^{\bullet} V^{*} \longrightarrow \mathbb{S}^{\bullet} V$$

such that when restricted to $\mathbb{S}_{\lambda} V \otimes \mathbb{S}_{\mu} V^{*}$ is

- the zero map if $\mu \not\subset \lambda$,

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- the zero map if $\mu \not\subset \lambda$,
- otherwise it is the restriction of the map

$$\tilde{\varphi} : \wedge_{\lambda'} V \otimes \wedge_{\mu'} V^* \longrightarrow \wedge_{\lambda'/\mu'} V$$

acting as a product of skew symmetric apolarity actions

$$\wedge^{\lambda'_i} V \otimes \wedge^{\mu'_i} V^* \longrightarrow \wedge^{\lambda'_i - \mu'_i} V.$$

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Proposition
The image of φ is contained in $\mathbb{S}_{\lambda/\mu} V$.

The apolarity action - III

For instance, consider $\lambda = (2,2)$ and $\mu = (1,1)$. Let

$$t = v_1 \wedge v_2 \otimes v_1 \wedge v_3 + v_1 \wedge v_3 \otimes v_1 \wedge v_2 \in \mathbb{S}_{(2,2)}\mathbb{C}^4$$

and let $s = x_1 \wedge x_2 \in \mathbb{S}_{(1,1)}\mathbb{C}^4$.

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and let $s = x_1 \wedge x_2 \in \mathbb{S}_{(1,1)}\mathbb{C}^4$. Then

$$\begin{aligned}
\varphi(t \otimes s) = & \\
&= \det \begin{pmatrix} x_1(v_1) & x_1(v_2) \\ x_2(v_1) & x_2(v_2) \end{pmatrix} v_1 \wedge v_3 + \det \begin{pmatrix} x_1(v_1) & x_1(v_3) \\ x_2(v_1) & x_2(v_3) \end{pmatrix} v_1 \wedge v_2 \\
&= v_1 \wedge v_3
\end{aligned}$$

which is an element of $\mathbb{S}_{(2,2)/(1,1)}\mathbb{C}^4$.

The subspace associated to a point - I

Definition Let $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ be a Flag variety $\mathbb{F}(k_1, \ldots, k_S; V)$ embedded with $\mathcal{O}(d_1, \ldots, d_s)$, and let p be a point of X-rank 1. Then p represents a flag

$$V_1 \subset \cdots \subset V_s \subset V.$$

Consider the orthogonal spaces $V_s^\perp \subset \cdots \subset V_1^\perp \subset V^*$.

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$$V_1 \subset \cdots \subset V_s \subset V.$$

Consider the orthogonal spaces $V_s^\perp \subset \cdots \subset V_1^\perp \subset V^*$. The *subspace* $I(p)$ *associated to* p is the vector subspace of $\mathbb{S}^\bullet(V^*)$ constructed in the following way:

• consider the generators of $V_s^\perp, \mathrm{Sym}^{d_s+1} V_{s-1}^\perp, \ldots, \mathrm{Sym}^{d_s+\cdots+d_2+1} V_1^\perp$

• use the maps $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$ to construct $I(p)$ step by step rescricting them to $(I(p) \cap \mathbb{S}_\lambda V) \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$

The subspace associated to a point - II

Let $p = (v_1 \wedge v_2)^{\otimes 2} \in X$, where X is $\mathbb{G}(2, \mathbb{C}^4)$ embedded with $\mathcal{O}(2)$. In this case we have only one subspace $V_1 = \langle v_1, v_2 \rangle$ and $V_1^{\perp} = \langle x_3, x_4 \rangle$.

One can check that via this definition we get

$$I \cap \mathbb{S}_{(1)}(\mathbb{C}^4)^* = \langle x_3, x_4 \rangle,$$
$$I \cap \mathbb{S}_{(2)}(\mathbb{C}^4)^* = \langle x_1x_3, x_2, x_3, x_3^2, x_3x_4, x_1x_4, x_2x_4, x_4^2 \rangle,$$
$$I \cap \mathbb{S}_{(1,1)}(\mathbb{C}^4)^* = \langle x_1 \wedge x_3, x_2 \wedge x_3, x_3 \wedge x_4, x_1 \wedge x_4, x_2 \wedge x_4 \rangle,$$

$I \cap \mathbb{S}_{(2,1)}(\mathbb{C}^4)^* = \langle$all the elements of the basis whose associated semi-std tableau is different from $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\rangle$,

$I \cap \mathbb{S}_{(2,2)}(\mathbb{C}^4)^* = \langle$all the elements of the basis whose associated semi-std tableau is different from $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\rangle$.

The apolarity lemma

Lemma (of apolarity) Let $f \in \mathbb{S}_\lambda V$ and let $p_1, \ldots, p_r \in \mathbb{S}_\lambda V$ points of X-rank 1. Then the following are equivalent

(1) there exist $c_1, \ldots, c_r \in \mathbb{C}$ such that $f = c_1 p_1 + \cdots + c_r p_r$,

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(2) we have the inclusion $I(p_1, \ldots, p_r) \subset f^\perp$, where $f^\perp$ is the subspace of $\mathbb{S}^\bullet V^*$ of el. which kill f via the Schur apolarity action.

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Idea of the proof. $(\Rightarrow)$ Assume that $f = c_1 p_1 + \dots + c_r p_r$. Then since every $I(p_i)$ kills p_i we get (2).

$(\Leftarrow)$ Assume that (2) holds. At first prove that $I(p_i) \cap \mathbb{S}_\lambda V = p_i^\perp \cap \mathbb{S}_\lambda V$. From this it follows that, since $I(p_1, \dots, p_r) \cap \mathbb{S}_\lambda V \subset f^\perp \cap \mathbb{S}_\lambda V$, we get $\langle f \rangle \subset \langle p_1, \dots, p_r \rangle$. $\square$

An example

Example - I

Consider the complete Flag variety $X = \mathbb{F}(1,2,3;\mathbb{C}^4)$ embedded with $\mathcal{O}(1,1,1)$ in $\mathbb{P}(\mathbb{S}_{(3,2,1)}\mathbb{C}^4)$.

We would like to compute the X-rank of the tensor

$$t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_3 - v_1 \wedge v_2 \wedge v_3 \otimes v_2 \wedge v_3 \otimes v_1.$$

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Suppose its X-rank is 1, i.e. it represents a flag $V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4$. Hence look for $V_3^\perp$, $\operatorname{Sym}^2 V_2^\perp$ and $\operatorname{Sym}^3 V_1^\perp$ inside $t^\perp$. Since

$$\ker(\varphi^{(3,2,1),(1)}) = \langle x_4 \rangle$$

we may assume that $V_3 = \{x_4 = 0\}$.

Example - II

Now we want to check if $\mathrm{Sym}^2 V_2^\perp \subset t^\perp$. Since $V_3^\perp \subset V_2^\perp$ we may assume that $V_2^\perp = \langle x_4, l \rangle$ and hence we must find l^2 in $ker(\varphi^{(3,2,1),(2)})$. Since

$$ker(\varphi^{(3,2,1),(2)}) = \langle x_1 x_4, x_2 x_4, x_3 x_4, x_4^2 \rangle$$

we conclude that there is no $l^2 \neq x_4^2$ in this kernel and hence t has not X-rank 1.

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Now we want to check if $\mathrm{Sym}^2 V_2^{\perp} \subset t^{\perp}$. Since $V_3^{\perp} \subset V_2^{\perp}$ we may assume that $V_2^{\perp} = \langle x_4, l \rangle$ and hence we must find l^2 in $ker(\varphi^{(3,2,1),(2)})$. Since

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we conclude that there is no $l^2 \neq x_4^2$ in this kernel and hence t has not X-rank 1. Suppose it has X-rank 2 and the associated flags are

$$V_1 \subset V_2 \subset \{x_4 = 0\},\ \ W_1 \subset W_2 \subset \{x_4 = 0\}.$$

We can note that if $V_2^{\perp} = \langle x_4, x_1 - x_3 \rangle$ and $W_2^{\perp} = \langle x_4, x_1 + x_3 \rangle$, then

$$\mathrm{Sym}^2 V_2^{\perp} \cap \mathrm{Sym}^2 W_2^{\perp} = \langle x_4 \rangle$$

is contained in $t^{\perp}$.

Example - III

Similarly one can check that given $V_1^\perp = \langle x_4, x_1 - x_3, x_2 \rangle$ and $W_1^\perp = \langle x_4, x_1 + x_3, x_2 \rangle$, then

$$\mathrm{Sym}^3 V_1^\perp \cap \mathrm{Sym}^3 W_1^\perp = \langle x_4^3, x_2 x_4^2, x_2^2 x_4, x_2^3 \rangle$$

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Example - III

Similarly one can check that given $V_1^\perp = \langle x_4, x_1 - x_3, x_2\rangle$ and $W_1^\perp = \langle x_4, x_1 + x_3, x_2\rangle$, then

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is contained in $t^\perp$. At this point we may try to decompose t as a sum of two points related to these flags

$$\begin{aligned} t = a(v_1 + v_3) \wedge v_2 \wedge v_3 \otimes (v_1 + v_3) \wedge v_2 \otimes (v_1 + v_3) + \\ + b(v_1 - v_3) \wedge v_2 \wedge v_3 \otimes (v_1 - v_3) \wedge v_2 \otimes (v_1 - v_3) \end{aligned}$$

and check that we have the equality for $a = -b = \frac{1}{2}$. Hence t has X-rank 2.

Conclusion

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Questions & work in progress:

- does there exist a way to build an apolarity with the same features of the classic and skew-symmetric case?
- definition of an algorithm which distinguishes tensors of border X-rank 2 where $X = (\mathbb{G}(k, V), \mathcal{O}(d))$.
- does this apolarity give informations about the dimension of secant varieties of rational homogeneous varieties?

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Thanks for the attention!