Spectral sequences in algebraic topology: computational aspects and new developments

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Given a **simplicial set** X, a chain complex $C_*(X)$ can be constructed such that the homology groups of X are defined as

$$H_n(X) := H_n(C_*(X))$$

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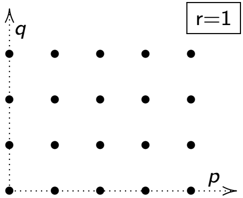

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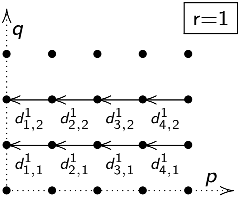

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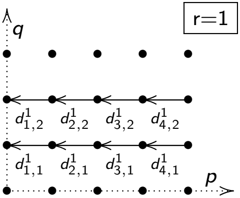

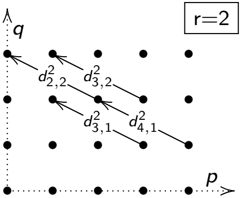

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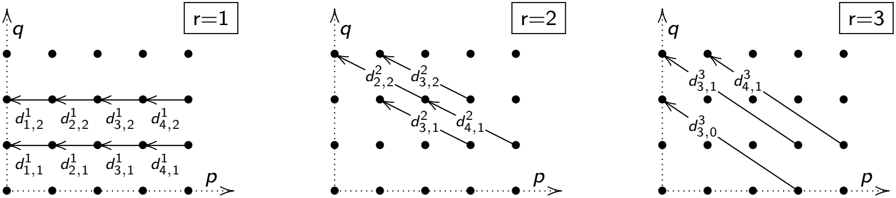

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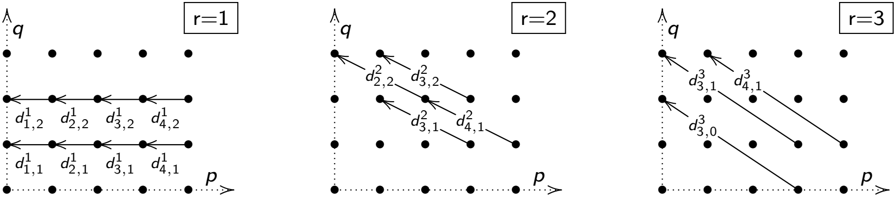

Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for each $r \geqslant 1$, one can define the **final groups** of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r \geqslant 1} E^r_{p,q}$.

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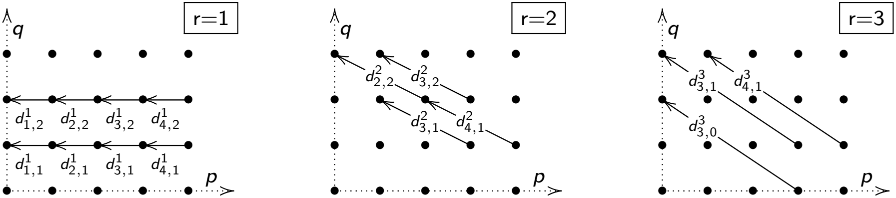

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- The **Adams spectral sequence** converges to the homotopy groups of a simplicial set.

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Theorem (Serre, 1951)

Let $G \hookrightarrow E \to B$ be a fibration *and suppose the base B is 1-reduced. There is a spectral sequence converging to $H_*(E)$ whose second page is given by $E^2_{p,q} = H_p(B; H_q(G))$.*

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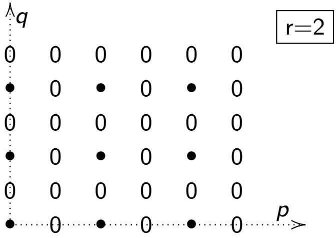

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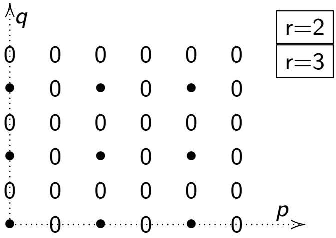

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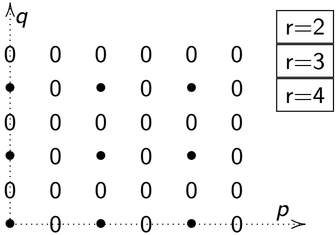

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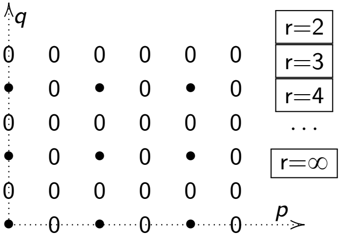

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They are not algorithms producing the desired H_*

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An **increasing filtration** F of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{N}}$ is a family of sub-chain complexes $\ldots \subseteq F_{p-1}C_* \subseteq F_pC_* \subseteq F_{p+1}C_* \subseteq \ldots$

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Theorem

Let C_* be a chain complex with a filtration. There exists a spectral sequence with

$$E^r_{p,q} = \frac{Z^r_{p,q} + F_{p-1}C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) + F_{p-1}C_{p+q}}$$

where $Z^r_{p,q}$ is $Z^r_{p,q} = \{a \in F_pC_{p+q} | \ d_{p+q}(a) \in F_{p-r}C_{p+q-1}\} \subseteq F_pC_{p+q}$, and $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the morphism induced by $d_{p+q} : C_{p+q} \to C_{p+q-1}$. This spectral sequence converges *to $H_*(C)$.*

S. MacLane. *Homology*. Springer, 1963.

Programs computing spectral seq. of filtered complexes

Remark: when the initial chain complex is of finite type, the groups $E^r_{p,q}$ (of all levels!) can be determined by means of diagonalization algorithms on some matrices **without knowing the differential maps**.

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By using the **effective homology** theory, implemented in the Kenzo system, it is also possible to determine spectral sequences of chain complexes which are not of finite type.

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- If a filtered complex C_* is of finite type, its spectral sequence can be determined by means of diagonalization algorithms on some matrices.
- Otherwise, a pair of **reductions** $C_* \Lleftarrow \hat{C}_* \Rrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered) is constructed, such that (thanks to some theoretical results) the spectral sequences of C_* and D_* are isomorphic after some level.

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Applications of our programs computing spectral sequences of filtered complexes:

- We can compute the spectral sequence associated with a bicomplex.
- We can compute the classical spectral sequences of Serre and Eilenberg–Moore, defined by means of filtered complexes, even when the spaces are not of finite type and (some) differential maps cannot be easily deduced.

Bousfield–Kan spectral sequence

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There does not exist a formal expression for the groups $E^r_{p,q}$'s as in the case of the spectral sequence associated with a filtered complex.

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Other **ingredients** have been necessary.

A. R. *Computing the first stages of the Bousfield–Kan spectral sequence*. Appl. Algebra Eng. Commun. Comput. 21(3), 227-248, 2010.

A. R., F. Sergeraert. *Programming before theorizing, a case study*. Proceedings ISSAC 2012, 289-296.

A. R., F. Sergeraert: *A Combinatorial Tool for Computing the Effective Homotopy of Iterated Loop Spaces*. Discrete and Computational Geometry 53(1), 1-15, 2015.

Spectral systems

Spectral systems

The notion of spectral sequence of a filtered complex was generalized by B. Matschke for a filtration indexed over a **poset** I, i.e. a collection of sub-chain complexes $\{F_i C_*\}_{i \in I}$ with $F_i C_* \subseteq F_j C_*$ if $i \leqslant j$.

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A **spectral system** (also called **generalized spectral sequence** or **higher spectral sequence**) is a set of groups, for all $z \leqslant s \leqslant p \leqslant b$ in I and for each degree n:

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B. Matschke. Successive Spectral Sequences. Preprint, 2013. http://arxiv.org/abs/1308.3187v1.

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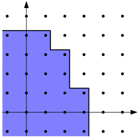

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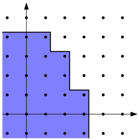
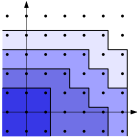
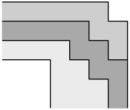


We denote $D(\mathbb{Z}^m)$ the collection of all downsets of $\mathbb{Z}^m$, which is a poset with respect to the inclusion $\subseteq$.





Serre spectral systems

We recall the definition of the Serre spectral sequence:

Theorem (Serre, 1951)

Let $G \hookrightarrow E \to B$ be a fibration *and suppose the base B is 1-reduced. There is a spectral sequence converging to $H_*(E)$ whose second page is given by $E^2_{p,q} = H_p(B; H_q(G))$.*

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This result has been generalized to towers of fibrations:

$$\begin{array}{ccccccc}
E_0 & \longrightarrow & \cdots & \longrightarrow & E_{m-1} & \longrightarrow & B \\
\uparrow & & & & \uparrow & & \\
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Serre spectral systems

Theorem (Matschke, 2013)

Consider a tower of m fibrations. There exists an associated spectral system over $D(\mathbb{Z}^m)$ with 2-page

$$S_n^*(P;m) \cong H_{p_m}(B; H_{p_{m-1}}(G_{m-1}; \ldots H_{p_1}(G_1; H_{p_0}(G_0)))),$$

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For $m = 2$, we have two fibrations:

$$\begin{array}{ccccc} E & \longrightarrow & N & \longrightarrow & B \\ \uparrow & & \uparrow & & \\ G & & M & & \end{array}$$

where the base B is 1-reduced. Then, there exists a $D(\mathbb{Z}^2)$-spectral system converging to $H_*(E)$ whose second page is given by

$$S_n^*(P;2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G))), \quad P = (p_1, p_2) \in \mathbb{Z}^2.$$

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A. Guidolin, A. R. *Effective Computation of Generalized Spectral Sequences*. Proceedings ISSAC 2018, 183-190.

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A. Guidolin, A. R. *Computing Higher Leray–Serre Spectral Sequences of Towers of Fibrations*. To appear in Foundations of Computational Mathematics.

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Associated invariant: **rank invariant**

$$\beta_n^{P,Q} := \dim_{\mathbb{F}} \mathrm{Im}(H_n(K_P) \to H_n(K_Q)), \qquad P, Q \in \mathbb{Z}^m, \quad P \leqslant Q.$$

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A. Guidolin, J. Divasón, A. R., F. Vaccarino. *Computing Multipersistence by Means of Spectral Systems*. Proceedings ISSAC 2019, 195-202.

A. Guidolin, J. Divasón, A. R., F. Vaccarino. *Computing invariants for multipersistence via spectral systems and effective homology*. Journal of Symbolic Computation 104, 724-753, 2021.

Spectral systems combining Serre and Eilenberg–Moore spectral sequences

We consider the following diagram of fibrations:

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D. Miguel, A. Guidolin, A. R., J. Rubio. *Towards a new spectral system combining Serre and Eilenberg-Moore spectral sequences*. Poster in MEGA 2021.

New Kenzo-SageMath interface

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J. Cuevas-Rozo, M. Marco-Buzunáriz, A. R. *Computing with Kenzo from Sage*. MEGA 2019, software presentation.

J. Cuevas-Rozo, J. Divasón, M. Marco-Buzunáriz, A. R. *A Kenzo interface for algebraic topology computations in SageMath*. ISSAC 2019, Best software demo award.

J. Cuevas-Rozo, J. Divasón, M. Marco-Buzunáriz, A. R. *Integration of the Kenzo system within SageMath for new Algebraic Topology Computations*. Mathematics 9(7), 722, 2021.

Demo

https://mybinder.org/v2/gh/ana-romero/mega2021-kenzo-sage/master?filepath=PresentationMEGA2021.ipynb