Experiments with Hilbert Schemes

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Context

Grothendieck (1961): There is a projective scheme $\mathrm{Hilb}^p(\mathbb{P}^m)$ parametrizing all $X \subseteq \mathbb{P}^m$ with Hilbert polynomial $p \in \mathbb{Q}[t]$.

Importance: moduli spaces (compactification/representability), deformation theory, source of higher-dimensional varieties, and computational benchmarks.

Effective methods provide insight (Bayer 1982), patterns (Reeves–Stillman 1997), examples (Cartwright–Erman–Velasco–Viray 2009), and verification (Hauenstein–Manivel–Szendrői 2021).

Challenge: Understand the geometry of $\mathrm{Hilb}^p(\mathbb{P}^m)$.

Lexicographic Points

The action of $\mathrm{PGL}(m+1)$ on $\mathbb{P}^m$ extends to $\mathrm{Hilb}^p(\mathbb{P}^m)$. A point is Borel-fixed if its stabilizer contains all upper triangular matrices. Each $X \subseteq \mathbb{P}^m$ specializes to a Borel-fixed point. The lexicographic point is an extremal one on every $\mathrm{Hilb}^p(\mathbb{P}^m)$.

Macaulay (1926): $\mathrm{Hilb}^p(\mathbb{P}^m) \neq \varnothing$ if and only if there exists a partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{N}^r$ such that $p(t) = \sum_{i=1}^r \binom{t+\lambda_i-i}{\lambda_i-1}$.

Hartshorne (1966): Nonempty $\mathrm{Hilb}^p(\mathbb{P}^m)$ are path connected.

Reeves–Stillman (1997): Each nonempty $\mathrm{Hilb}^p(\mathbb{P}^m)$ has a generically smooth irreducible component.

Celebrated Examples

- $X \subseteq \mathbb{P}^m$ is a linear subspace if and only if $p(t) = \binom{t+\lambda_1-1}{\lambda_1-1}$, so $r=1$ implies $\mathrm{Hilb}^p(\mathbb{P}^m) = \mathrm{Gr}(\lambda_1 - 1, \mathbb{P}^m)$.
- $X \subseteq \mathbb{P}^m$ is a hypersurface of degree r if and only if $p(t) = \sum_{i=1}^{r} \binom{t+m-i}{m-1}$, so $\lambda = (m^r)$ implies $\mathrm{Hilb}^p(\mathbb{P}^m) = \mathbb{P}^{\binom{r+m}{m}-1}$.

PIENE–SCHLESSINGER (1985): For $\lambda = (2^3, 1^1)$, $\mathrm{Hilb}^{3t+1}(\mathbb{P}^3)$ has two components: twisted cubic curves and plane cubics union a point.

CHEN–COSKUN–NOLLET (2011): For $\lambda = (2^2, 1^1)$, $\mathrm{Hilb}^{2t+2}(\mathbb{P}^3)$ has two components: two skew lines and plane conic union a point.

Cautionary Tales

Mumford (1962): An irreducible component of $\mathrm{Hilb}^{14t-23}(\mathbb{P}^3)$ is generically nonreduced (singular at every point).

Ellia–Hirschowitz–Mezzetti (1992): Irreducible components of $\mathrm{Hilb}^{dt+1-g}(\mathbb{P}^3)$ are not bounded by a polynomial in d and g.

Vakil (2006): Every singularity type appears on some $\mathrm{Hilb}^{p}(\mathbb{P}^4)$.

Antagonistic Aim: Identify those Hilbert schemes $\mathrm{Hilb}^{p}(\mathbb{P}^m)$ with comparatively simply geometry: smooth, irreducible, mild singularities, small number of irreducible components, …

Characterizing Smoothness

THEOREM (Skjelnes–Smith 2020): $\mathrm{Hilb}^p(\mathbb{P}^m)$ is smooth if and only if one of the following holds:

(1) $m = 2 \geqslant \lambda_1$

(2) $m \geqslant \lambda_1$ and $\lambda_r \geqslant 2$,

(3) $\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$ where $r \geqslant 2$ and $m \geqslant \lambda_{r-1} \geqslant 1$,

(4) $\lambda = (m^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r-3 \geqslant s \geqslant 0$ and $m-1 \geqslant \lambda_{r-s-2} \geqslant 3$,

(5) $\lambda = (m^{r-s-5}, 2^{s+4}, 1)$ where $r-5 \geqslant s \geqslant 0$,

(6) $\lambda = (m^{r-3}, 1^3)$ where $r \geqslant 3$,

(7) $\lambda = (m+1)$ or $r = 0$.

Discovery Process

Staal (2020): Cases (2) and (3) classify $\mathrm{Hilb}^p(\mathbb{P}^m)$ with a unique Borel-fixed point, namely the lexicographic point.

Skjelnes–Smith (2020): In cases (2) and (3), $\mathrm{Hilb}^p(\mathbb{P}^m)$ is a projective bundle over a partial flag variety.

Corollary: For all partitions λ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 1$, $\mathrm{Hilb}^p(\mathbb{P}^m)$ is simply a partial flag variety.

Experimentation: A computer search suggests that cases (4) and (5) are the only "missing" smooth ones. One new family of Borel-fixed points seems to explain non-smoothness.

Validation

KEY: This new family corresponds to a singular point lying on the lexicographic component of $\mathrm{Hilb}^p(\mathbb{P}^m)$.

A POSTERIORI: In cases (2)–(5), the Hilbert schemes have at most 2 Borel-fixed points. Ramkumar (2019) already analyzes the $\mathrm{Hilb}^p(\mathbb{P}^m)$ with 2 Borel-fixed points.

SKJELNES–SMITH (2020): In cases (4)–(6), $\mathrm{Hilb}^p(\mathbb{P}^m)$ is *birational* to the product of a projective bundle over a partial flag variety and $\mathbb{P}^m$. A general point corresponds to the disjoint union of a "residual flag" and an isolated point.

Further Experiments?

Heuristic: The number of parts in the integer partition λ equal to 1 appears to govern the "geometric complexity" of the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^m)$. When this number less than 8, what is the intersection graph for the irreducible components of $\mathrm{Hilb}^p(\mathbb{P}^m)$?

Potentially Accessible Problems:

- When $\mathrm{Hilb}^p(\mathbb{P}^m)$ is reducible, must it have an irreducible component that is smaller than the lexicographic one?
- Can one classify those Hilbert schemes with exactly two smooth irreducible components?