Mustafin Degenerations

Between applied and arithmetic geometry

Marvin Anas Hahn
(with Binglin Li and Annette Werner)

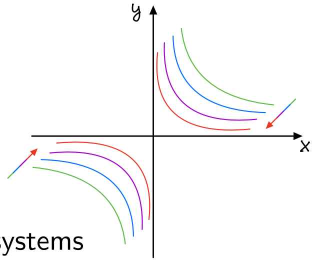
Max-Planck-Institut für
Mathematik
in den Naturwissenschaften

Effective Methods in Algebraic Geometry

June 2021

Introduction und context

Degenerations: Central technique in the study of algebraic varieties.



Mustafin degenerations:

- May be computed in computer algebra systems
- Have an interesting combinatorial structure

Introduction und context

Applications in several different areas of algebraic and arithmetic geometry.

1976 **Mustafin**: Introduced – Uniformisation in higher dimensions

2001 **Faltings**: Shimura varieties

2005 **Keel and Tevelev**: Chow quotients of Grassmannians

2011 **Cartwright, Häbich, Sturmfels and Werner**: Combinatorial interface with tropical geometry

2019 **He and Zhang:** Brill–Noether theory

Introduction und context

Today:

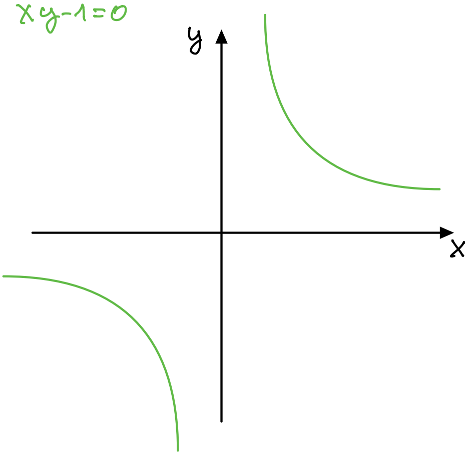
- Review some of the theory surrounding Mustafin degenerations.
- Give some classification results
- Show on a novel and curious connection to computer vision.

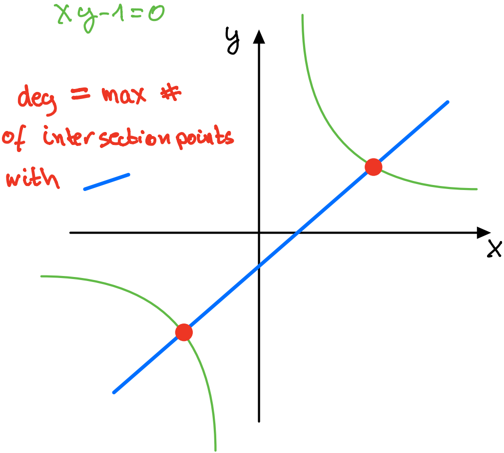
Contents

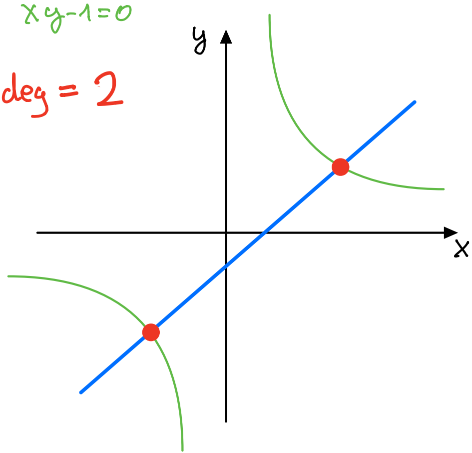
1. What is a degeneration?
2. Bruhat–Tits buildings and convex hulls
3. Classification of Mustafin degenerations
4. Families of multiview varieties

Contents

1. What is a degeneration?
2. Bruhat–Tits buildings and convex hulls
3. Classification of Mustafin degenerations
4. Families of multiview varieties

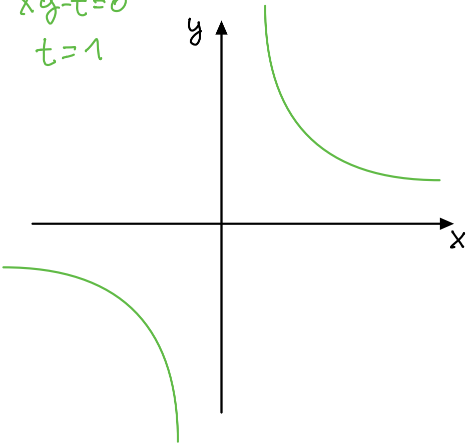

xy-1=0
y
x

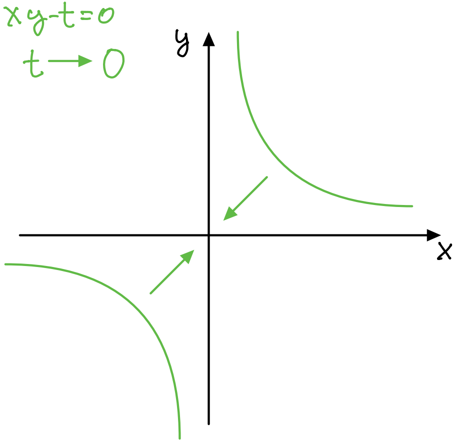

$xy-1=0$
y
x
deg = max #
of intersection points
with

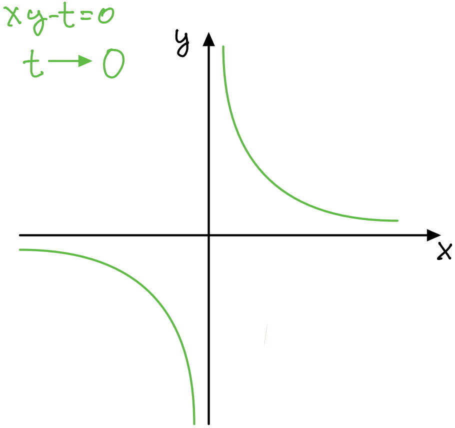

xy-1=0
deg = 2
y
x

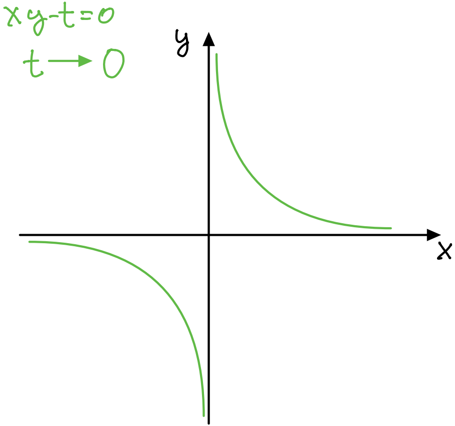
$xy - t = 0$

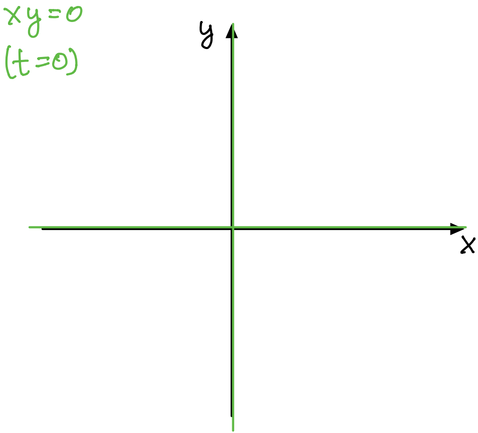
$t = 1$

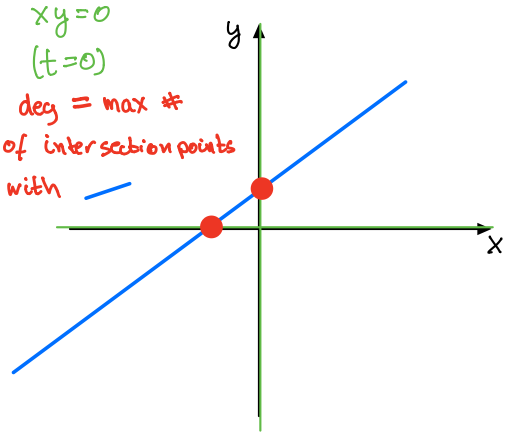


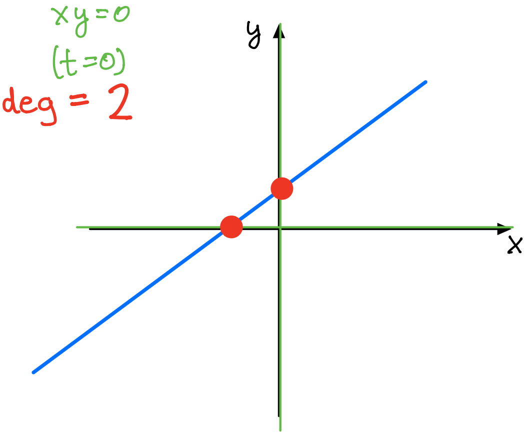

xy-t=0
t → 0
y
x


xy-t=0
t → 0
y
x


xy-t=0
t ⟶ 0
y
x

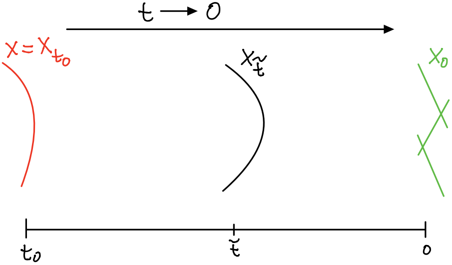

xy=0
(t=0)
y
x


$xy=0$
$(t=0)$
deg = max #
of intersection points
with
y
x


xy=0
(t=0)
deg = 2
y
x

Definition

A limiting process of a variety X, with limit X_0, that preserves geometric information, is called *degeneration from* X *to* X_0.



For us: Degenerations are flat and proper schemes over a discrete valuation ring.

Central question: Existence and construction of degenerations with given properties

Mustafin degenerations give an attractive and conceptual combinatorial framework.

Contents

1. What is a degeneration?
2. Bruhat–Tits buildings and convex hulls
3. Classification of Mustafin degenerations
4. Families of multiview varieties

Bruhat–Tits building

Mustafin degenerations are degenerations induced by combinatorial data in so-called **Bruhat–Tits buildings** that are central in the theory of algebraic groups.

Here: Focus on the building induced by $\mathrm{GL}(V)$, where V is a vector space over a discretely valued field.

To a valued field K one may associate a "coefficient field" k.
The field k is called residue field.

$$K \xrightarrow{\text{residue field}} k$$

Example

- We consider

$$K = \mathbb{C}((t)) = \left\{ \sum_{n \geq l} a_n t^n \mid a_n \in \mathbb{C},\, l \in \mathbb{Z} \right\}.$$

Thus, we obtain

$$k \cong \mathbb{C}.$$

Example

- Let $K = \mathbb{Q}_p$ the $p-$adic numbers for a prime p. For all $q \in \mathbb{Q}_p$, we have

$$q = \sum_{n \geq l} a_n p^n \quad \text{with} \quad a_n \in \{0, \ldots, p-1\}, \, l \in \mathbb{Z}.$$

Thus, we obtain

$$k \cong \mathbb{F}_p.$$

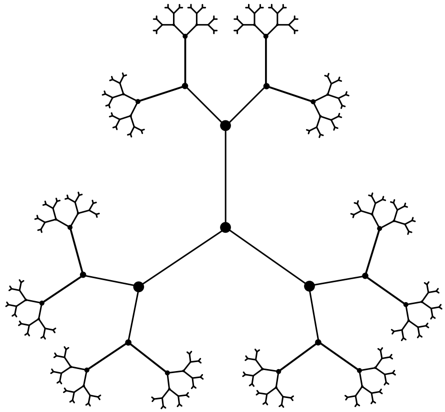
Bruhat–Tits building

Let V be a vector space over a discretely valued field K with $\dim(V) = d$.

Definition

The Bruhat–Tits building $\mathfrak{B}_d$ induced by $\mathrm{GL}(V)$ is the geometric realisation of a simplicial complex.
The vertex set is given by equivalence classes of invertible matrices $g \in \mathrm{GL}(V)$.

Bruhat–Tits building



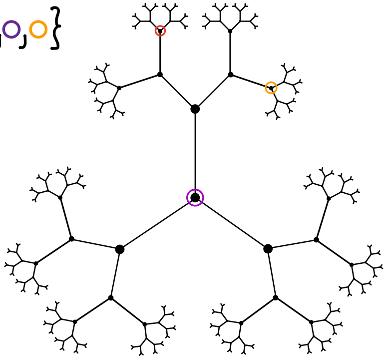
Local realisation of the building $\mathfrak{B}_2$ *for* $K = \mathbb{Q}_2$

Bruhat–Tits building

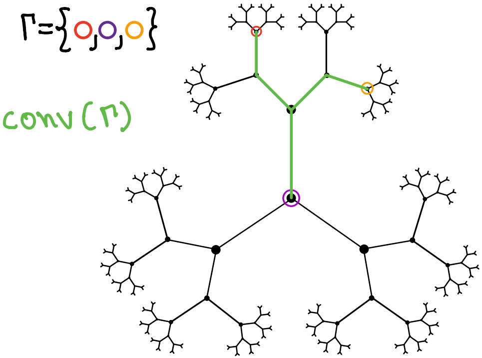
Definition

A finite set of vertices $\Gamma \subset \mathfrak{B}_d$ is called *point configuration*.

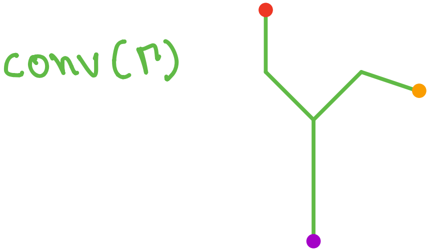
$\Gamma = \{\circ, \circ, \circ\}$



Analogously to polytopes: **Convex hulls** $\operatorname{conv}(\Gamma)$ of point configurations Γ.



Analogously to polytopes: **Convex hulls** $\mathrm{conv}(\Gamma)$ of point configurations Γ.



Theorem (Faltings'01)

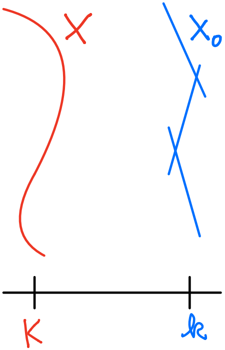
For any point configuration $\Gamma \subset \mathfrak{B}_d$, we have $\operatorname{conv}(\Gamma)$ is a finite simplicial complex.

Contents

1. What is a degeneration?
2. Bruhat–Tits buildings and convex hulls
3. Classification of Mustafin degenerations
4. Families of multiview varieties

Definition

A Mustafin degeneration $X(\Gamma)$ is a degeneration of $X \subset \mathbb{P}(V)$ induced by a point configuration $\Gamma \subset \mathfrak{B}_d$.
The limit X_0 is a variety over k.



Theorem (H.–Li'17)

Let $X = \mathbb{P}(V)$, $\Gamma \subset \mathfrak{B}_d$ a point configuration and X_0 the limit of the induced Mustafin degeneration. Then, the geometry of X_0 is determined by $\mathrm{conv}(\Gamma)$.

$$\begin{tikzcd} \Gamma \arrow[rr] \arrow[d] & & \mathrm{conv}(\Gamma) \arrow[dll, "\text{Complete classification}"] \\ X(\Gamma) & & \end{tikzcd}$$

Theorem (H.–Li'17)

Let $X = \mathbb{P}(V)$, $\Gamma \subset \mathfrak{B}_d$ a point configuration and X_0 the limit of the induced Mustafin degeneration. Then, the geometry of X_0 is determined by $\mathrm{conv}(\Gamma)$.

$$\begin{tikzcd}
\Gamma \arrow[rr] \arrow[d] & & \mathrm{conv}(\Gamma) \arrow[dll, "\substack{\text{Complete}\\\text{classification}}"] \\
X(\Gamma) & &
\end{tikzcd}$$

In H.–Werner ('19) and H. ('20), we generalise these results for X a curve.

Contents

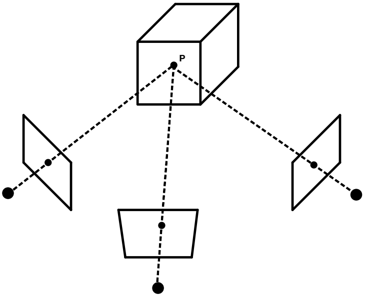
1. What is a degeneration?
2. Bruhat–Tits buildings and convex hulls
3. Classification of Mustafin degenerations
4. Families of multiview varieties

Multi-view varieties

In multi–view geometry, one considers the following computer vision problem:

Question

What are the properties of a real world scene, given several images of it?



For n images, the question is modelled by:

- Every camera i corresponds to a projection from three-space to two-space, i.e. a matrix $A_i \in \mathrm{Mat}(3 \times 4)$ with $\mathrm{rank}(A_i) = 3$ and map

$$A_i \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

- One then considers for $\underline{A} = (A_1, \ldots, A_n)$ the map

$$f_{\underline{A}} \colon \mathbb{P}^3 \xdashrightarrow{(A_1,\ldots,A_n)} \left(\mathbb{P}^2\right)^n$$

Definition

We call $M_{\underline{A}} = \overline{\mathrm{Im}(f_{\underline{A}})}$ the associated *multi-view variety*.

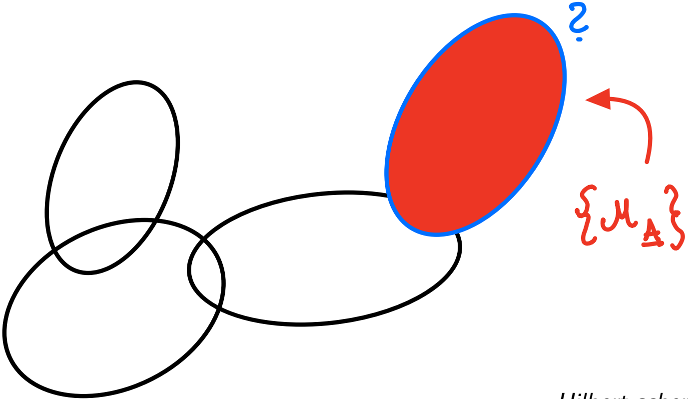
Multi-view varieties

Theorem (Aholt–Sturmfels–Thomas'13)

The space of multi-view varieties

$$\{M_{\underline{A}} \mid A_i \in \mathrm{Mat}(3 \times 4)\,\mathrm{with}\,\mathrm{rank}(A) = 3\,\mathrm{and\ generic}\}$$

lies dense in an irreducible component of the respective Hilbert scheme.



Hilbert scheme

Theorem (Aholt–Sturmfels–Thomas'13)

The space of multi-view varieties

$$\{M_{\underline{A}} \mid A_i \in \mathrm{Mat}(3 \times 4) \, \mathrm{with} \, \mathrm{rank}(A) = 3 \, \mathrm{and} \; \mathrm{generic}\}$$

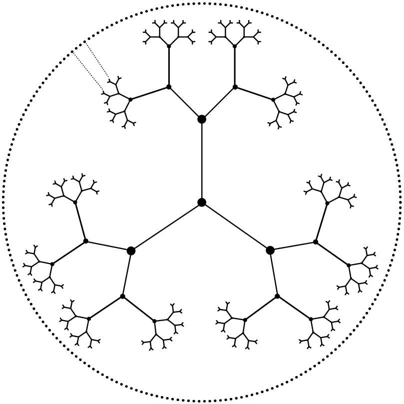
lies dense in an irreducible component of the respective Hilbert scheme.

Question

What does the boundary look like?

Theorem (Werner'05)

One may compactify the Bruhat–Tits building $\mathfrak{B}_d$ by full rank matrices of size $e \times d$ for $e \leq d$.



Multi-view varieties

Theorem (Werner'05)

One may compactify the Bruhat–Tits building $\mathfrak{B}_d$ by full rank matrices of size $e \times d$ for $e \leq d$.

- A point configuration Γ in the boundary of $\overline{\mathfrak{B}_4}$ induces a degeneration $\mathcal{MV}(\Gamma)$ of multi-view varieties that we call *generalised Mustafin degeneration*.
- The limit $\mathcal{MV}(\Gamma)_0$ for $t \to 0$ lies in the boundary.

Hope: The entire boundary is reached by this construction.

Multi-view varieties

Theorem (H.'20)

Let Γ be a point configuration in the compactified Bruhat–Tits building $\overline{\mathfrak{B}_4}$. Then, we obtain a partial classification of the irreducible components of $\mathcal{MV}(\Gamma)_0$.
Finally, we show that a very singular point of the Hilbert scheme is reached.