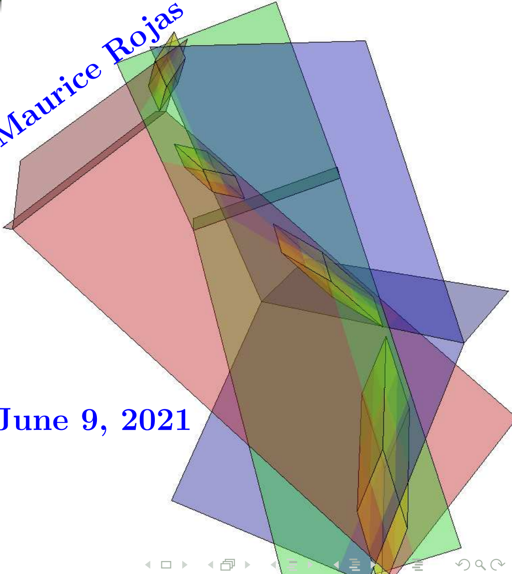
Counting Real Roots in Polynomial-Time for Systems Supported on Circuits

J. Maurice Rojas



June 9, 2021

1 Motivation & Background

1. Motivation & Background

2. Fewnomials and Number Theory

3. Faster Real Root Counting for Circuit Systems

We'd like to solve...

...systems like:

$$\begin{pmatrix}
2x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60}+x_1^{76}x_2^{240}x_4^{41}x_5+x_1^{74}x_2^{179}x_3^{25}x_5^{57}+x_1^{25}x_2^{203}x_3^{44}x_4+x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68}-37137x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25}-\frac{9}{2}x_3^{166}x_4^{68}x_5^{343}, \\
x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60}+2x_1^{76}x_2^{240}x_4^{41}x_5+x_1^{74}x_2^{179}x_3^{25}x_5^{57}+x_1^{25}x_2^{203}x_3^{44}x_4+x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68}-24849x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25}-\frac{21}{4}x_3^{166}x_4^{68}x_5^{343}, \\
x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60}+x_1^{76}x_2^{240}x_4^{41}x_5+2x_1^{74}x_2^{179}x_3^{25}x_5^{57}+x_1^{25}x_2^{203}x_3^{44}x_4+x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68}-21009x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25}-\frac{21}{4}x_3^{166}x_4^{68}x_5^{343}, \\
x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60}+x_1^{76}x_2^{240}x_4^{41}x_5+x_1^{74}x_2^{179}x_3^{25}x_5^{57}+2x_1^{25}x_2^{203}x_3^{44}x_4+x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68}-20769x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25}-\frac{21}{4}x_3^{166}x_4^{68}x_5^{343}, \\
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\end{pmatrix}$$

quickly.

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$$x_1^{36}x_2^{194}x_3^{50}x_4^{82}x_5^{60} + 2x_1^{76}x_2^{240}x_4^{41}x_5 + x_1^{74}x_2^{179}x_3^{25}x_5^{57} + x_1^{25}x_2^{203}x_3^{44}x_4 + x_1^{20}x_2^{167}x_3^{64}x_4^{12}x_5^{68} - 24849x_1^{58}x_2^{194}x_3^{24}x_4^{36}x_5^{25} - \frac{21}{4}x_3^{166}x_4^{68}x_5^{343},$$
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But how?

More importantly, why?

Polynomials Equations and Complexity Theory

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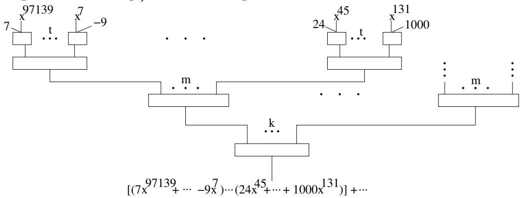

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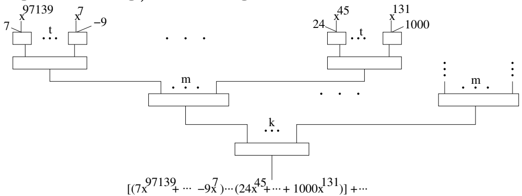

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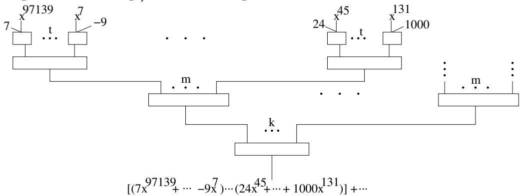

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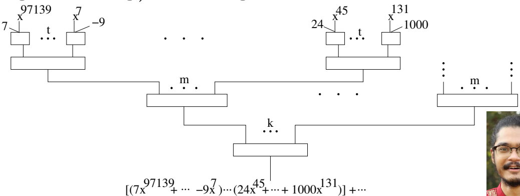

Fundamental Idea: Theorems about existence of roots are close to the **P** vs. **NP** Problem. Theorems about the deeper structure of polynomials are close to derandomization, i.e., the **P** vs. **BPP** Problem [Koiran, '11; Dutta, Saxena, Thierauf, '20].

Descartes' Rule, Biochemistry, and Learning Theory...




Recent work on chemical reaction networks

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But what about *exact counting*?

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- But what about systems?

Counting Roots for $(n+1)$-nomial $n \times n$ Systems is Easy, but...

Bonus Exercise. *Given* $[c_{i,j}] \in \{-H, \ldots, H\}^{n\times(n+1)}$,

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in $\mathbb{R}^n_+$, $(\mathbb{R}^*)^n$, *and* $\mathbb{R}^n$ *in time polynomial in* $n + \log(dH)$...

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$$\boxed{\begin{matrix} c_{1,1}x^{a_1}+\cdots+c_{1,n+1}x^{a_{n+1}}=0 \\ \vdots \\ c_{n,1}x^{a_1}+\cdots+c_{n,n+1}x^{a_{n+1}}=0 \end{matrix}}$$

in $\mathbb{R}^n_+$, $(\mathbb{R}^*)^n$, *and* $\mathbb{R}^n$ *in time polynomial in* $n+\log(dH)$...

- *Smith Factorization* helps,

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Over 245 million ***complex*** **roots!**

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[Bihan, Rojas, Sottile '07]: $\exists$ *systems with* $\left\lfloor \frac{n+k-1}{\min\{n,k-1\}} \right\rfloor^{\min\{n,k-1\}}$ *positive roots.*

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[Bertrand, Bihan, Sottile '06]: Tight *bound for* $k = 2$ *of* $\boxed{n+1}$.

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Theorem. [Bürgisser, Ergür, Tonelli-Cueto, 2019] *For an* $(n+k)$*-nomial* $n \times n$ *system with independent standard real Gaussian coefficients*

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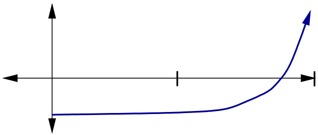
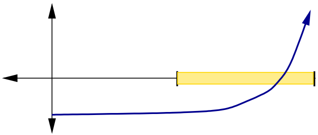


$$u^D \overset{?}{\substack{> \\ = \\ <}} cv^D$$



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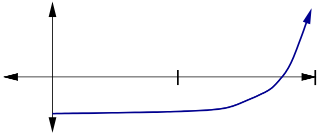
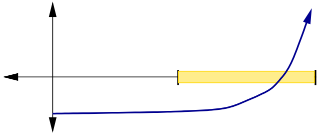


Trouble?



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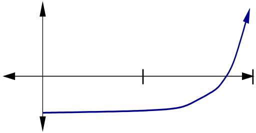
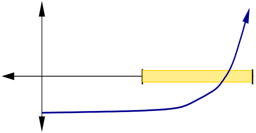


$$\boxed{D\log|u| \;\overset{?}{\substack{>\\=\\<}}\; \log|c| + D\log|v|}$$



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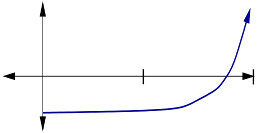

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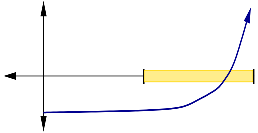

[Baker, 1966] If $a_i, b_i \in \mathbb{Z}$ with $A := \max_i \log |a_i|$, $B := \max_i \log |b_i|$, and $\Lambda := \sum_{i=1}^{n} b_i \log a_i$, then $\Lambda \neq 0 \Longrightarrow |\log |\Lambda|| = O(A)^n \log B$.

Real Univariate Trinomials?

• Deciding the sign of a trinomial $f \in \mathbb{Q}[x_1]$ at $z \in \mathbb{Q}$ in time $(\text{size}(f) + \text{height}(z))^{O(1)}$ is an open problem!

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- Despite nice progress by [Jindal, Sagraloff, 2017] on *coarse* approximants, real root counting in time $(t \log(dH))^{O(1)}$ is still an open problem!

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•Sufficiently refined versions of the *abc-Conjecture* can reduce the complexity to polynomial in n...

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By Gauss-Jordan Elimination applied to the linear combinations of monomials, our original 5×5 example can be reduced to

$$\begin{aligned}
x_1^{36}x_2^{194}x_3^{-116}x_4^{14}x_5^{-283} &= 16384cx_1^{58}x_2^{194}x_3^{-142}x_4^{-32}x_5^{-318} + \frac{1}{4} \\
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Next: Observe that the set of exponent vectors is a *circuit in the sense of combinatorics*: It is the union of a point and the vertex set of a simplex...

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provided some additional sign conditions are met...

Note: The exponents are usually *much* larger!

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Examine the graph of the linear combination of logarithms...

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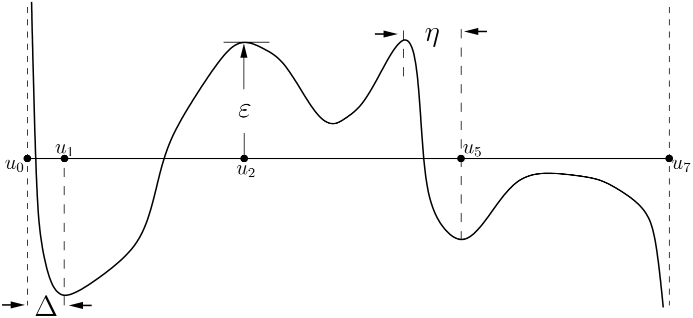

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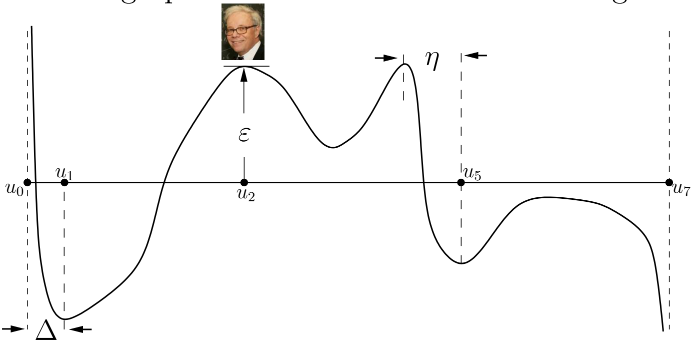

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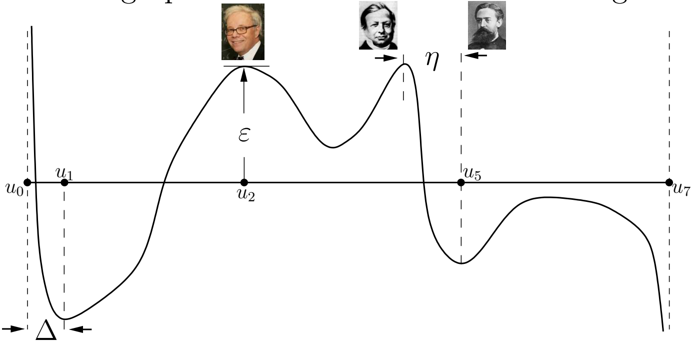

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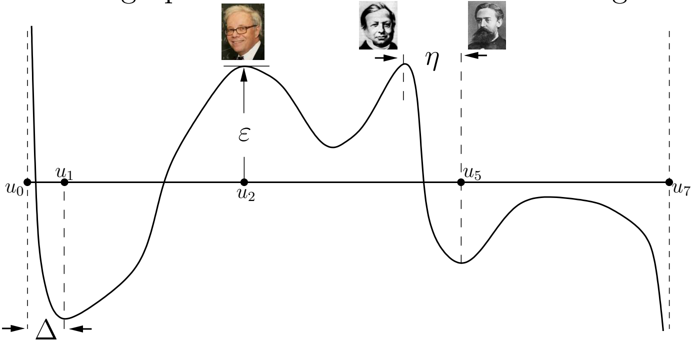

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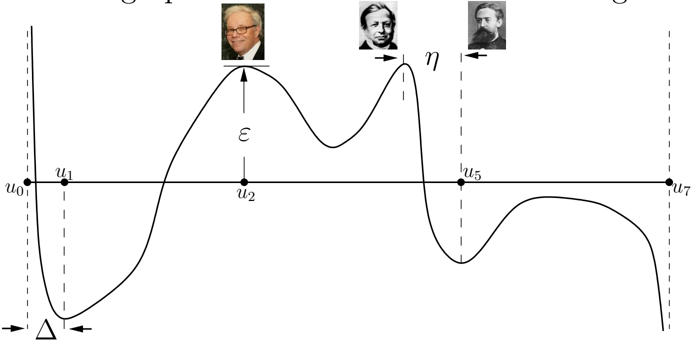

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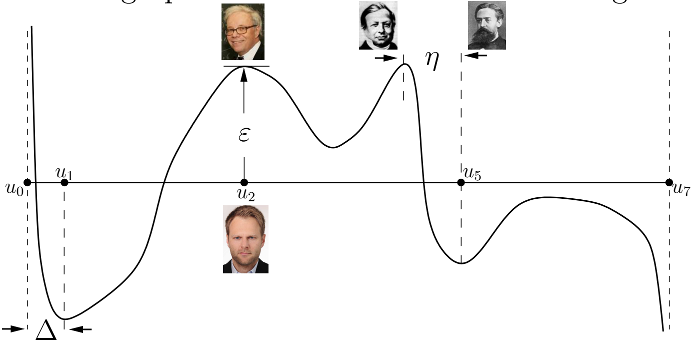

Baker-*Wustholtz* gives height of peaks... Bounds of Liouville and Markov control root spacing... Then use Rolle's Theorem, AGM Iteration for accurate logs, and Sturm-Habicht sequences to isolate critical points!...

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Even though there *are* $(n+1)$-nomial $n \times n$ systems,

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While we can now counting real roots in time $(\log(dH))^{O(n)}$, there is growing evidence that we can attain complexity $(n \log(DH))^{O(1)}$ *on average* [Deng, Ergür, Paouris, Rojas, 2021]   ...

♡ Thank you for your attention!

See `www.math.tamu.edu/~rojas` for preprints and further info...