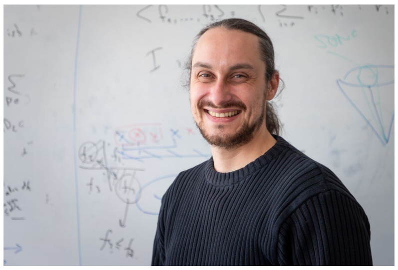
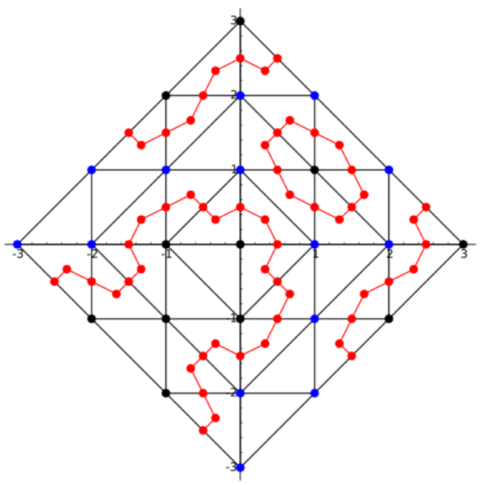
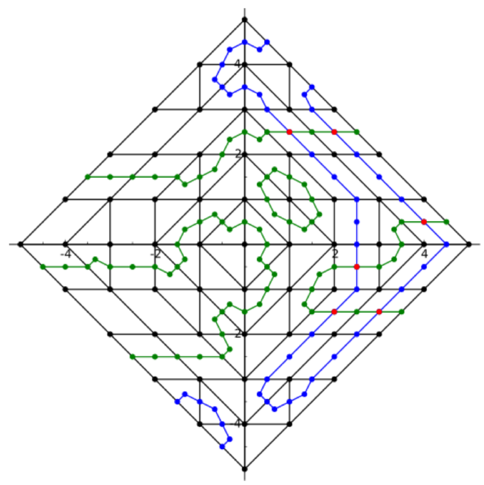
two patchworked people and curves







Given two equations $f \in \mathbb{R}[x,y]$ and $g \in \mathbb{R}[x,y]$, decide if I can create a patchworking that is isotopic to the common real zero set of f and g.

Why? ⓪ Love of geometry.

① Suppose f and g has both m terms. How many real zeros are there in

$Z(f) \cap Z(g)$?

In general all we know is $O(2^{m^2})$.

For patchworked ones: $4 \cdot m^2$. :)

② Computational complexity

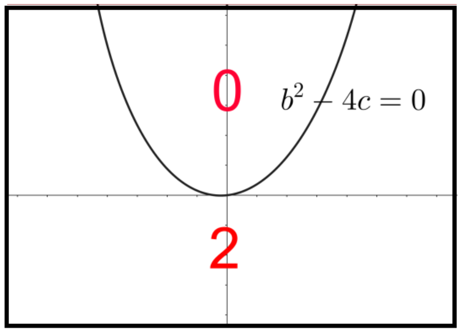
(not for two variables) in general real root finding is HARD.

Design algorithms that can sense tractable equations.

(3) Practical purposes + Diverse tool box

We don't know which applications create hard equations for real zero finding, and which applications seems to create easy equations.

It might get interesting, so this is probably a good time to close this video and pick a better one from netflix.



$x^2 + bx + c = 0$

from your high school algebra class.

Becomes a page filling equation very quickly!

Here are some ideas from GKZ:

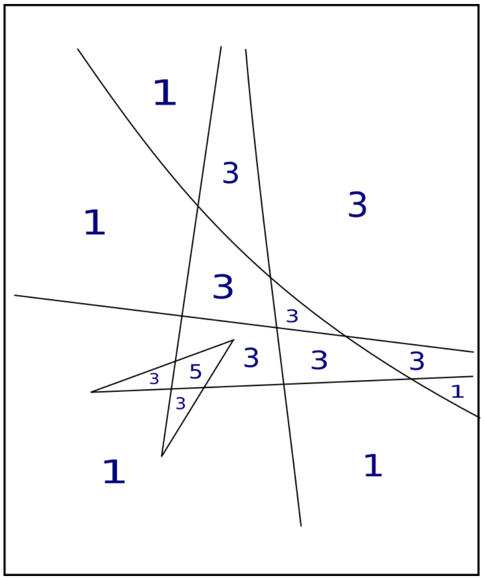
① Δ_A be the discriminant corresponding to sparse support set A.

$$A = \{a_1, a_2, \cdots, a_m\} \subseteq \mathbb{Z}^n$$

∇_A be the discriminant variety on $(\mathbb{C}^*)^m$

② $\mathrm{Log} : (\mathbb{C}^*)^m \longrightarrow \mathbb{R}^m$

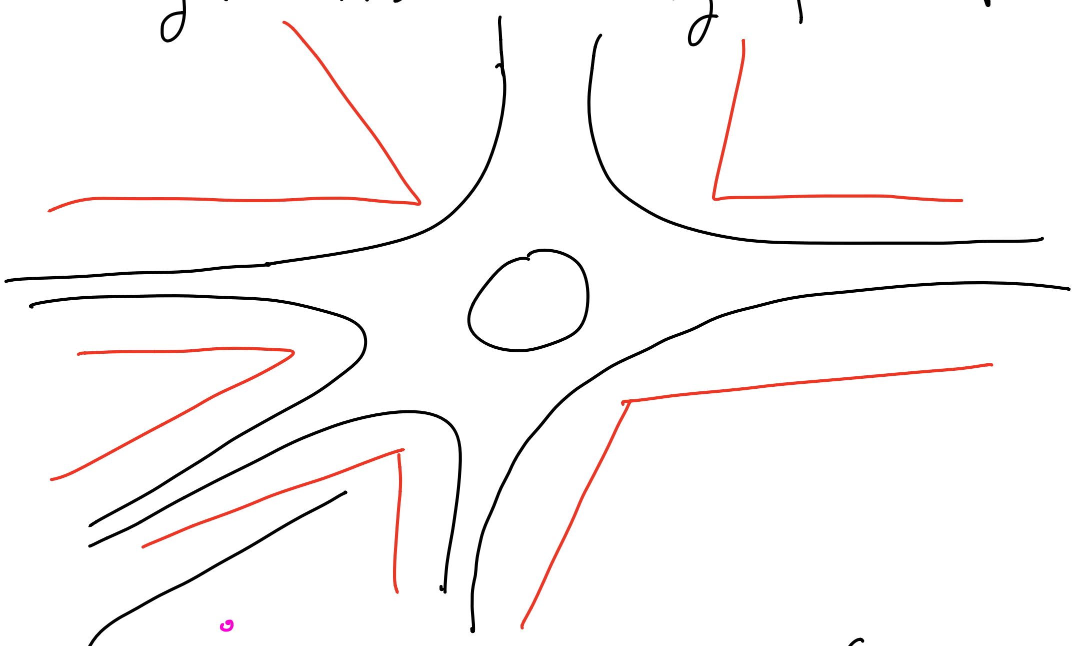
$$(z_1, \cdots, z_m) \longrightarrow (\log|z_1|, \cdots, \log|z_m|)$$

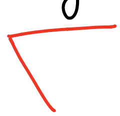


$\mathrm{Log}(\nabla_A)$ starts looking nice!

③ In general,

$\mathrm{Log}(Z(f))$ for any equation f



convex regions with a copy of normal cone  included.

4 For ∇_A, there is more (GKZ + Passare & Tsikh)

$$A = \{a_1, a_2, \ldots, a_m\} \subseteq \mathbb{Z}^n.$$

$$\hat{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ 1 & 1 & & 1 \end{bmatrix} \qquad (n+1) \times m$$

$B \quad m \times (m-n-1) \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$$\underbrace{\text{change of variables}}^{n} + \overbrace{\text{scaling}}^{1} = n+1$$

B captures "homogeneities" in ∇_A.

$$\boxed{\text{I care about } \operatorname{Log}(D_A(\mathbb{R})).}$$

$$\boxed{\text{I can even do } B^T \operatorname{Log}(D_A(\mathbb{R}))}$$

why?

$$\hat{A} B = 0 \Rightarrow B^T \hat{A}^T = 0$$

the action of $(\mathbb{C}^*)^n$ and scaling is represented by $\hat{A}^T$.

Now:

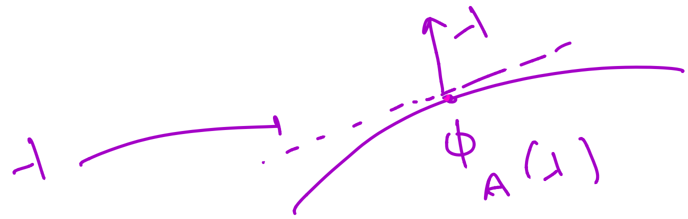
$$\phi_A : (\mathbb{R}^*)^{m-n-1} \longrightarrow (\mathbb{R}^*)^{m-n-1}$$

$$\boxed{\phi_A(\lambda) = \sum_{i=1}^{m} b_i \log |\langle b_i, \lambda \rangle|.}$$

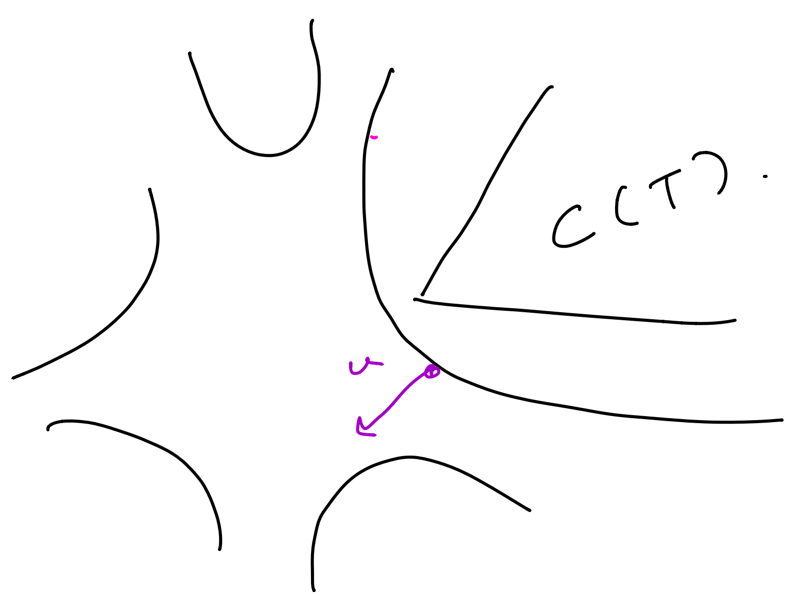
$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad b_i \in \mathbb{Z}^{m-n-1}$$

1-) $\boxed{B^T \operatorname{Log}(\nabla_A(\mathbb{R})) \subseteq \phi_A((\mathbb{R}^*)^{m-n-1})}$

2-) $\boxed{\text{normal direction at } \phi_A(\lambda) \text{ is } \lambda.}$



proved by Kapranov in 91.



$u \in C(T)^{\circ}.$

just by convexity.

we know B^T (secondary cone)

B^T (mixed cell cone).

is in there!

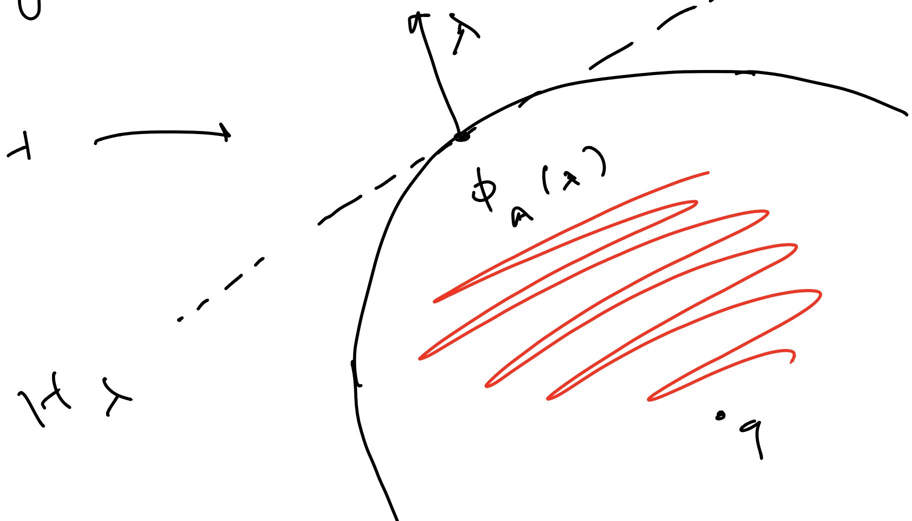
Say $M(T)$ = mixed cell cone of a triangulation T.

$$u \in \left(B^T M(T)\right)^{\circ}$$

$$\langle u, B^T v\rangle = \langle Bu, v\rangle \geq 0$$

$$\Rightarrow \quad Bu \in M(T)^{\circ}.$$

Summary:



$$H_\lambda := \{ x \in \mathbb{R}^{m-n-1} : \langle \lambda, x \rangle = \langle \lambda, \phi_A(\lambda) \rangle \}$$

we realized $\lambda \in M(\tau)^\circ$.

to check if $q \in$ [shaded region] .

check

$$\langle \lambda, q \rangle > \langle \lambda, \phi_A(\lambda) \rangle$$

for all $\lambda \in M(\tau)^\circ$.

$$\phi_A(\lambda) = \sum_{i=1}^{m} b_i \cdot \log |\langle b_i, \lambda \rangle|$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad b_i \text{ are rows.}$$

$$\langle \lambda, \phi_A(\lambda) \rangle = \sum_{i=1}^{m} \langle b_i, \lambda \rangle \log |\langle b_i, \lambda \rangle|$$

Entropy of $B\lambda$!

Recall:

$$(1 \ 1 \ \ldots \ 1) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = 0$$

that is, $$\sum_{i=1}^{m} b_i = 0$$

We have:

$$|\langle \phi_A(\lambda), \lambda \rangle| < \log(m) \cdot \|B\lambda\|_1 .$$

$B\lambda \in M(T)^\circ$. (remember??).

So?

- $\boxed{f = (f_1, f_2, \dots, f_n)}$ input polynomial with support sets $\boxed{(A_1, A_2, \dots, A_n)}$. coefficients $\boxed{(C_1, C_2, \dots, C_n)}$.
- $\boxed{A = \text{Cayley polytope}(A_1, \dots, A_n)}$. $\boxed{C = (C_1, C_2, \dots, C_n)}$.
- Create a triangulation $\boxed{T}$ of $\boxed{A}$ using $\boxed{\text{Log}(C)}$ as a lifting function.

- Check if

$$\langle \underline{\mathrm{Log}(C)}, \underline{u} \rangle \geq \boxed{\log(mn) \cdot \|u\|_1}$$

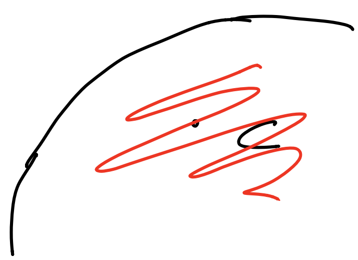
for generators of $M(T)^\circ$.

(circuit inequalities)

(computed by Jensen's algorithm)

- If coefficient vector C

has



good news, start from Viro's

patchworking of T with f

track only real zeros to f

by numerical path trackers.