Algorithms for distance computations between 3-R$\underline{P}$R configurations

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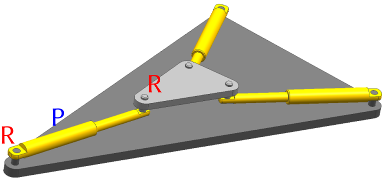
Introduction



3-RPR planar parallel manipulator

- 3-degrees of freedom (2-translational and one rotational).
- Actuated by Prismatic joints (P), while Revolute joints (R) are passive.

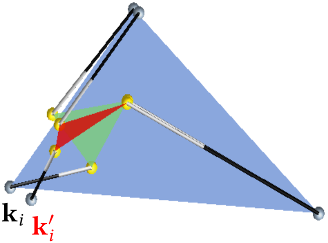
Singularity condition (V_3) for 3-RPR

A configuration is singular if and only if the carrier lines of the three legs intersect in a common point or are parallel.

Review on distance metrics

According to Nawratil [7], the distance between configurations $\mathcal{K}$ and $\mathcal{K}'$ of a 3-R<u>P</u>R manipulator, which can even differ in their geometry (shape of platform/base & length of the legs), can be given as

$$d(\mathcal{K}, \mathcal{K}')^2 = \frac{1}{6}\sum_{i=1}^{6} ||\mathbf{k}_i' - \mathbf{k}_i||^2$$



where $\mathbf{k}_i$ and $\mathbf{k}_i'$ are the vectors of the six anchor points in the two configurations $\mathcal{K}$ and $\mathcal{K}'$.
This metric can be used to compute singularity distance as the global minimizer of an optimization problem.

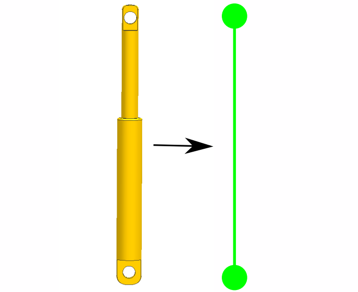
General problem formulation

Optimization problem: Computation of the closest point (w.r.t. the metric d) on the singularity variety (V_3) to the given non-singular manipulator configuration.
The corresponding Lagrange function L reads

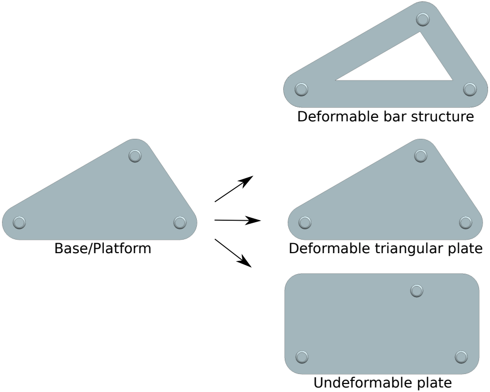
$$L: \; d^2 - \lambda V_3 = 0$$

We compute the critical points of the Lagrange function L numerically by using the homotopy continuation algorithms implemented in the software via Bertini [2].

Structural elements

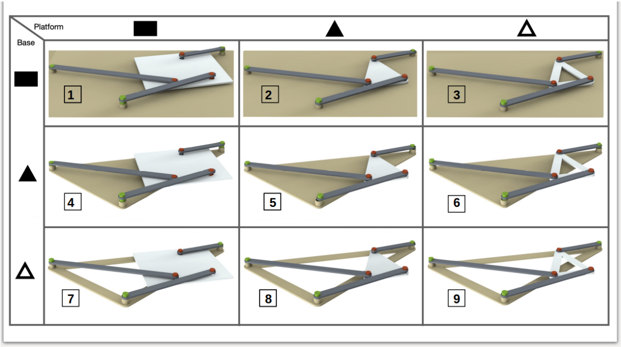


Prismatic joint ⇒ Deformable line-segment



The combination of these different structural elements result in nine interpretations of a 3-R$\underline{\text{P}}$R manipulator.

Interpretation table



How one can perform transformations

Structural elements made of deformable material ($|$, ▲, △) $\Rightarrow$ affine transformation.
Platform/base consists of undeformable material (indicated by ▬) $\Rightarrow$ Eucledian motion.

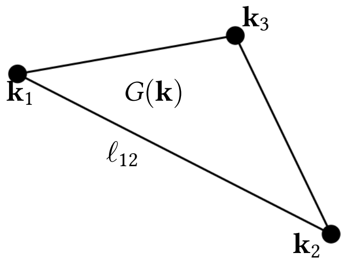
Limitations of Nawratil's distance metric

- It does not take in to account how the vertices are connected combinatorially.
- It does not consider the base and platform different design options respectively.

To overcome these limitations, according to [6] we will consider 3-R$\underline{\text{P}}$R planar parallel manipulator as a planar framework in the Euclidean plane $\mathbb{E}^2$ and perform an algebraic approach to concepts from rigidity theory.

Notation used for 3-R$\underline{\text{P}}$R as frameworks

- An abstract graph G fixing the combinatorial structure.
- Edge lengths between $\mathbf{k}_1$ and $\mathbf{k}_2$ are denoted by $\ell_{ij} \in \mathbb{R}_{>0}$ with $i < j$.



- We denote framework's configuration by $G(\mathbf{k})$
- Configuration of knots $\mathbf{k} := (\mathbf{k}_1, \ldots \mathbf{k}_6)$ is composed of the 2-dimensional coordinate vectors $\mathbf{k}_i$ of the knots $K_i (i = 1, \ldots 6)$.

Classification of metrics

To measure the distance between two configurations of a 3-R$\underline{\text{P}}$R, one can distinguish two differnt types of metrics:

Extrinsic metric

The metric is based on the embedding of the framework into the Euclidean plane. E.g. Nawratil's distance metric [7].

Intrinsic metric

The metric is based on the inner geometry of the framework (i.e. length of bars and shape of plates). E.g. for interpretation 1 it is the 3-dimensional joint space (cf. Zein [8]).

Motivation for the research

- The intrinsic metric is suited for computing the singularity distance. (discussed later on).

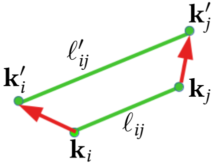
In contrast, our motivation for dealing with extrinsic metrics is twofold; they can be used to

- Compute singularity-free spheres in the embedding space.
- Quantify the change in shape implied by variations of the inner metric (sensitivity analysis, e.g. Caro et al. [3]).

On the other hand, for extrinsic metric we could solve the system of equations using symbolic approaches e.g., (Gröbner basis method, resultant based elimination) but for the case of intrinsic metric symbolic approaches are not promising.

Extrinsic metrics between structural elements

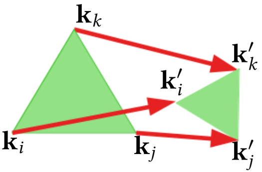
Distance between oriented line-segments according to Chen and Pottmann [4].



By considering a similarity transformation, the squared distance between two oriented line-segments $\ell_{ij} = (\mathbf{k}_i, \mathbf{k}_j)$ and $\ell'_{ij} = (\mathbf{k}'_i, \mathbf{k}'_j)$ can be computed as:

$$d(\ell_{ij}, \ell'_{ij})^2 = \frac{1}{3}\left[\|\mathbf{k}_i - \mathbf{k}'_i\|^2 + \|\mathbf{k}_j - \mathbf{k}'_j\|^2 + (\mathbf{k}_i - \mathbf{k}'_i)^T(\mathbf{k}_j - \mathbf{k}'_j)\right].$$

Extending the idea of Chen and Pottmann



By considering an affine mapping, the squared distance between two triangles $\blacktriangle_{ijk} = (\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k)$ and $\blacktriangle'_{ijk} = (\mathbf{k}'_i, \mathbf{k}'_j, \mathbf{k}'_k)$ can be computed as:

$$d(\blacktriangle_{ijk}, \blacktriangle'_{ijk})^2 = \frac{1}{6}\Bigg[\sum_{x=i,j,k} \|\mathbf{k}_x - \mathbf{k}'_x\|^2 +$$

$$(\mathbf{k}_i - \mathbf{k}'_i)^T(\mathbf{k}_k - \mathbf{k}'_k) + (\mathbf{k}_i - \mathbf{k}'_i)^T(\mathbf{k}_j - \mathbf{k}'_j) + (\mathbf{k}_k - \mathbf{k}'_k)^T(\mathbf{k}_j - \mathbf{k}'_j)\Bigg].$$

Intrinsic metric - Strain energy between triangular plates

According to [6] the elastic strain energy U for the deformation of triangular plate $\blacktriangle_{ijk}$ to the deformed triangular plate $\blacktriangle'_{ijk}$ can be calculated as

$$U(\blacktriangle_{ijk}, \blacktriangle'_{ijk}) = Vol_{\blacktriangle_{ijk}} \frac{1}{2} \mathbf{e}^T \mathbf{M}\, \mathbf{e} \quad \text{with} \quad \mathbf{e} = \left(\epsilon_x, \epsilon_y, 2\gamma_{xy}\right)^T$$

where ϵ_x, ϵ_y, are the Green-Lagrange (GL) normal strains respectively, and γ_{xy} the GL shear strain.

$Vol_{\blacktriangle_{ijk}}$ denotes the volume of the undeformed triangular plate and $\mathbf{M}$ corresponds to the planar stress/strain constitutive matrix.

Intrinsic metric - Strain energy between bars

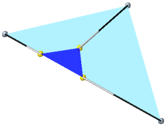
According to [6] the elastic Green-Lagrange strain energy between an undeformed bar ℓ_{ij} and the deformed bar ℓ'_{ij} can be computed as

$$U(\ell_{ij}, \ell'_{ij}) = \frac{A}{8\ell_{ij}^3}\left(\ell'^2_{ij} - \ell^2_{ij}\right)^2$$

where A is the cross sectional area of the undeformed bar.

Case study

Let us consider interpretation $\boxed{3}$ where the base is made with undeformed material (▬), and the platform is made with deformable material i.e. pin-jointed triangular bar-structure ($\triangle$).



We are considering a 1-parametric motion (parameter t) which is discretized into a user defined number of poses.

Extrinsic metric

The extrinsic distance function is

$$d_{\blacksquare}^{\triangle}(\mathbf{k},\mathbf{k}')^2 = \frac{1}{6}\sum_{(ij)\in\mathcal{J}} d\left(\ell_{ij},\ell'_{ij}\right)^2$$

where $\mathcal{J} = \{14, 25, 36, 45, 46, 56\}$.

The squared distance equals the mean of the squared distances between corresponding structural elements.

The corresponding Lagrange optimization problem for interpretation $\boxed{3}$ reads as follows:

$$L:\ \ d_{\blacksquare}^{\triangle}(\mathbf{k},\mathbf{k}')^2 - \lambda V_3 - \mu M = 0$$

where $M = 0$ is the additional side constraint restricting the deformation of the base to the Euclidean motions.

Remark 1: Computations are based on the point based approach [5].

Intrinsic metric

The strain energy density function for interpretation $\boxed{3}$ is

$$D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}') = \frac{1}{AL} \sum_{(ij) \in \mathcal{J}} U(\ell_{ij}, \ell'_{ij})$$

where L is the total length $L = \sum_{(ij) \in \mathcal{J}} \ell_{ij}$ and $\mathcal{J} = \{14, 25, 36, 45, 46, 56\}$. Thus the strain energy density function equals the sum of strain energies between corresponding structural elements that is divided by the framework's volume.

Remark 2: By replacing the leg lengths by the distance of corresponding knots yields the dependence of $D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}')$ on $\mathbf{k}$ and $\mathbf{k}'$ respectively.

Further discussion on intrinsic metric

- $G(\mathbf{k})$ is a given undeformed configuration.
- A configuration $G(\mathbf{k}')$ is *undeformed configuration* $\Leftrightarrow D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}') = 0$.
- A configuration $G(\mathbf{k}')$ is *deformed configuration* $\Leftrightarrow D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}') > 0$.

For the computation of the closest singular configuration in terms of the intrinsic metric, there is no need of a Lagrange function due to the following result of [6].

The critical points of $D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}')$ which correspond to deformed configurations are singular. It can be shown [6] that the closest singularity corresponds to a saddle point.

Separation of saddle configurations

- Compute all the critical points using homotopy algorithm for $D^{\triangle}_{\blacksquare}(\mathbf{k}, \mathbf{k}') = 0$.
- The undeformed configurations are at the local minima of the total elastic strain energy density (but not vice versa).
- Critical points which are no local extrema, correspond to so-called *saddle configurations.*
- This separation can be done by *second partial derivative test.*

How to verify if $G(\mathbf{k})$ is deformed into $G(\mathbf{k}')$?

Let us assume that $G(\mathbf{k}')$ which belongs to the set of saddle points, yields the minimal value of $D^{\triangle}_{\blacksquare}$, where $G(\mathbf{k})$ is the given undeformed configuration. Let us consider the below formulation

$$L_{ij}^{'} = \ell_{ij}^{'2} + m(\ell_{ij}^{2} - \ell_{ij}^{'2}) \quad \text{with} \quad m \in [0, 1] \tag{1}$$

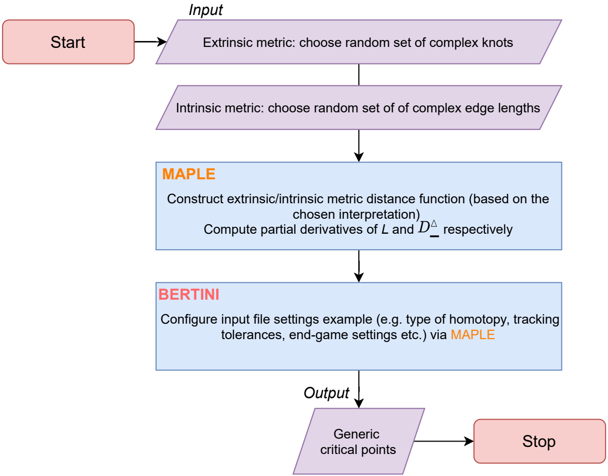
where $L_{ij}{}'$ is a path between given undeformed squared length ℓ_{ij} of the given configuration and deformed squared length $\ell_{ij}{}'$ of the saddle configuration.

Eq. (1) implies a user-homotopy (parameter m) where we track the solution for $G(\mathbf{k})$ ($m = 1$) which yields $G(\mathbf{k}_m)$. If the condition

$$G(\mathbf{k}_m)\Big|_{m=0} = G(\mathbf{k}')$$

holds, then $G(\mathbf{k}')$ is the closest singular configuration.

Algorithm's Step 0: Source configuration



Keypoints

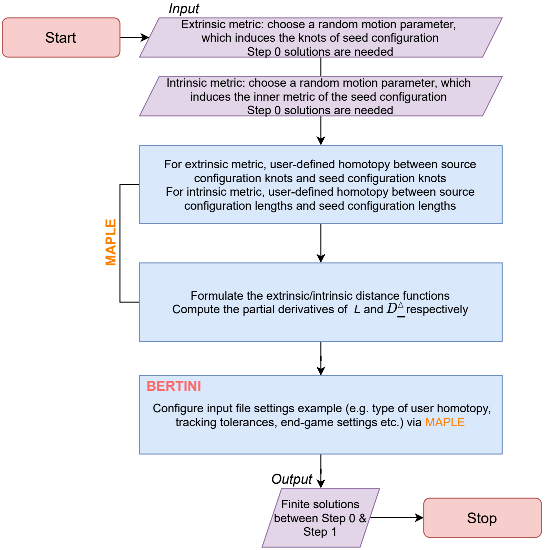
Remark 3: Note that solutions obtained from Step 0 are generic critical points, only have to be computed once (for each of the different extrinsic and intrinsic metrics).

Limitation: Bertini is a numerical software, and users do not know the exact number of solutions for the system.

Remark 4: The coefficient of partial derivatives are scaled between 0 and 1 for good accuracy.

Remark 5: For step 0, the number of paths to be tracked by standard homotopy is given as n^N, where n is the degree of equations and N is the number of equations.

Algorithm's Step 1: Seed configuration



Keypoints

Remark 6: For the user-defined homotopy of Step 1, we use the simplest possible path in $\mathbb{C}^{12}$ and track the solutions of Step 0.

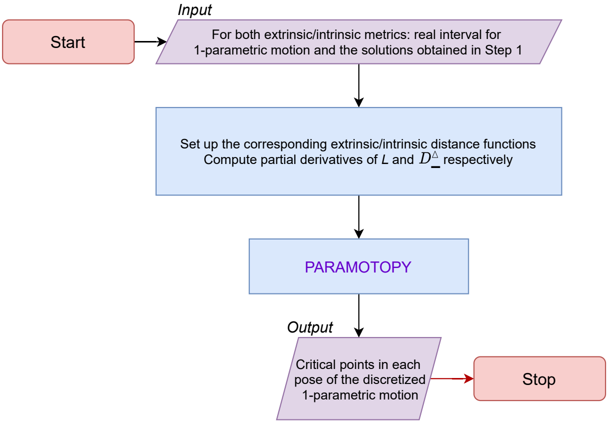
Remark 7: In the case of an intrinsic metric, while doing user-homotopy, the motion parameter appears inside the square root.

E.g. $j_i = \sqrt{C_i^2 + t\,(\ell_i^2 - C_i^2)}$ where is t is the motion parameter.

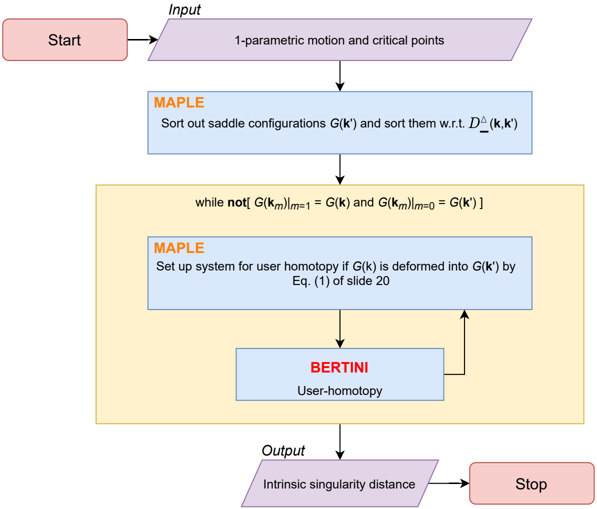
Parameter homotopy approach only works if the parameterized system is analytic.

We solved the issue of **Remark 7** by *doubling technique*. This approach doubles the system of equations, and we have to pay the price of computational time and extra solutions.

Algorithm's Step 2: Paramotopy



Post processing of saddle configurations



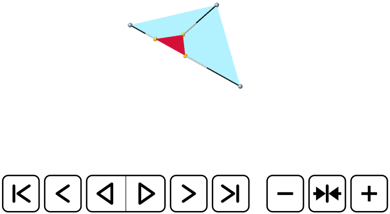
Computational results - comparison

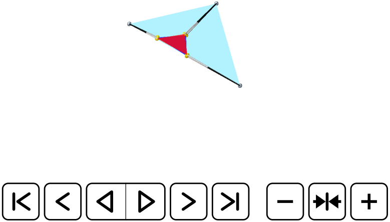
For interpretation $\boxed{3}$ for Step 0 (Source configuration), the numerical example is taken from [7].

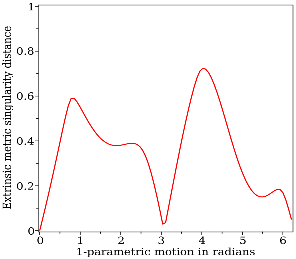
	Extrinisic metric	Intrinsic metric
Type of homotopy	Regeneration	Multi-homogeneous
Generic solutions	80	18 238
Total paths tracked	9 567	46 656
Number of equations	12	12
Gröbner basis	80	–

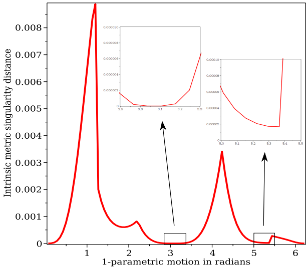
Note: The number of paths that have to be tracked for extrinsic metric using multi-homogeneous homotopy are 2 361 960.

Computational results - comparison









Conclusions

- We presented the elements required to construct extrinsic and intrinsic metrics for different interpretations of 3-R$\underline{\text{P}}$R manipulator.
- We solved an optimization problem using the metric constructions and computed the singularity distances along a 1-parametric motion for interpretation $\boxed{3}$.
- We developed a novel open-source software interface between Maple, BERTINI and Paramotopy [1].
- The interface algorithm's advantage is that we computed the generic critical points for all the presented 9 interpretations using intrinsic and extrinsic metrics.
- In this way, users need to run only Step 1 and Step 2 by defining base/platform configuration knots and a 1-parametric motion to compute the closest singularity-distances for 3-R$\underline{\text{P}}$R manipulators.

Acknowledgement

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Thank you for your attention

Questions?

References

[1] Dan Bates, Danielle Brake, and Matt Niemerg. "Paramotopy: Parameter homotopies in parallel". In: *International Congress on Mathematical Software*. Springer. 2018, pp. 28–35.

[2] Daniel J. Bates et al. *Bertini: Software for Numerical Algebraic Geometry*. Available at bertini.nd.edu with permanent doi: dx.doi.org/10.7274/R0H41PB5.

[3] Stéphane Caro, Nicolas Binaud, and Philippe Wenger. "Sensitivity analysis of 3-RPR planar parallel manipulators". In: *Journal of mechanical design* 131.12 (2009).

[4] Horng-Yang Chen and Helmut Pottmann. "Approximation by ruled surfaces". In: *Journal of computational and Applied Mathematics* 102.1 (1999), pp. 143–156.

[5] Aditya Kapilavai and Georg Nawratil. "On Homotopy Continuation Based Singularity Distance Computations for 3-RPR Manipulators". In: *New Trends in Mechanism and Machine Science*. Springer International Publishing, 2020, pp. 56–64. ISBN: 978-3-030-55061-5.

[6] Georg Nawratil. "On the snappability and singularity-distance of frameworks with bars and triangular plates". In: *arXiv preprint arXiv:2003.09904* (2020).

[7] Georg Nawratil. "Singularity Distance for Parallel Manipulators of Stewart Gough Type". In: *Advances in Mechanism and Machine Science*. Springer, 2019, pp. 259–268.

[8] Mazen Zein, Philippe Wenger, and Damien Chablat. "Singularity Surfaces and Maximal Singularity-Free Boxes in the Joint Space of Planar 3-RPR Parallel Manipulators". In: 2007, pp. 1–6.