Real-fibered morphisms of real del Pezzo surfaces

Joint with Mario Kummer and Cèdric Le Texier

$X/_{\mathbb{C}}$ non-singular alg variety of dim n

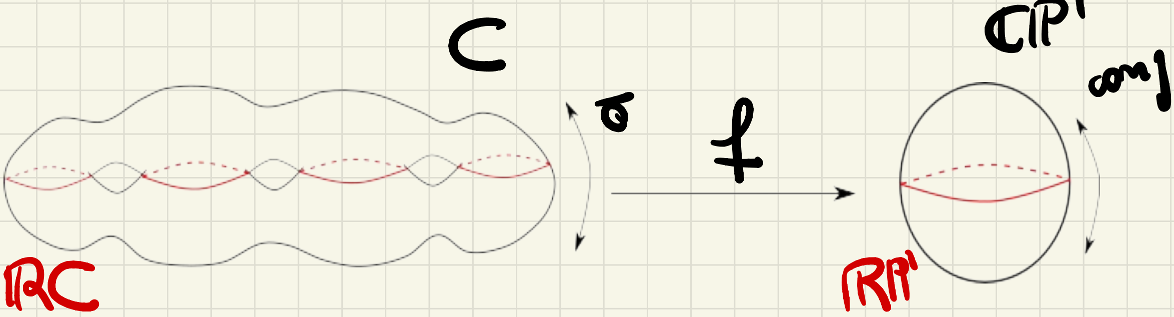
$\sigma : X \longrightarrow X$ real structure

$\mathrm{fix}(\sigma) = \mathbb{R}X$ real part

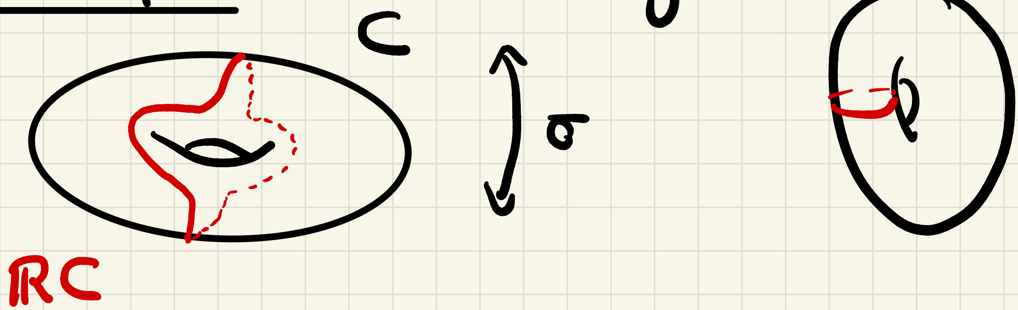
DEF: $f : X \circlearrowleft \sigma \longrightarrow \mathbb{CP}^n \circlearrowleft \mathrm{conj}$ real morphism, we say that f is <u>real-fibered</u> if

$$f^{-1}(\mathbb{RP}^n) = \mathbb{R}X$$

$m=1$ Real algebraic curves



example: real curve $g=1$



THEOREM (ALFHORS '50): $\exists\, f: C \to \mathbb{C}P^1$ real-fibered iff $\underbrace{\mathbb{C}C \setminus \mathbb{R}C}_{C \text{ separating}}$ is disconnected

$(\Leftrightarrow)$

$[\mathbb{R}C]$ trivial in $H_1(C; \mathbb{Z}/2\mathbb{Z})$

- $\forall\,(C, \sigma)$ separating

$l = \#$ c.c. of $\mathbb{R}C$ $\quad l \equiv g(C)+1 \bmod 2$

- $l = g(C)+1$

(Harnack-Klein inequality)

then C is maximal

Any real maximal curve is separating.

- Rokhlin, Klein, Arnold, Alfhors, Huisman, Gabard, Kummer-Shaw, Mikhalkin-Orevkov

. . . .

From $n=1$ to $n\geq 2$

$n=1$	$\exists\, C \to \mathbb{CP}^1$ real-fibered	C is separating
		$[\mathbb{R}C]$ trivial in $H_1(C;\mathbb{Z}/2\mathbb{Z})$
$n\geq 2$	(X,σ) $\exists\, f: X \to \mathbb{CP}^n$ real fibered, can we characterise X?	$\cancel{(\Leftrightarrow)}$
		Viro '06: (X,σ) bounds in complexification
		$[\mathbb{R}X]$ trivial in $H_n(X;\mathbb{Z}/2\mathbb{Z})$

$M \geq 2$ RIGIDITY

(X, σ) $\exists$ $f: X \to \mathbb{CP}^n$ real-fibred

Kummer-Shamovic '15:

$f|_{\mathbb{R}X}\ \mathbb{R}X \to \mathbb{RP}^n$ UNRAMIFIED

1) $\boxed{\mathbb{R}X \simeq \bigsqcup_s S^n \sqcup \bigcup_r \mathbb{RP}^n}$

2) $\deg f = 2s + r$

Rk: (X, σ) $\exists\ f: \tilde{X} \xrightarrow{p_1, p_2} \mathbb{CP}^m$ real fibred

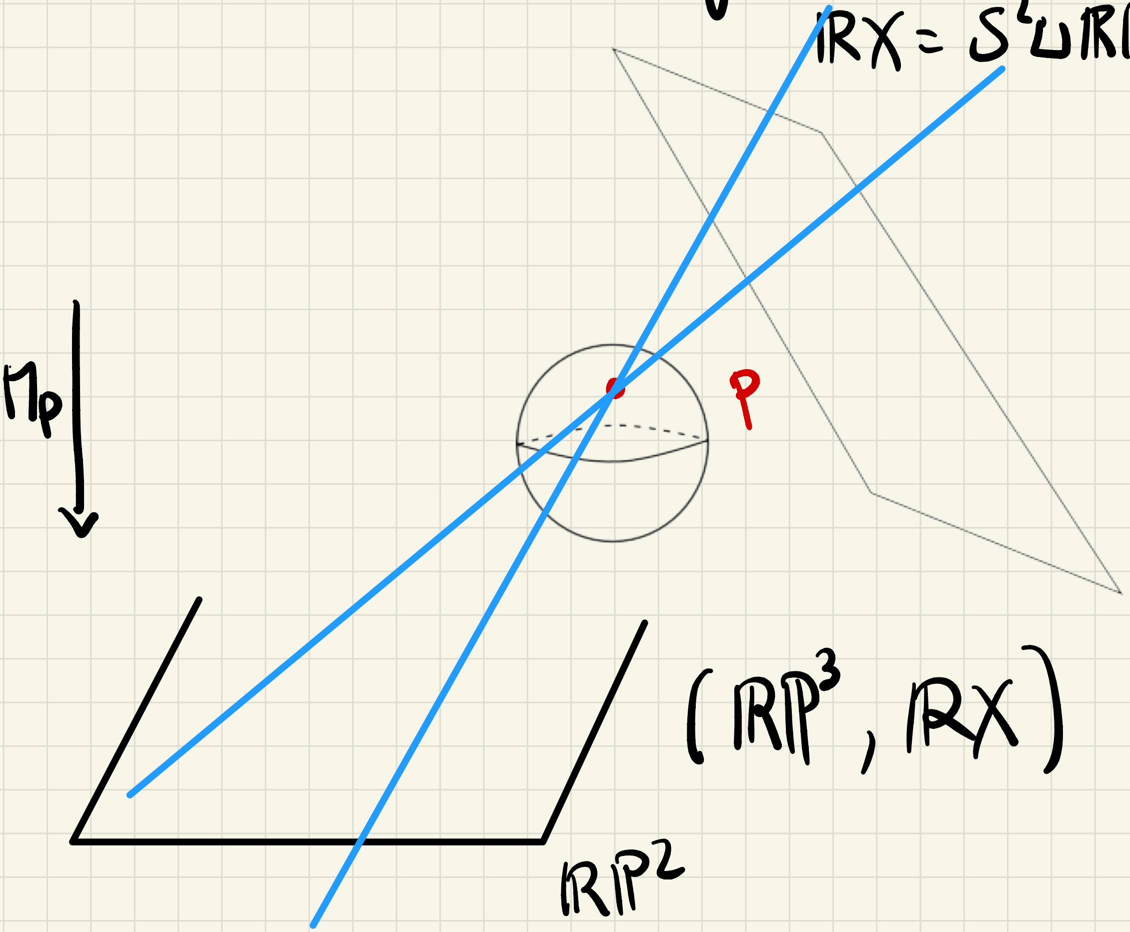
$\tilde{f}: Bl_{p_1, p_2}(X) \longrightarrow \mathbb{CP}^n$ real-fibred

p_1, p_2 cx conj

non-finite

example X cubic surface $\subseteq \mathbb{CP}^3$

$\mathbb{R}X = S^2 \sqcup \mathbb{RP}^2$



$(\mathbb{RP}^3, \mathbb{R}X)$

$\pi_p|_X \; X \longrightarrow \mathbb{CP}^2$ real-fib. of deg 3

DEF: $f: X \to \mathbb{CP}^m$ real fibered if $f: X \overset{i}{\hookrightarrow} \mathbb{CP}^n \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^m$ we say f is <u>hyperbolic</u> wrt E. X is <u>hyperbolic</u>.

(real very ample divisor $X \overset{i}{\hookrightarrow} \mathbb{CP}^n$)
($E \subset \mathbb{CP}^n$ $E \cap X = \emptyset$
$\pi_E|_X \; X \to \mathbb{CP}^m$ real fibered)

Del Pezzo surfaces

- Real classification known (COMESSATTI 1914-1913)
- $\exists\ (X,\sigma)$ del Pezzo s.t.
$$\mathbb{R}X \cong \bigsqcup_{s} S^2 \sqcup \bigsqcup_{r} \mathbb{R}\mathbb{P}^2$$

DEF: X non-sing alg surf.
we say that is del Pezzo
$-K_X$ is ample.

$$\deg X = (-K_X)^2 = d$$

$$0 \leq d \leq 9$$

<u>RK</u> : • $d=9$ $X=\mathbb{CP}^2$ $\mathbb{R}X=\mathbb{RP}^2$

• $d=8$ $X=\mathbb{CP}^1\times\mathbb{CP}^1$ $\sigma:(w,\eta)\longmapsto(\bar{\eta},\bar{w})$
$\mathbb{R}X=S^2$

• For some degrees d, $\forall X$ of deg d there are some $\sigma: X\to X$ s.t. $\mathbb{R}X\neq\bigsqcup_s S^2\sqcup\bigsqcup_r \mathbb{RP}^2$

• $1\leq d\leq 7$ there are (X,σ) s.t. $\mathbb{R}X$ has the right topology admits only real-fibered morphism $f:X\to\mathbb{CP}^2$ which are <u>non-finite</u>

<u>TH (KLTM) 1:</u> the classification of <u>finite</u> real-fibered morphisms of real del Pezzo surface X of deg d with $\mathbb{R}X\simeq\bigsqcup_s S^2\sqcup\bigsqcup_r\mathbb{RP}^2$ is as follows:

(X,σ) $\mathbb{R}X \cong \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$ $d = \deg X$

X	d	s	r	# real-fibered	HYPERBOLIC	h
X, $p \in \mathbb{R}X$	4	2	0	1 $-K_X$	✓	4
$Bl_p X$, $q \in \mathbb{R}Bl_pX$	3	1	1	1 $-K_X$	✓	3
$Bl_{p,q} X$	2	0	2	1 $-K_X$	✗	
	2	3	0	2 ↻ GEYSER	✓	
	2	4	0	1 $-2K_X$	✓	6
	1	2	1	4 2 pairs ↻ BERTINI	✓	
	1	3	1	2 ↻ BERTINI	✓	
	1	4	1	1 $-3K_X$	✓	6

Hyperbolic: $X \overset{i}{\hookrightarrow} \mathbb{CP}^h \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^2$

$h = s+2$ $r \in \{0,1\}$

• $\deg X = 2$ $\quad \mathbb{R}X = \mathbb{RP}^2 \sqcup \mathbb{RP}^2$

$$f : X \xrightarrow{2-1} \mathbb{CP}^2 \hookleftarrow Q \text{ plane quatic}$$

$$\mathbb{R}Q = \emptyset$$

$$\boxed{\text{STRATEGY HYPERBOLIC CASES:}}$$

(X, σ) del Pezzo surfaces $\mathbb{R}X \cong \sqcup_s S^2 \sqcup \sqcup_r \mathbb{RP}^2$

• If X admits $f : X \to \mathbb{CP}^2$ real-fibered FINITE then, let D real ample divisor ass. to f

• $\max\{1, r\} \leq D \cdot K_X + 4$

$\leq r + 2S$

• $r \equiv D \cdot K_X \mod 4$

- FINITE # of possible divisors which may give real fibered morphisms

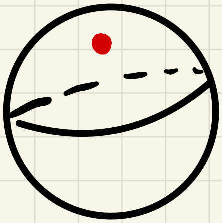
- very ampleness from Numerical criteria (di Rocco '96)

- look from lin. sub $E \subseteq \mathbb{CP}^n$ from which to project and obtain

$$\pi_E|_X : X \longrightarrow \mathbb{CP}^2$$

real fibred

$\mathbb{C}P^n \quad n > 3$



We were inside or outside of a sphere

$\rightsquigarrow$ LINKING NUMBERS to deal with this problem.