On the regularity of Cactus Schemes

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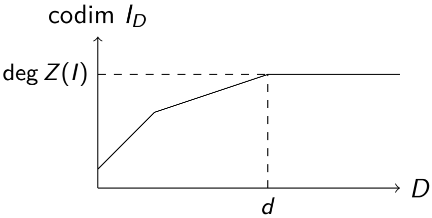
The problem

Geometric formulation

Among the zero-dimensional schemes Z apolar to a given degree-d form F, is it true that those of minimal degree are d-regular?

Algebraic formulation

Does the Hilbert function of a zero-dimensional ideal I, which is apolar to a given degree-d form F, stabilize in degree d?



Apolarity

Setting

$\mathbb{k} = \overline{\mathbb{k}}$, $\mathrm{char}(\mathbb{k}) = 0$, $\mathcal{S} = \mathbb{k}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} \mathcal{S}_d$.

Apolar ideal

The **apolar ideal** to $F \in \mathcal{S}_d$ is

$$F^\perp = \{H \in \mathcal{S} \mid H(\delta)(F) = 0\}$$

Example

In $\mathbb{k}[X, Y, Z]$ we have

$$(X^3 + X^2Y)^\perp = \langle X^3 - 3X^2Y, Y^2, Z \rangle.$$

Apolarity

Apolar schemes

A zero-dimensional scheme Z is said to be **apolar** to F if

$$I(Z) \subseteq F^{\perp}.$$

Cactus schemes

The **cactus rank** of F is the minimum degree of an apolar scheme of F. We call **cactus scheme** a scheme apolar to F that computes its cactus rank.

Example

The cactus rank of $X^3 + X^2Y$ is 2, and a cactus scheme is defined by the ideal

$$\langle Y^2, Z\rangle \subsetneq (X^3 + X^2Y)^{\perp}.$$

Generalized Additive Decomposition

Generalized additive decomposition (GAD)

Let $F \in \mathcal{S}_d$ and let $L_1, \ldots, L_s \in \mathcal{S}_1$ be different linear forms. A **generalized additive decomposition** (GAD) of F **supported at** $(L_1, \ldots, L_s)$ is an expression

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i, \quad \text{where } 0 \leq k_i \leq d, \text{ for all } i \in \{1, \ldots, s\},$$

where L_i does not divide G_i, for each $i \in \{1, \ldots, s\}$.

Generalized Additive Decomposition

Example

Let us indicate the supports with the blue color.

	GAD
$X^3 + X^2Y = (X)^3 \cdot 1 + (Y) \cdot X^2$	✓
$= (X)^3 \cdot 1 + (X)^2 \cdot Y$	×
$= (X)^2 \cdot (X + Y)$	✓
$= (X) \cdot (X^2 + XY)$	×
$= (X - Y)^0 \cdot (X^3 + X^2Y)$	✓
$= (X + Z)^0 \cdot (X^3 + X^2Y)$	✓

Natural scheme apolar to F at L

We associate a 0-dimensional scheme to a GAD [1, 2]:

Natural apolar scheme

Given a linear form $L \in \mathcal{S}_1$, we denote the de-homogenization of F with respect to L by f_L. The **affine natural scheme apolar to F at L** is defined by

$$Z^a_{F,L} = V(f_L^\perp),$$

and its homogenization $Z_{F,L}$ with respect to L is called the **natural scheme apolar to F at L**.

Fact [2, Corollary 4]

$Z_{F,L}$ is apolar to F.

Scheme evincing a GAD

Scheme evincing a GAD

We say that the scheme

$$Z = Z_1 \cup \ldots \cup Z_s, \qquad \text{with } Z_i = Z_{L_i^{d-k_i} G_i, L_i}$$

evinces the GAD

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i.$$

The **size** of the GAD is

$$\sum_{i=1}^{s} \deg(Z_i).$$

Scheme evincing a GAD

Example

Let us consider the (valid) GADs of $X^3 + X^2 Y$:

i) $(X)^3 \cdot 1 + (Y) \cdot X^2$ ii) $(X)^2 \cdot (X + Y)$

iii) $(X - Y)^0 \cdot (X^3 + X^2 Y)$ iv) $(X + Z)^0 \cdot (X^3 + X^2 Y)$

The schemes evincing them are

		Size	Reg
i)	$Z_{X^3,X} \cup Z_{X^2 Y,Y} = V(\langle Y, Z\rangle \cap \langle X^3, Z\rangle) = V(\langle X^3 Y, Z\rangle)$	4	3
ii)	$Z_{X^2(X+Y),X} = V(\langle Y^2, Z\rangle)$	2	1
iii)	$Z_{X^3+X^2Y,X-Y} = V(\langle (X+Y)^4, Z\rangle$	4	3
iv)	$Z_{X^3+X^2Y,X+Z} = V(\langle (X-Z)^2(X-3Y-Z), Y^2\rangle$	6	3

Regularity of schemes evincing GADs

Theorem

Let $F \in \mathcal{S}_d$ and Z be a scheme evincing one of its GADs. Then Z is regular in degree d.

Recall: regularity in degree d

The Hilbert function of $I(Z)$ stabilizes to

$$\dim(\Bbbk(\mathbf{X})/I(Z))_d = \deg(Z).$$

Sketch of the proof

Theorem

Let $F \in \mathcal{S}_d$ and Z be a scheme evincing one of its GADs. Then Z is regular in degree d.

Idea of the proof

- Local case: $Z_{L^{d-k}Q,L}$ is contained in the k-fat point supported at L.
- Merge local cases: inverse systems corresponding to different supports are linearly independent.
- To do it: read their elements as generalized eigenvectors, common to the same multiplication operators.

Corollaries

By [1, Theorem 3.7], the set of forms of degree d with a GAD of minimal size r coincides with the set of forms with cactus rank equal to r. Hence

Corollary

For every $F \in \mathcal{S}_d$ there exists a cactus scheme of F that is regular in degree d.

Example

The natural scheme apolar to $X^3 + X^2Y$ at X

$$V(\langle Y^2, Z \rangle)$$

is a cactus scheme that is regular in degree 1, hence in degree $d = 3$.

Bases to be tested in apolar decomposition algorithms

In [3, Section 6] we presented an algorithm to recover a GAD of minimal size for $F \in \mathcal{S}_d$, but we needed testing bases of $A = \Bbbk[\mathbf{X}]/I$ of degree up to $\sim r$ (the cactus rank).

By Corollary 1, only bases with degree up to d need to be tested.

Example

If we are dealing with a tensor in $\Bbbk[X, Y, Z]$ of degree $d = 4$ and rank $r = 7$, we do not need to test bases like

$$[1, Y, Z, Y^2, Y^3, Y^4, Y^5]$$

anymore.

Interpolation polynomials

Let I be a zero-dimensional ideal supported at $\{P_j\}_{1\leq j\leq s}$, namely its primary decomposition is

$$I = \bigcap_{1\leq j\leq s} Q_j, \qquad \sqrt{Q_j} = \mathfrak{m}_{P_j}.$$

We can always construct [4, Section 3] special interpolation polynomials $\{u_i\}_{1\leq i\leq s}$ such that

$$\begin{cases} u_i(P_j) = \delta_{i,j}, & \\ u_i^2 \equiv u_i & \in \Bbbk[\mathbf{X}]/I, \\ \sum_{i=1}^s u_i \equiv 1 & \in \Bbbk[\mathbf{X}]/I. \end{cases}$$

By construction, the degree of these u_i's may be assumed to be lower than the regularity of I, which in our setting is bounded by d.

On the regularity of *every* non-redundant scheme

Non-redundant schemes

A scheme Z apolar to F is called **non-redundant** if there are no proper subschemes $Z' \subsetneq Z$ apolar to F.

Notice: cactus implies non-redundancy.

Claim

Every non-redundant scheme Z apolar to $F \in \mathcal{S}_d$ is regular in degree d.

Idea of the proof: given $I \subseteq F^{\perp}$, we produce $I \subseteq J \subseteq F^{\perp}$ that evinces a GAD of F.

On the regularity of *every* non-redundant scheme

An extended example

Let us consider

$$F = X^3 + 3X^2Y + 3X^2Z + 3XY^2 + 12XYZ + Y^3 + 3Y^2Z \in \mathcal{S}_3,$$

and

$$\begin{aligned} I &= \langle Y, Z\rangle^3 \cap \langle X - Y, Z\rangle^2 \\ &= \langle X^2Y^3 - 2XY^4 + Y^5, XY^2Z - Y^3Z, YZ^2, Z^3\rangle. \end{aligned}$$

We have

$$I \subseteq F^\perp.$$

We want

$$I \subseteq J \subseteq F^\perp \quad \text{evincing a GAD of } F.$$

On the regularity of *every* non-redundant scheme

An extended example

We fill the Hankel matrix H_F of F in order to have $I \subseteq \ker H_F$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 & h_1 & h_2 & h_3 & h_4 \\ 1 & 2 & 0 & 1 & 0 & 0 & h_2 & h_3 & h_4 & h_5 \\ 1 & 1 & 1 & h_1 & h_2 & h_3 & h_6 & h_7 & h_8 & h_9 \\ 2 & 1 & 0 & h_2 & h_3 & h_4 & h_7 & h_8 & h_9 & h_{10} \\ 0 & 0 & 0 & h_3 & h_4 & h_5 & h_8 & h_9 & h_{10} & h_{11} \\ 1 & h_1 & h_2 & h_6 & h_7 & h_8 & h_{12} & h_{13} & h_{14} & h_{15} \\ 1 & h_2 & h_3 & h_7 & h_8 & h_9 & h_{13} & h_{14} & h_{15} & h_{16} \\ 0 & h_3 & h_4 & h_8 & h_9 & h_{10} & h_{14} & h_{15} & h_{16} & h_{17} \\ 0 & h_4 & h_5 & h_9 & h_{10} & h_{11} & h_{15} & h_{16} & h_{17} & h_{18} \end{bmatrix}$$

On the regularity of *every* non-redundant scheme

An extended example

We fill the Hankel matrix H_F of F in order to have $I \subseteq \ker H_F$:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 & \frac{h_{57}+7}{8} & 1 & 0 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & \frac{h_{57}+7}{8} & 1 & 0 & \frac{h_{57}+3}{4} & 1 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{h_{57}+7}{8} & 1 & 0 & 1 & 0 & \frac{3h_{57}+5}{8} & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{h_{57}+3}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h_{57}+3}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h_{57}+3}{4} & 0 & 0
\end{bmatrix}$$

On the regularity of *every* non-redundant scheme

An extended example

We fill the Hankel matrix H_F of F in order to have $I \subseteq \ker H_F$: $h_{57} = 1$

$$H_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

On the regularity of *every* non-redundant scheme

An extended example

The kernel of this matrix is

$$J' = \langle XY^2 - Y^3, Z^2 \rangle \supseteq I.$$

$V(J')$ evinces a GAD for an extension G of F:

$$G = \frac{1}{120}\left(30(X)^4 YZ + (X+Y)^5(X+Y+6Z)\right).$$

We obtain by derivation $\partial_X^3 G = F$ a GAD of F, which is evinced by a scheme defined by $V(J)$ for some $J \supseteq J'$ (in our example: $J = J'$):

$$V(J) \text{ evinces } F = 6XYZ + (X+Y)^2(X+Y+3Z).$$

Bibliography

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