Positive Ulrich Sheaves

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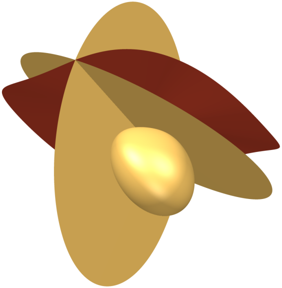

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Ulrich sheaves

Definition Let $f : X \to Y$ be a morphism of schemes. A coherent sheaf $\mathcal{F}$ on X is called *f-Ulrich* if $f_*\mathcal{F} \cong \mathcal{O}_Y^r$ for some $r > 0$.

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Example Let $X \subset \mathbb{P}^n$ be a variety of dimension d. Let $E \subset \mathbb{P}^n$ be a linear subspace of dimension $n-d-1$ such that $E \cap X = \emptyset$ and consider the linear projection $\pi_E : X \to \mathbb{P}^d$. Let $\mathcal{F}$ be a coherent sheaf on X.

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- $\mathcal{F}$ is π_E-Ulrich,
- $H^k(X, \mathcal{F}(-i)) = 0$ for $1 \leq i \leq d$ and all k,
- the associated graded module over the polynomial ring has a linear free resolution of length $n - d$.

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In this case we just say that $\mathcal{F}$ is Ulrich.

Boij–Söderberg theory

Let $X \subset \mathbb{P}^n$ be a smooth projective variety.

Problem

Classify the vector bundles $\mathcal{F}$ on X.

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Let $\pi : X \to \mathbb{P}^d$ some finite surjective linear projection. Since π is affine, we have $H^i(\mathcal{F}, X) = H^i(\pi_*\mathcal{F}, \mathbb{P}^d)$ for all coherent sheaves $\mathcal{F}$ on X.

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If $\mathcal{F}$ is Ulrich on X and $\mathcal{G}$ is a vector bundle on $\mathbb{P}^d$, then
$H^i(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}, X) = H^i(\pi_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^d}} \mathcal{G}, \mathbb{P}^d) = H^i(\mathcal{G}^r, \mathbb{P}^d)$.

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Does every projective variety have an Ulrich sheaf?

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- Complete intersections (Herzog–Ulrich–Backelin)
- Curves (Eisenbud–Schreyer)
- Determinantal varieties (Bruns–Römer–Wiebe)

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Let $f : X \to Y$ be a finite surjective morphism of projective varieties over a field K. Let $\mathcal{F}_1, \mathcal{F}_2$ be torsion-free coherent sheaves on X. Consider a morphism of sheaves on X:

$$\psi : \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \to f^! \mathcal{O}_Y.$$

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Thus ψ induces a K-bilinear form $\overline{\psi} : V_1 \times V_2 \to K$ where V_i is the space of global sections of $\mathcal{F}_i$.

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If $\overline{\psi}$ is a perfect pairing and $\dim(V_1) \geq \deg(f) \cdot \operatorname{rank}(\mathcal{F}_1)$, then $\mathcal{F}_1$ is f-Ulrich.

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Remark

If $\mathcal{F}$ is f-Ulrich, then the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^! \mathcal{O}_Y) \to f^! \mathcal{O}_Y$$

satisfies the assumptions of the theorem.

Positive Ulrich sheaves

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Now $K = \mathbb{R}$.
If $\mathcal{F}_1 = \mathcal{F}_2$ and $\overline{\psi}$ is symmetric and positive definite, then we say that $\mathcal{F}_1$ is a *positive symmetric f-Ulrich sheaf.*

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Real-fibered morphisms

Theorem

Let $f : X \to Y$ be a morphism of real varieties. If there is a positive (symmetric or hermitian) f-Ulrich sheaf, then f is real-fibered in the sense that $f^{-1}(Y(\mathbb{R})) = X(\mathbb{R})$.

Real-fibered morphisms

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Problem

Which real-fibered morphisms admit a positive Ulrich sheaf?

Hyperbolic polynomials

Definition A homogeneous form $h \in \mathbb{R}[x_0, \ldots, x_n]_d$ of degree d is called *hyperbolic with respect to* $e \in \mathbb{R}^{n+1}$ if $h(te - a) \in \mathbb{R}[t]$ has only real zeros for all $a \in \mathbb{R}^{n+1}$.

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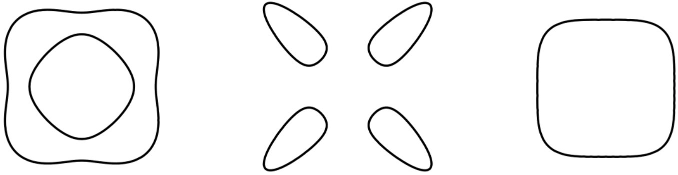

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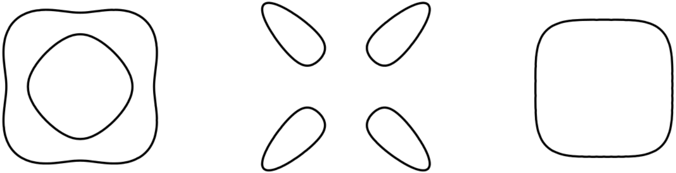

- Let h be hyperbolic with respect to e and $X = Z(h) \subset \mathbb{P}^n$. Then the linear projection $\pi_e : X \to \mathbb{P}^{n-1}$ from center e is real-fibered.

Positive Ulrich sheaves and determinantal representations

Let $h \in \mathbb{R}[x_0, \ldots, x_n]_d$ be hyperbolic with respect to $e = (1, 0, \ldots, 0)$ and cosider the zero set X of h in $\mathbb{P}^n$. Let $\pi_e : X \to \mathbb{P}^{n-1}$ be the linear projection from e. There is a positive symmetric/hermitian π_e-Ulrich sheaf $\mathcal{F}$ of rank r if and only if $h^r = \det(x_0 I - (x_1 A_1 + \ldots + x_n A_n))$ for some real symmetric/hermitian matrices A_i. We say that h^r has a *definite symmetric/hermitian determinantal representation*.

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Sketch.

Take the representing matrix $A = A(x_1, \dots, x_n)$ of the map

$$\mathcal{O}_{\mathbb{P}^{n-1}}^{dr} \cong f_*\mathcal{F} \xrightarrow{\cdot x_0} f_*\mathcal{F}(1) \cong \mathcal{O}_{\mathbb{P}^{n-1}}^{dr}(1)$$

defined by multiplication with x_0. Then $h^r = \det(x_0 I - A)$. $\square$

Positive Ulrich sheaves and sums of squares

Let $p \in \mathbb{R}[x_0, \ldots, x_n]_{2d}$ be homogeneous of degree $2d$ and cosider the zero set X of $y^2 - p$ in $\mathbb{P}(d, 1, \ldots, 1)$. Let $f : X \to \mathbb{P}^n$ be the projection on the x-variables.

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- p is a sum of squares if and only if there is a positive hermitian f-Ulrich sheaf.

Positive Ulrich sheaves and Weil divisors

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If $2D = R + (g)$, *then there is a positive symmetric* f*-Ulrich sheaf of rank one.*

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If $D + \overline{D} = R + (g)$, then there is a positive hermitian f-Ulrich sheaf of rank one.

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- Let $Q' = 2N + (s)$ for some divisor N and $s \in \mathbb{C}(C)$.
- Then $R = Q + \overline{Q} = 2nP + Q' + \overline{Q'} = 2(N + \overline{N} + nP) + (s \cdot \overline{s})$.

Positive Ulrich sheaves on curves

Corollary (Helton–Vinnikov 2007)

Every hyperbolic polynomial $h \in \mathbb{R}[x, y, z]_d$ has a definite symmetric determinantal representation.

Positive Ulrich sheaves on del Pezzo surfaces

A *del Pezzo surface* is a smooth projective surface whose anticanonical class $-K_X$ is ample.

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Example

- Let $h \in \mathbb{R}[x_0, x_1, x_2, x_3]_3$ such that its zero set $X \subset \mathbb{P}^3$ is smooth. Then X is a del Pezzo surface embedded via the anticanonical linear system.

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Example

- Let $h \in \mathbb{R}[x_0, x_1, x_2, x_3]_3$ such that its zero set $X \subset \mathbb{P}^3$ is smooth. Then X is a del Pezzo surface embedded via the anticanonical linear system.
- Let $p \in \mathbb{R}[x_0, x_1, x_2]_4$ such that the zero set X of $y^2 - p$ in $\mathbb{P}(2, 1, 1, 1)$ is smooth. Then X is a del Pezzo surface and the projection $f : X \to \mathbb{P}^2$ onto the x-coordinates corresponds to its anticanonical linear system.

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Lemma

Let $f : X \to \mathbb{P}^2$ be a finite real fibered morphism from a del Pezzo surface X such that the corresponding linear system is anticanonical. There are disjoint (complex) lines L_1, L_2 on X such that with $D = L_2 - L_1 - K$ we have $D + \overline{D} = R + (s)$ for some $s \in \mathbb{R}(X)$ nonnegative on X.

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Riemann–Roch for surfaces:

$$\ell(D) + \ell(K-D) = \frac{1}{2} D.(D-K) + 1 + p_a + s(D) \geq \frac{1}{2} D.(D-K) + 1$$

$= K.K = \deg(f)$. Further $(-K).(K-D) = -2\deg(f) < 0$ thus $\ell(K-D) = 0$.

Positive Ulrich sheaves on del Pezzo surfaces

Corollary (Hilbert 1888)

Every nonnegative polynomial $p \in \mathbb{R}[x, y, z]_4$ *is a sum of (three) squares.*

Corollary (Buckley–Košir 2007)

Every hyperbolic polynomial $h \in \mathbb{R}[w, x, y, z]_3$ *has a definite hermitian determinantal representation.*