Quadratic Isogeny Primes

github.com/barinderbanwait/quadratic_isogeny_primes
arxiv.org/abs/2101.02673 - submitted

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Isogenies

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ϕ is said to be K-rational if it is compatible with the G_K-action on E_1 and E_2; that is, if the following diagram commutes for all $\sigma \in G_K$:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "\sigma"'] & E_2 \arrow[d, "\sigma"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

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Slogan

You can identify an isogeny with its kernel.

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"Understand rational isogenies."

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Since we can identify isogenies with their kernels, which are finite abelian groups, which break up as a direct sum of cyclic groups, the above goal reduces to

Reduced Goal

"Understand rational isogenies with cyclic kernel."

Call these cyclic K-isogenies.

The Dream made precise

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A priori these could be infinite sets.

The Theorems of Mazur and Kenku

Barry C. Mazur

Monsur A. Kenku

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Theorem (Mazur, 1978)

$$\mathrm{IsogPrimeDeg}(\mathbb{Q}) = \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}.$$

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Theorem (Kenku, 1982)

$$\mathrm{IsogCyclicDeg}(\mathbb{Q}) = \{1 \leq N \leq 19\} \cup \{21, 25, 27, 37, 43, 67, 163\}.$$

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Theorem (B., 2021)

Assuming GRH, we have the following.

$$\mathrm{IsogPrimeDeg}(\mathbb{Q}(\sqrt{7})) = \mathrm{IsogPrimeDeg}(\mathbb{Q})$$
$$\mathrm{IsogPrimeDeg}(\mathbb{Q}(\sqrt{-10})) = \mathrm{IsogPrimeDeg}(\mathbb{Q})$$
$$\mathrm{IsogPrimeDeg}(\mathbb{Q}(\sqrt{5})) = \mathrm{IsogPrimeDeg}(\mathbb{Q}) \cup \{23, 47\}$$

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Let K be a quadratic field which is not imaginary quadratic of class number 1. Then there is an algorithm which computes a superset of $\mathrm{IsogPrimeDeg}(K)^*$ *as the union of three sets:*

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$$\mathrm{IsogPrimeDeg}(K) \subseteq \mathrm{PreTypeOneTwoPrimes}(K) \cup \mathrm{TypeOnePrimes}(K) \\ \cup \mathrm{TypeTwoPrimes}(K).$$

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Remark

If K is imaginary quadratic of class number one, then $\mathrm{IsogPrimeDeg}(K)$ *is infinite because of complex multiplication.*

Preview of the Main Calling Function

```python
def get_isogeny_primes(K, aux_prime_count, bound=1000, loop_only_j=True):

    # Start with some helpful user info

    print("\nFinding isogeny primes for {}\n".format(K))
    print("Number of auxiliary primes is {}\n".format(aux_prime_count))

    # Get and show TypeOnePrimes

    type_1_primes = get_type_1_primes(K, aux_prime_count=aux_prime_count,
                                      loop_only_j=loop_only_j)
    print("type_1_primes = {}\n".format(type_1_primes))

    # Get and show PreTypeOneTwoPrimes

    pre_type_one_two_primes = get_pre_type_one_two_primes(K,
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    print("pre_type_2_primes = {}\n".format(pre_type_one_two_primes))

    # Get and show TypeTwoPrimes

    type_2_primes = get_type_2_primes(K, bound=bound)
    print("type_2_primes = {}\n".format(type_2_primes))

    # Put them all together and sort the list before returning
    candidates = set.union(set(type_1_primes),
                           set(pre_type_one_two_primes),
                           set(type_2_primes))
    candidates = list(candidates)
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`github.com/barinderbanwait/quadratic_isogeny_primes`

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We'll have a live-demo of the command-line tool after giving an overview of the algorithm.

PreTypeOneTwoPrimes

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Isogenies of prime degree over number fields

FUMIYUKI MOMOSE
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Received 19 August 1993; accepted in final form 24 December 1993

In 1993, Momose classified isogenies into three types.

Momose's Classification of Isogenies into three types

Theorem (Momose)

Let K be a number field. Then there exists a constant $C_0 = C_0(K)$ such that for any prime $p > C_0$, and for any elliptic curve admitting a K-rational p-isogeny, the isogeny character λ falls into one of the following three types:

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PreTypeOneTwoPrimes

Fumiyuki Momose

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Slogan

If K is a quadratic field which is not imaginary quadratic of class number one, then there is a finite set of primes PreTypeOneTwoPrimes(K) outside of which the isogeny character is of Type 1 or 2.

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but we show that it's possible to take the primes dividing a handful of explicitly computable integers.

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Theorem (Momose, Theorem B)

Let K be a quadratic field which is not an imaginary quadratic field of class number 1. Then $\mathrm{IsogPrimeDeg}(K)$ is finite.

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Theorem (Momose, Theorem B)

Let K be a quadratic field which is not an imaginary quadratic field of class number 1. Then IsogPrimeDeg(K) *is finite.*

Henceforth, when we say an *isogeny-finite* K, we will mean K as above.

Fumiyuki Momose

Drilling into Momose's proof I - Finitely many ϵs

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LEMMA 1. *Assume that k is a Galois extension of* $\mathbf{Q}$ *and that the rational prime p is unramified in k. Then for a fixed prime $\mathfrak{p}$ of k lying over p, we have integers a_σ, $0 \leqslant a_\sigma \leqslant 12$, for $\sigma \in \mathrm{Gal}(k/\mathbf{Q})$ such that*

$$\lambda^{12}((\alpha)) \equiv \alpha^{\varepsilon} \pmod{\mathfrak{p}}$$

for $\varepsilon = \Sigma_\sigma a_\sigma \sigma$ and $\alpha \in k^\times$ prime to p.

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For quadratic K, we can identify ϵ as a pair $(a, b) := a + b\sigma$, for σ the non-trivial Galois element.

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For quadratic K, we can identify ϵ as a pair $(a, b) := a + b\sigma$, for σ the non-trivial Galois element.

REMARK 1. The integers $a_{\mathfrak{P}}$'s take the values $0, 12; 4, 8$ (only if the modular invariant $j(E) \equiv 0 \pmod{\mathfrak{p}}$ and $p \equiv 2 \pmod 3$); 6 (only if $j(E) \equiv 1728 \pmod{\mathfrak{p}}$ and $p \equiv 3 \pmod 4$) (cf. [Ma1], Chap. 3; [Ma2]).

```python
# The PreTypeOneTwo epsilons, with their types
EPSILONS_PRE_TYPE_1_2 = {

    (0,12): 'quadratic',
    (12,0): 'quadratic',

    (0,4): 'quartic',
    (0,8): 'quartic',
    (4,0): 'quartic',
    (4,4): 'quartic',
    (4,8): 'quartic',
    (4,12): 'quartic',
    (8,0): 'quartic',
    (8,4): 'quartic',
    (8,8): 'quartic',
    (8,12): 'quartic',
    (12,4): 'quartic',
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    (0,6) : 'sextic',
    (6,0) : 'sextic',
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Note that the three pairs $(0,0), (12,12), (6,6)$ are not declared here, because ...

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Type 1. λ^{12} *or* $(\lambda\theta_p^{-1})^{12}$ *is unramified.* $\Leftarrow \epsilon = (0,0)$ *or* $(12,12)$

Type 2. $\lambda^{12} = \theta_p^6$ *and* $p \equiv 3 \pmod 4$. $\Leftarrow \epsilon = (6,6)$

Type 3. *K contains the Hilbert class field H_L of an imaginary quadratic field. The rational prime p splits in L:*

$$p\mathcal{O}_L = \mathfrak{p}\bar{\mathfrak{p}}.$$

For any prime $\mathfrak{q}$ of K prime to $\mathfrak{p}$,

$$\lambda^{12}(\mathrm{Frob}_{\mathfrak{q}}) = \alpha^{12} \pmod{\mathfrak{p}}$$

for any $\alpha \in K^\times$ with $\alpha\mathcal{O}_L = \mathrm{Nm}_{k/L}(\mathfrak{q})$.

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To make this explicit ...

For every other ϵ, find the possible isogeny primes which have an isogeny character acting via ϵ.

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$$C(\epsilon, \mathfrak{q}) := \mathrm{lcm}(\{\mathrm{Nm}_{K(\beta)/\mathbb{Q}}(\alpha^{\epsilon} - \beta^{12h_K}) \mid \beta \text{ is a Frobenius root over } \mathbb{F}_{\mathfrak{q}}\}).$$

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for Aux a finite set of good **auxiliary primes**.

A quick glance at the implementation

```
def get_AB_primes(K, q, epsilons, q_class_group_order):

    output_dict_AB = {}
    alphas = (q ** q_class_group_order).gens_reduced()
    assert len(alphas) == 1, "q^q_class_group_order not principal, which is very bad"
    alpha = alphas[0]
    rat_q = ZZ(q.norm())
    assert rat_q.is_prime(), "somehow the degree 1 prime is not prime"
    for eps in epsilons:
        alpha_to_eps = group_ring_exp(alpha,eps)
        A = (alpha_to_eps - 1).norm()
        B = (alpha_to_eps - (rat_q ** (12 * q_class_group_order))).norm()
        output_dict_AB[eps] = lcm(A,B)
    return output_dict_AB
```

```
for frob_poly in frob_polys_to_loop:
    if frob_poly.is_irreducible():
        frob_poly_root_field = frob_poly.root_field('a')
        _, K_into_KL, L_into_KL, _ = K.composite_fields(frob_poly_root_field, 'c', bot
    else:
        frob_poly_root_field = IntegerRing()
    roots_of_frob = frob_poly.roots(frob_poly_root_field)
    betas = [r for r,e in roots_of_frob]

    for beta in betas:
        if beta in K:
            for eps in epsilons:
                N = (group_ring_exp(alpha, eps) - beta ** (12*q_class_group_order)).ab
                N = ZZ(N)
                output_dict_C[eps] = lcm(output_dict_C[eps], N)
        else:
            for eps in epsilons:
                N = (K_into_KL(group_ring_exp(alpha, eps)) - L_into_KL(beta ** (12*q_c
                N = ZZ(N)
                output_dict_C[eps] = lcm(output_dict_C[eps], N)
return output_dict_C
```

TypeTwoPrimes

Condition CC ...

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Condition CC (Momose + ϵ)

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Let K be an isogeny-finite quadratic field, and E/K an elliptic curve admitting a K-rational p-isogeny, with p of Type 2. Let q be a rational prime $< p/4$ such that $q^2 + q + 1 \not\equiv 0 \pmod{p}$. Then the following implication holds:

if q splits or ramifies in K, then q does not split in $\mathbb{Q}(\sqrt{-p})$.

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Let K be an isogeny-finite quadratic field, and E/K an elliptic curve admitting a K-rational p-isogeny, with p of Type 2. Let q be a rational prime $< p/4$ such that $q^2 + q + 1 \not\equiv 0 \pmod{p}$. Then the following implication holds:

$$\text{if } q \text{ splits or ramifies in } K, \text{ then } q \text{ does not split in } \mathbb{Q}(\sqrt{-p}).$$

Dorian Goldfeld

Appendix. *An Analogue of the Class Number One Problem*

By D. Goldfeld

1. Let K be an algebraic number field of finite degree over $\mathbf{Q}$ with discriminant k, and let S be a finite set of rational primes. Define $\mathcal{N}(K, S)$ to be the set of rational integers N satisfying the conditions:

$-N$ is a discriminant of a quadratic field and for all primes $l \notin S$, $l < |N|/4$, if l splits completely in K, then l doesn't split in $\mathbf{Q}(\sqrt{-N})$.

In the case that K is equal to $\mathbf{Q}$ or a quadratic field, we shall show that $\mathcal{N}(K, S)$ is a finite set. The method of proof, however, is ineffective and all that can be deduced is

… Generalises Mazur's Claim

Claim. If the above case occurs then for all odd primes $p < N/4$ *we have* $\left(\frac{p}{N}\right) = -1$.

Barry C. Mazur receives National Medal of Science from US President Barack H. Obama

… Generalises Mazur's Claim

Claim. If the above case occurs then for all odd primes $p < N/4$ we have $\left(\frac{p}{N}\right) = -1$.

To conclude our theorem, we shall now prove that the above *claim* implies that $\mathbf{Q}(\sqrt{-N})$ has class number 1 and hence (by Baker-Stark-Heegner [3, 37, 38]) we have $N = 11, 19, 43, 67$, or 163 (ignoring the genus 0 cases).

Since $N \equiv -1 \bmod 4$, quadratic reciprocity applied to (7.1) implies that for $2 < p < N/4$, p remains prime in $\mathbf{Q}(\sqrt{-N})$.

Thus all ideals I of odd norm $< N/4$ are principal in the ring of integers of $\mathbf{Q}(\sqrt{-N})$. To be sure, if we had the stronger assertion that *all* ideals of norm $< N/4$ were principal, then $\mathbf{Q}(\sqrt{-N})$ would have class number 1 by Minkowski's theorem: the absolute value of the discriminant of $\mathbf{Q}(\sqrt{-N})$ is N; the Minkowski constant is $2/\pi$; and $2/\pi \cdot \sqrt{N} < N/4$ for $N \geqq 11$. We shall prove this stronger assertion. If 2 does not split in $\mathbf{Q}(\sqrt{-N})$, there is nothing to prove. Suppose, then, that 2 does split, in which case $N \equiv -1$ or $7 \bmod 16$. We must show that one (and hence both) of the primes of norm 2 are principal. If $N \equiv -1 \bmod 16$, consider the element $\alpha = (3 + \sqrt{-N})/2$. One sees that the norm of α is twice an odd number; hence $(\alpha) = \mathfrak{p} \cdot I$ where $\mathfrak{p}$ is one of the primes of norm 2, and I is an "odd" ideal, with norm $(9 + N)/8$. Since $N \geqq 11$, the norm of I is less than $N/4$, and therefore I is principal. Consequently so is $\mathfrak{p}$. If $N \equiv 7 \bmod 16$, take the element $\alpha = (1 + \sqrt{-N})/2$, and repeat the above argument.

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Theorem 7.9. *Under GRH, there are effectively computable absolute constants c_2, c_3, and c_4 such that we can take in Theorems 6.4 and 5.16*

$$\prod_{\ell \in S_K} \ell \leqslant \exp\left(c_2^{n_K} \cdot \left(R_K \cdot n_K^{r_K} + h_K^2 \cdot (\log \Delta_K)^2\right)\right)$$

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Question

Can we remove the "effectively computable absolute constants"?

Proposition (B.)

Assume GRH. Let K be an isogeny-finite quadratic field, and E/K an elliptic curve possessing a K-rational p-isogeny, for p a Type 2 prime. Then p satisfies

$$p \leq (16 \log p + 16 \log(12\Delta_K) + 26)^4.$$

In particular, there are only finitely many primes p as above.

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Strategy

Check all primes up to this bound for whether they satisfy condition CC or not.

```python
def get_type_2_primes(K, bound=None):
    """Compute a list containing the type 2 primes"""

    # First get the bound
    if bound is None:
        bound = get_type_2_bound(K)
        print("type 2 bound = {}".format(bound))

    # We need to include all primes up to 25
    # see Larson/Vaintrob's proof of Theorem 6.4
    output = set(prime_range(25))

    for p in pari.primes(25, bound):
        p_int = Integer(p)
        if p_int % 4 == 3:  # Type 2 primes necessarily congruent to 3 mod 4
            if satisfies_condition_CC(K,p_int):
                output.add(p_int)
    return output
```

```
blockSize=100000;
export(blockSize)

checktypetwo(pBeg) =
{
    my(p,cond);
    forprime(p = pBeg*blockSize, (pBeg+1)*blockSize-1,
        cond=custom_congruence_condition(p,D);
        if(cond,print_satisfiesCC(p)));
}
export(checktypetwo)

howMany=floor(typetwobound/blockSize);
parapply(checktypetwo,[0..howMany]);
```

Live Demo

IsogPrimeDeg($\mathbb{Q}(\sqrt{5})$)

To determine:

$$\{23, 29, 31, 41, 47, 53, 59, 61, 71, 73, 79\}.$$

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i.e. for each p in this set, determine whether the modular curve $X_0(p)$ admits any non-cuspidal $\mathbb{Q}(\sqrt{5})$-rational points.

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```
Type ? for help.  Type <Ctrl>-D to quit.
> R<x> := PolynomialRing(Rationals());
> K<a> := NumberField(R![-1, -1, 1]);
> N:=23;
> X := SmallModularCurve(N,K);
> RationalPoints(X : Bound:=10);
{@ (1 : 0 : 0), (1 : 1 : 0), (-3 : 23*a - 26 : 1), (-3 : -23*a - 3 : 1) @}
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All of these primes are such that $\mathrm{genus}(X_0(p)) \leq 5$.

Quadratic Points of Low-genus modular curves

Peter J. Bruin

Hyperelliptic modular curves $X_0(N)$ and isogenies of elliptic curves over quadratic fields, 2015

Filip Najman

Ekin Özman

Quadratic points on modular curves, 2019

Samir Siksek

Josha Box

Quadratic points on modular curves with infinite Mordell-Weil group, 2021

Summary

Using their results, we can rule out the other values to conclude that

$$\mathrm{IsogPrimeDeg}(\mathbb{Q}(\sqrt{5})) = \mathrm{IsogPrimeDeg}(\mathbb{Q}) \cup \{23, 47\}.$$

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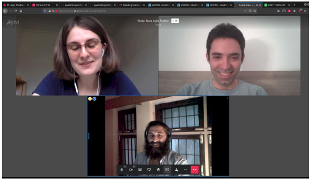
See the final section of the paper for the details.

Further Avenues

Working on determining IsogCyclicDeg(K) for certain Ks with Oana Adascalitei in Boston, USA, and Filip Najman in Zagreb, Croatia.



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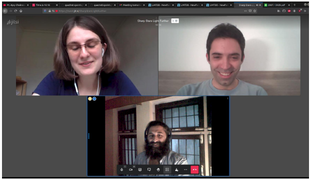

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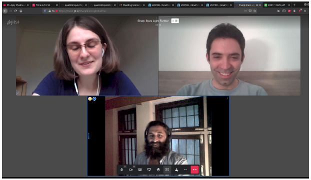
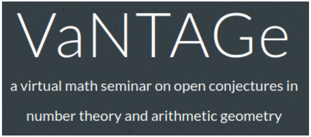


Working on extending the methods to cubic and higher degree number fields with Maarten Derickx in Den Haag, The Netherlands.

I'll be giving a live demo of our latest algorithm on a cubic field at my VaNTAGe seminar talk on **June 29th**:
`https://sites.google.com/view/vantageseminar`



Thanks for listening!

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