Combinatorial Differential Algebra of x^p

Rida Ait El Manssour and Anna-Laura Sattelberger

Abstract. We link n-jets of the affine monomial scheme defined by x^p to the stable set polytope of some perfect graph. We prove that, as p varies, the dimension of the coordinate ring of the scheme of n-jets as a $\mathbb{C}$-vector space is a polynomial of degree $n+1$, namely the Erhart polynomial of the stable set polytope of that graph. One main ingredient for our proof is a result of Zobnin who determined a differential Gröbner basis of the differential ideal generated by x^p. We generalize Zobnin's result to the bivariate case. We study (m,n)-jets, a higher-dimensional analog of jets, and relate them to regular unimodular triangulations of the $m \times n$-rectangle.

Contents

Introduction

Differential Algebra—an infinite version of polynomial algebra in a sense—studies polynomial partial differential equations with tools from Commutative Algebra. Differential Algebraic Geometry studies varieties that are defined by a system of polynomial PDEs. An upper bound for the number of components of such a variety was recently constructed in [14]. Differential Algebraic Geometry comes with an own version of the Nullstellensatz, the *differential* Nullstellensatz, relating points of a differential variety with formal power series solutions of the defining system of equations. Lower and upper bounds for the effective differential Nullstellensatz are provided in [11]. In this article, we transfer the combinatorial flavor of Commutative Algebra [16] to Differential Algebra and undertake first steps in *Combinatorial Differential Algebra*. We present a case study of the fat point x^p on the affine line.

Denote by $C_{p,n}$ the ideal in $R_n = \mathbb{C}[x_0,\ldots,x_n]$ generated by the coefficients of $f_{p,n} = (x_0 + x_1 t + \cdots + x_n t^n)^p$, read as polynomial in the variable t with coefficients in R_n. The affine scheme defined by $C_{p,n}$ is the scheme of n-jets of the fat point x^p on the affine line. B. Sturmfels suggested to investigate the following question.

Question 2.1. *For fixed $n \in \mathbb{N}$, is the sequence $\left(\dim_{\mathbb{C}}(R_n/C_{p,n})\right)_{p\in\mathbb{N}}$ a polynomial in p of degree $n+1$?*

2020 *Mathematics Subject Classification.* 05E40, 13P10, 12H05 (primary), 52B20 (secondary).

Key words and phrases. Differential ideal, Gröbner basis, multivariate jet, perfect graph, regular triangulation, lattice polytope.

The point of departure are experimental observations. A first main result of this article is the proof that this question has a positive answer.

One main tool for the proof is a result from Differential Algebra. The object of study is the differential ring $\mathbb{C}[x^{(\infty)}] = (\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots], \partial)$, i.e., the polynomial ring in the countably infinitely many variables $\{x^{(k)}\}_{k\in\mathbb{N}}$ with the differential ∂ acting as $\partial(x^{(k)}) = x^{(k+1)}$ and $\partial|_{\mathbb{C}} \equiv 0$. An ideal I in $\mathbb{C}[x^{(\infty)}]$ is a *differential ideal* if $\partial(I) \subseteq I$. Zobnin [24] proved that the singleton $\{x^p\}$ is a differential Gröbner basis of the differential ideal generated by x^p with respect to any β-ordering. Denote by $I_{p,n}$ the differential ideal generated by x^p and $x^{(n)}$. Then the map

$$R_n/C_{p,n} \xrightarrow{\cong} \mathbb{C}[x^{(\infty)}]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!}x^{(k)}$$

is an isomorphism and Zobnin's result can be used to investigate $C_{p,n}$. An investigation of the leading monomials of $C_{p,n}$ then reveals the following.

Proposition 2.5. *As p varies, $\dim_{\mathbb{C}}(R_n/C_{p,n})$ is polynomial of degree $n+1$. It is the Erhart polynomial of the convex polytope*

$$P_n := \left\{(u_0, \ldots, u_n) \in (\mathbb{R}_{\geq 0})^{n+1} \middle| u_i + u_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n-1\right\}$$

evaluated at $p-1$, i.e., it counts the lattice points of the polytope P_n dilated by $p-1$.

A study of jet schemes of monomial ideals was also undertaken in [9]. Therein, it is shown that jet schemes of monomial ideals are in general not monomial, but their reduced subschemes are. A study of the multiplicity of jet schemes of simple normal crossing divisors was undertaken by C. Yuen in [23]. In [22], she introduced *truncated m-wedges*, a two-dimensional analog of jets, studying differentials in two variables whose orders add up to m at most. In Definition 1.9, we introduce another generalization of jets to higher dimensions, namely *(m,n)-jets*, allowing for derivatives in the variables up to order m and n, respectively.

We extend our studies of $\dim_{\mathbb{C}}(R_n/C_{p,n})$ to the case of two independent variables and give a link to regular unimodular triangulations. For the theory of triangulations, we refer our readers to [5, 21]. We study the *partial* differential ring $\mathbb{C}[x^{(\infty,\infty)}] := \left(\mathbb{C}[x^{(k,\ell)}]_{k,\ell\in\mathbb{N}}, \partial_s, \partial_t\right)$ in two independent variables s, t and consider the differential ideal $I_{p,(m,n)}$ generated by $x^p, x^{(m,0)}$, and $x^{(0,n)}$. Denote by $C_{p,(m,n)}$ the ideal in $\mathbb{C}[\{x_{k,\ell}\}_{0\leq k\leq m, 0\leq \ell\leq n}]$ generated by the coefficients of

$$f_{p,(m,n)} := \left(\sum_{k=0}^{m}\sum_{\ell=0}^{n} x_{k,\ell}t^k s^\ell\right)^p,$$

read as bivariate polynomial in s and t. We refer to the affine scheme associated to $C_{p,(m,n)}$ as the scheme of *(m,n)-jets* of x^p. The ideals $I_{p,(m,n)}$ and $C_{p,(m,n)}$ then are related just as in the univariate case.

For a triangulation T of the $m \times n$-rectangle and fixed p, we define *T-orderings* on the truncated partial differential ring $\mathbb{C}[x^{(\leq m, \leq n)}]$ as those monomial orderings for which the leading monomials of $\{(x^p)^{(k,\ell)}\}_{k=0,\ldots,mp,\ell=0,\ldots,np}$ are supported on the triangles of T. Note that this is in contrast to the usual occurrence of regular triangulations in Combinatorial Commutative Algebra, where the leading monomials are supported on *non*-faces (see for instance Sturmfels' correspondence [5, Theorem 9.4.5]). We consider the placing triangulation $T_{m,2}$ of the point configuration

$$[(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), \ldots, (m,0), (m,1), (m,2)].$$

This is a regular unimodular triangulation of the $m\times 2$-rectangle induced by the vector $(1,2,2^2,\dots,2^{2m+1})$ in the lower hull convention.

We formulate the following conjectural generalization of Zobnin's result to the partial differential ring in two independent variables.

Conjecture 1.14. *For all $m,p\in\mathbb{N}$, $\{(x^p)^{(k,\ell)}\}_{0\le k\le mp, 0\le \ell\le 2p}$ is a Gröbner basis of the differential ideal generated by x^p in the truncated partial differential ring $\mathbb{C}[x^{(\le m,\le 2)}]$ w.r.t. any $T_{m,2}$-ordering.*

As pointed out in Proposition 1.15, we have computational evidence that this conjecture holds true. This theorem is the main ingredient for the following proposition.

Proposition 2.6. *For $m\le 12$ and $p\le 5$, the number $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$ is the Erhart polynomial of the $3(m+1)$-dimensional lattice polytope*

$$P_{(m,2)} := \Big\{(u_{00},u_{01},u_{02}\dots,u_{m0},u_{m1},u_{m2})\in(\mathbb{R}_{\ge 0})^{3(m+1)}\big| u_{k_1,l_1}+u_{k_2,l_2}+u_{k_3,l_3}\le 1 \\ \textit{for all indices s.t. } \{(k_1,l_1),(k_2,l_2),(k_3,l_3)\} \textit{ is a triangle of } T_{m,2}\Big\}$$

evaluated at $p-1$.

In Section 3, we study regular unimodular triangulations of the $m\times n$-rectangle. We consider the weighted degree reverse lexicographical ordering on $\mathbb{C}[\{x_{k,\ell}\}_{0\le k\le m,0\le\ell\le n}]$ for vectors inducing those triangulations in the upper hull convention. We show that for some of them, the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of the ideal $C_{p,(m,n)}$.

We end our article with an outlook to future work. Our results suggest to further develop *Combinatorial Differential Algebra*.

Acknowledgments. We thank Bernd Sturmfels for suggesting to work on the problem and for insightful discussions. We are grateful to Alex Fink and Gleb Pogudin for useful discussions and to Taylor Brysiewicz, Lars Kastner, Marta Panizzut, and Paul Vater for their great support with computations in `polymake` [8] and `TOPCOM` [19].

1. Differential ideals and jets

1.1. One independent variable.

In this section, we repeat basics from differential algebra and give a link to the theory of jet schemes. For further background on differential algebra, we refer the reader to the books [12, 20].

Consider the polynomial ring $\mathbb{C}[x,x^{(1)},x^{(2)},\dots]$ in the countably infinitely many variables $\{x^{(k)}\}_{k\in\mathbb{N}}$, where $x:=x^{(0)}$. Denote by $\mathbb{C}[x^{(\infty)}]$ the differential ring

$$\mathbb{C}[x^{(\infty)}] := \Big(\mathbb{C}[x,x^{(1)},x^{(2)},\dots],\partial\Big),$$

where, denoting $x=x^{(0)}$, the differential is given as $\partial(x^{(k)})=x^{(k+1)}$ and $\partial|_{\mathbb{C}}\equiv 0$.

Definition 1.1. An ideal $I\lhd\mathbb{C}[x^{(\infty)}]$ is called *differential ideal* if $\partial(I)\subseteq I$. For a subset J of $\mathbb{C}[x^{(\infty)}]$, $\langle J\rangle^{(\infty)}$ denotes the differential ideal generated by J.

We denote by $I_{p,n}:=\langle x^p,x^{(n)}\rangle^{(\infty)}$ the differential ideal in $\mathbb{C}[x^{(\infty)}]$ generated by x^p and $x^{(n)}$ and by $\mathbb{C}[x^{(\le n)}]$ the *truncated differential ring* $\mathbb{C}[x^{(\infty)}]/\langle x^{(n+1)}\rangle^{(\infty)}$.

For $n\in\mathbb{N}$, denote by

$$R_n := \mathbb{C}[x_0,\dots,x_n]$$

the polynomial ring in $n+1$ variables with coefficients in the complex numbers. Consider $f_{p,n} = (x_0 + x_1 t + \cdots + x_n t^n)^p \in R_n[t]$. By the multinomial theorem,

$$f_{p,n} = \sum_{k_0+\cdots+k_n=p} \binom{p}{k_0, k_1, \ldots, k_n} x_0^{k_0} \cdots x_n^{k_n} t^{k_1+2k_2+\cdots+nk_n},$$

where

$$\binom{p}{k_0, k_1, \ldots, k_n} = \frac{p!}{k_0! \cdots k_n!}.$$

Denote by $C_{p,n} \triangleleft R_n$ the ideal generated by the coefficients of $f_{p,n}$. This ideal defines the scheme of n-jets of the affine scheme $\operatorname{Spec}(\mathbb{C}[x]/\langle x^p \rangle)$. Up to constants, the coefficient of t^k in $f_{p,n}$ recovers the k-th derivative of the monomial x^p, giving rise to the following relation between the differential ideal $I_{p,n}$ and the ideal $C_{p,n}$ in the polynomial ring R_n.

Proposition 1.2. *The following map is an isomorphism of $\mathbb{C}$-algebras:*

$$R_n/C_{p,n} \xrightarrow{\cong} \mathbb{C}[x^{(\infty)}]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}.$$

Proof. Notice that $(x^p)^{(k)}$ is given as follows:

$$(x^p)^{(k)} = \sum_{j_0+\cdots+j_{p-1}=k} \binom{k}{j_0, \ldots, j_{p-1}} x^{(j_0)} \cdots x^{(j_{p-1})}.$$

Let us consider its image in the truncated differential ring $\mathbb{C}[x^{(\leq n)}]$. We denote by i_ℓ the multiplicity of ℓ in the multiset $\{j_0, \ldots, j_{p-1}\}$, so that $i_0 + \cdots + i_n = p$ and $i_1 + 2i_2 + \cdots + n i_n = k$. Let $y_i := x^{(i)}$ for all $0 \leq i \leq n$. Then

$$(x^p)^{(k)} = \sum_{j_0+\cdots+j_{p-1}=k} \binom{k}{j_0, \ldots, j_{p-1}} y_0^{i_0} \cdots y_n^{i_n}.$$

In the previous sum, there are some repeated terms: for each $\{j_0, \ldots, j_{p-1}\}$ by exchanging the order of j_i and respecting the numbers $i_0, \ldots, i_n$, we get the same term. We have $\binom{p}{i_0}$ possibilities to choose i_0 many places for 0 in the multiset $\{j_0, \ldots j_{p-1}\}$. We have $\binom{p-i_0}{i_1}$ possibilities to choose i_1 many places for 1 from the remaining places in the set $\{j_0, \ldots, j_{p-1}\}$. We continue like this and obtain

$$(x^p)^{(k)} = \sum_{(i_0,\ldots,i_n)\in I} \binom{p}{i_0, \ldots, i_n} \cdot \frac{k!}{(0!)^{i_0} \cdots (n!)^{i_n}} \cdot y_0^{i_0} \cdots y_n^{i_n},$$

where $I = \{(i_0, i_1, \ldots, i_p) \mid i_0 + \cdots + i_n = p \text{ and } i_1 + \cdots + n i_n = k\}$. Denote by φ the following homomorphism of rings:

$$\varphi\colon\ \mathbb{C}[x^{(\leq n)}] \to R_n/C_{p,n}, \quad x^{(k)} \mapsto k! \cdot x_k.$$

This homomorphism maps $(x^p)^{(k)}$ to the coefficient of t^k in the polynomial $f_{p,n}$ multiplied by $k!$. The kernel of φ is the ideal generated by $\{(x^p)^{(k)}\}_{k\in\mathbb{N}}$. Thus,

$$\mathbb{C}[x^{(\infty)}]/I_{p,n+1} \cong \mathbb{C}[x^{(\leq n)}]/\langle\{(x^p)^{(k)} \mid k \in \mathbb{N}\}\rangle \cong R_n/C_{p,n},$$

concluding the proof. $\square$

Remark 1.3. Proposition 1.2 follows from [17, Proposition 5.12] applied to the ideal generated by x^p. To make this article self-contained, we decided to provide a proof. $\triangle$

Following [18, 24], we now repeat the concept of *differential Gröbner bases.* For that, the monomial orderings have to be compatible with ∂ in the following sense.

Definition 1.4. A monomial ordering $\prec$ on $\mathbb{C}[x^{(\infty)}]$ is called *admissible* if it satisfies the following properties for all monomials $M_1, M_2,$ and M_3:

(i) $1 \prec M_1$ if $M_1 \neq 1$.
(ii) $M_1 \prec M_2$ implies $M_1 M_3 \prec M_2 M_3$.
(iii) $M_1 \prec \operatorname{lm}(\partial M_1)$ if $M_1 \neq 1$.
(iv) $M_1 \prec M_2$ implies $\operatorname{lm}(\partial M_1) \prec \operatorname{lm}(\partial M_2)$.

Example 1.5. The degrevlex ordering is an admissible ordering. We order the variables as $x < x^{(1)} < x^{(2)} < \ldots$ If $M = x_m^{i_m} \cdots x_n^{i_n}$, where $m = \min\{k \mid i_k \neq 0\}$ and x_k is identified with $x^{(k)}$, then $\operatorname{lm}(\partial M) = x_m^{i_m - 1} x_{m+1}^{i_{m+1}+1} \cdots x_n^{i_n}$, which implies $M \prec \operatorname{lm}(\partial M)$. If $M_1 = x_m^{i_m} \cdots x_n^{i_n} \prec M_2 = x_{m'}^{i_{m'}} \cdots x_{n'}^{i_{n'}}$, where $m = \min\{k \mid i_k \neq 0\}$ and $m' = \min\{k \mid i'_k \neq 0\}$, then $\operatorname{lm}(\partial M_1) = x_m^{i_m-1} x_{m+1}^{i_{m+1}+1} \cdots x_n^{i_n} \prec \operatorname{lm}(\partial M_2) = x_{m'}^{i_{m'}-1} x_{m'+1}^{i_{m'+1}+1} \cdots x_{n'}^{i_{n'}}$. △

Definition 1.6. Fix an admissible monomial ordering $\prec$ on $\mathbb{C}[x^{(\infty)}]$ and let $I \lhd \mathbb{C}[x^{(\infty)}]$ be a differential ideal. A subset of polynomials $G \subseteq I$ s.t. $\langle G \rangle^{(\infty)} = I$ is a *differential Gröbner basis of* I if $\{\partial^k(g) \mid k \in \mathbb{N}, g \in G\}$ is an algebraic Gröbner basis of $I \lhd \mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots]$ w.r.t. $\prec$.

Zobnin studied the differential ideal $\langle x^p \rangle^\infty$ and proved the following.

Theorem 1.7 ([24])**.** *The singleton $\{x^p\}$ is a differential Gröbner basis of $\langle x^p \rangle^{(\infty)}$ w.r.t. the reverse lexicographical ordering.*

Remark 1.8. Zobnin proved this result for so called *β-orderings*, i.e., monomial orderings on $\mathbb{C}[x^{(\infty)}]$ for which the leading monomial of $(x^p)^{(k)}$ is of the form $(x^{(i)})^a (x^{(i+1)})^{p-a}$ (see [13]). Since, in this article, we do not need the statement in its full generality, we just point out that the reverse lexicographical ordering is such a β-ordering. Note moreover that $(x^p)^{(k)}$ is bihomogeneous w.r.t. the vectors $(1, 1, 1, \ldots)$ and $(0, 1, 2, 3, \ldots)$, i.e., every monomial summand $\prod_i (x^{(i)})^{a_i}$ in $(x^p)^{(k)}$ verifies $\sum_i a_i = p$, and $\sum_i i a_i = k$. △

1.2. **Two independent variables.** In this subsection, we generalize Proposition 1.2 to two independent variables. We denote by $\mathbb{C}[x^{(\infty,\infty)}]$ the partial differential ring

$$\mathbb{C}[x^{(\infty,\infty)}] := \left(\mathbb{C}[x^{(k,\ell)}]_{k,\ell \in \mathbb{N}}, \partial_s, \partial_t\right)$$

in the two independent variables s, t and the commuting differentials ∂_s, ∂_t acting as

$$\partial_s(x^{(k,\ell)}) = x^{(k+1,\ell)}, \quad \partial_t(x^{(k,\ell)}) = x^{(k,\ell+1)}, \quad \partial_s|_{\mathbb{C}} \equiv \partial_t|_{\mathbb{C}} \equiv 0.$$

For $m, n \in \mathbb{N}$, denote by $I_{p,(m,n)}$ the differential ideal $\langle x^p, x^{(m,0)}, x^{(0,n)} \rangle^{(\infty,\infty)}$ in $\mathbb{C}[x^{(\infty,\infty)}]$ generated by $x^p, x^{(m,0)}$, and $x^{(0,n)}$.

Denote by $R_{m,n}$ the polynomial ring in the $(m+1)(n+1)$ many variables $\{x_{k,\ell}\}_{0 \le k \le m, 0 \le \ell \le n}$ and let $f_{p,(m,n)}$ be the bivariate polynomial

$$f_{p,(m,n)} := \left(\sum_{k=0}^{m} \sum_{\ell=0}^{n} x_{k,\ell} t^k s^\ell\right)^p \in R_{m,n}[s,t].$$

By the multinomial theorem,

$$f_{p,(m,n)} = \sum_{\sum i_{k,\ell}=p} \left[\binom{p}{i_{0,0},\ldots,i_{m,n}} \cdot \prod_{k,\ell} x_{k,\ell}^{i_{k,\ell}} s^{k\cdot i_{k,\ell}} \cdot t^{\ell \cdot i_{k,\ell}} \right],$$

where $(k,\ell) \in \{0,\ldots,m\} \times \{0,\ldots n\}$ and $i_{k,\ell} \in \mathbb{N}$ for all (k,ℓ). Let $C_{p,(m,n)} \lhd R_{m,n}$ denote the ideal generated by the the coefficients of $f_{p,(m,n)}$.

Definition 1.9. We refer to $\mathrm{Spec}(R_{m,n}/C_{p,(m,n)})$ as the scheme of (m,n)*-jets* of the affine monomial scheme defined by x^p.

Proposition 1.10. *The following map is an isomorphism of* $\mathbb{C}$*-algebras:*

$$R_{m,n}/C_{p,(m,n)} \xrightarrow{\cong} \mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m+1,n+1)}, \quad x_{k,\ell} \mapsto \frac{1}{k!\ell!} \cdot x^{(k,\ell)}.$$

Proof. By the multinomial theorem,

$$f_{p,(m,n)} = \sum_{\sum i_{k,\ell}=p} \left[\binom{p}{i_{0,0},\ldots,i_{m,n}} \cdot \prod_{k,\ell} x_{k,\ell}^{i_{k,\ell}} s^{k\cdot i_{k,\ell}} \cdot t^{\ell \cdot i_{k,\ell}} \right].$$

The coefficient $f_{a,b}$ of $s^a t^b$ in $f_{p,(m,n)}$ is given as

$$f_{a,b} = \sum_{(i_{k,\ell})\in I} \left[\binom{p}{i_{0,0},\ldots,i_{m,n}} \cdot \prod_{k,\ell} x_{k,\ell}^{i_{k,\ell}} \right],$$

where $I = \{(i_{k,\ell})_{k,\ell} \mid \sum_{k=0}^m (k \sum_{\ell=0}^n i_{k,\ell}) = a,\ \sum_{\ell=0}^n (\ell \sum_{k=0}^m i_{k,\ell}) = b,\ \sum i_{k,\ell} = p\}$. By the symmetry of the second derivatives, we obtain

$$\begin{aligned}
(x^p)^{(a,b)} &= \left((x^p)^{(a,0)}\right)^{(0,b)} \\
&= \left(\sum_{\sum_{i=0}^m k_i = p,\, k_1+2k_2+\cdots mk_m=a} \binom{p}{k_0,\ldots,k_m} \frac{a!}{(0!)^{k_0}\cdots(m!)^{k_m}} \cdot (x^{(0,0)})^{k_0} \cdots (x^{(m,0)})^{k_m} \right)^{(0,b)} \\
&= \sum_{k_0,\ldots,k_m} \binom{p}{k_0,\ldots,k_m} \frac{a!}{(0!)^{k_0}\cdots(m!)^{k_m}} \sum_{\ell_0+\cdots+\ell_{p-1}=b} \binom{b}{\ell_0,\ldots,\ell_{p-1}} \\
&\qquad \prod_{0\le i\le m} x^{(i,\ell_{k_0+\ldots+k_{i-1}})} \cdots x^{(i,\ell_{k_0+\ldots+k_i-1})}.
\end{aligned}$$

For all $0 \le i \le m$ and $0 \le s \le n$, let j_i^s be the multiplicity of s in the multiset $\{l_{k_0+\cdots+k_{i-1}},\ldots,l_{k_0+\cdots+k_i-1}\}$. Thus $k_i = \sum_{s=0}^n j_i^s$, $\sum_{i,s} j_i^s = p$, $\sum_{i,s} i j_i^s = a$, and $\sum_{i,s} s j_i^s = b$. Let J denote the set of all those $(j_i^s)_{s,i}$. Then $(x^p)^{(a,b)}$ equals

$$\begin{aligned}
&\sum_{(j_i^s)\in J} \left[\frac{p!}{k_0!\cdots k_m!} \frac{a!}{(0!)^{k_0}\cdots(m!)^{k_m}} \frac{b!}{(0!)^{\sum j_i^0}\cdots(n!)^{\sum j_i^n}} \frac{k_0!}{j_0^0!\cdots j_0^n!} \cdots \frac{k_m!}{j_m^0!\cdots j_m^n!} \prod_{i,s} \left(x^{(i,s)}\right)^{j_i^s} \right] \\
&= \sum_{(j_i^s)\in J} \left[\binom{p}{i_{0,0}\cdots i_{m,n}} a!b! \prod_{i,s} \frac{\left(x^{(i,s)}\right)^{j_i^s}}{i!s!} \right],
\end{aligned}$$

concluding the proof. □

In order to generalize Theorem 1.7 to *partial* differential rings, we first generalize the concept of β-orderings. We denote by $\mathbb{C}[x^{(\le m,\le n)}]$ the truncated differential ring $\mathbb{C}[x^{(\infty,\infty)}]/\langle x^{(m,0)}, x^{(0,n)}\rangle^{(\infty,\infty)}$.

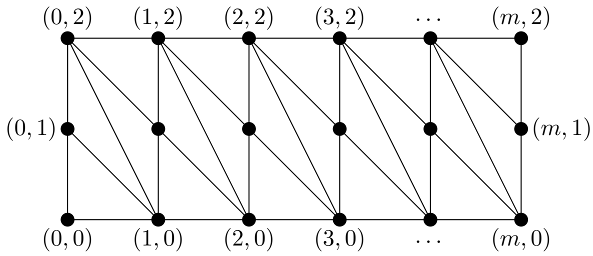

FIGURE 1. The placing triangulation $T_{m,2}$ of the $m \times 2$-rectangle of the point configuration $[(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),\dots,(m,0),(m,1),(m,2)]$.

Definition 1.11. Fix $m, n, p \in \mathbb{N}$ and a triangulation T of the $m \times n$-rectangle. A monomial ordering $\prec$ on $\mathbb{C}[x^{(\leq m, \leq n)}]$ is a T*-ordering* if the leading monomial of each $(x^p)^{(k,\ell)}$, $0 \leq k \leq mp$, $0 \leq \ell \leq np$, is supported on a triangle of T.

Remark 1.12. By identifying $x^{(k,\ell)}$ with $k!\ell!x_{k,\ell}$, one equivalently defines a T*-ordering* as a monomial ordering on $R_{m,n} = \mathbb{C}[\{x_{k,\ell}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}]$ s.t. the leading monomial of each coefficient of $f_{p,(m,n)} \in R_{m,n}[s,t]$ is supported on a triangle of T. △

Denote by $T_{m,2}$ the unimodular triangulation of the $m \times 2$-rectangle depicted in Figure 1. This is the placing triangulation of the point configuration

$$[(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),\dots,(m,0),(m,1),(m,2)].$$

Note that the vector $(1, 2, 2^2, \dots, 2^{3m+1})$ induces the triangulation $T_{m,2}$ in the *lower* hull convention, hence $T_{m,2}$ is a regular triangulation. Denote by $T_{m,n}$ the placing triangulation of $[(0,0),\dots,(0,n),(1,0),\dots,(1,n),\dots,(m,0),\dots,(m,n)]$. It consists of m copies of the triangulation in Figure 2.

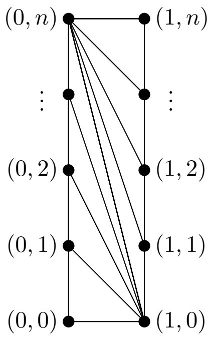

FIGURE 2. The regular placing triangulation $T_{1,n}$ of the $1 \times n$-rectangle for the point configuration $[(0,0),(0,1),\dots,(0,n),(1,0),\dots,(1,n)]$

Proposition 1.13. *For all* $0 \leq k \leq mp$, *and* $0 \leq \ell \leq np$, $(x^p)^{(k,\ell)}$ *has a unique monomial summand supported on a triangle of* $T_{m,n}$. *Moreover, the reverse lexicographical ordering* $\prec$ *on* $\mathbb{C}[x^{(0,0)}, x^{(0,1)}, \dots, x^{(0,n)}, \dots, x^{(m,0)}, \dots, x^{(m,n)}]$ *is a* T*-ordering for* $T = T_{m,n}$ *for all* p, *where we order the variables as* $x^{(0,0)} < x^{(0,1)} < \dots < x^{(m,n)}$.

Proof. Consider $(x^p)^{(k,\ell)}$ and let us suppose that it has a monomial summand supported on a triangle of T of the form $x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c$. Suppose that there exists a monomial $M = \prod_j x_{j,0}^{i_{j,0}} \cdots x_{j,n}^{i_{j,n}}$ in $(x^p)^{(k,\ell)}$ such that $x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c \prec M$. Since all monomial summands in $(x^p)^{(k,\ell)}$ have the same degree, it follows that $i_{h,n} \leq a$, $i_{h,0} = \dots = i_{h,n-1} = 0$, and $i_{j,0} = \dots = i_{j,n} = 0$ for all $j < h$. Moreover, the following identities hold:

$$
\begin{aligned}
a+b+c &= \sum_{j\geq h} i_{j,0}+\cdots+i_{j,n} = p,\\
ha+(h+1)b+(h+1)c &= \sum_{j\geq h} j(i_{j,0}+\cdots+i_{j,n}) = k,\\
na+sb+(s+1)c &= \sum_{j\geq h} i_{j,1}+\cdots+ni_{j,n} = \ell.
\end{aligned}
\tag{1}
$$

Then from the second line in (1), we obtain

$$(a-i_{h,n})+(h+1)\big(\sum_{j\geq h}(i_{j,0}+\cdots+i_{j,n})-p\big)+\sum_{j\geq h+2}(j-h-1)(i_{j,0}+\cdots+i_{j,n})=0.$$

Thus $M = x_{h,n}^{i_{h,n}} x_{h+1,0}^{i_{h+1,0}}\cdots x_{h+1,n}^{i_{h+1,n}}$, $i_{h,n}=a$, and $i_{h+1,0}+\ldots+i_{h+1,n}=b+c$. Since $x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c \prec M$, we have $i_{h+1,s}\leq b$, and for all $r<s, i_{h+1,r}=0$. Then from the third equality we have

$$s(b+c)+c = s(i_{h+1,s}+\cdots+i_{h+1,n})+i_{h+1,s+1}+\cdots+(n-s)i_{h+1,n}.$$

Thus $b=i_{h+1,s}$ and $c=i_{h+1,s+1}$. We conclude that $M=x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c$.

Now suppose there exists a monomial summand of $(x^p)^{(k,\ell)}$ that is supported on a triangle of $T_{m,n}$ of the form $x_{h,s}^a x_{h,s+1}^b x_{h+1,0}^c$ and suppose that there exists a monomial M such that $x_{h,s}^a x_{h,s+1}^b x_{h+1,0}^c \prec M$. Then $i_{h,s}\leq a$, $i_{h,r}=0$ for all $r<s$, and

$$
\begin{aligned}
a+b+c &= \sum_{j\geq h} i_{j,0}+\cdots+i_{j,n} = p,\\
ha+hb+(h+1)c &= \sum_{j\geq h} j(i_{j,0}+\cdots+i_{j,n}) = k,\\
sa+(s+1)b &= \sum_{j\geq h} i_{j,1}+\cdots+ni_{j,n} = \ell.
\end{aligned}
\tag{2}
$$

Suppose $a+b<i_{h,s}+\cdots+i_{h,n}$. Then $b<i_{h,s+1}+\cdots+i_{h,n}$. By the third line in (2),

$$s(a+b)+b = s(i_{h,s}+\cdots+i_{h,n})+(i_{h,s+1}+\cdots+(n-s)i_{h,n})+\sum_{j\geq h+1} i_{j,1}+\cdots+ni_{j,n}, \tag{3}$$

which is a contradiction to our assumption. From the second line in (2) we then obtain

$$(a+b-(i_{h,s}+\cdots+i_{h,n}))+(h+1)\left(\sum_{j\geq h}(i_{j,0}+\cdots+i_{j,n})-p\right)+\sum_{j\geq h+2}(j-h-1)(i_{j,0}+\cdots+i_{j,n})=0.$$

Thus $a+b=i_{h,s}+\cdots+i_{h,n}$, and $c=i_{h+1,0}+\cdots+i_{h+1,n}$. Therefore, from (3) we conclude that $a=i_{h,s}, b=i_{h,s+1}$, and $c=i_{h+1,0}$ which means $M=x_{h,s}^a x_{h,s+1}^b x_{h+1,0}^c$.

We proved that if $(x^p)^{(k,\ell)}$ contains a monomial summand supported on a triangle of $T_{m,n}$, then this monomial is its leading monomial. Therefore, for every $0\leq k\leq mp$, $0\leq \ell\leq np$, $(x^p)^{(k,\ell)}$ has at most one monomial summand that is supported on a triangle of $T_{m,n}$. The triangles of $T_{m,n}$ are given by $\{(j,n),(j+1,s),(j+1,s+1)\}$ and $\{(j+1,0),(j,s),(j,s+1)\}$ for $0\leq j\leq m-1$ and $s=0,\ldots,n-1$. The number of monomials of degree p that are supported on these triangles is $(mp+1)(np+1)$. Indeed, we have $2nm$ triangles, $(3n+1)m+n$ edges, and $(n+1)(m+1)$ vertices on $T_{m,n}$. The number of monomials which can be formed by the $2nm$ triangles containing all three corresponding variables is $2nm\cdot\#\{a+b+c=p\,|\,a,b,c>0\} = 2nm\frac{(p-1)(p-2)}{2}$. The edges give rise to $((3n+1)m+n)(p-1)$ monomials of degree p in which both variables appear. The vertices give rise to $(n+1)(m+1)$ monomials of degree p containing only this variable. Then we have $2nm\frac{(p-1)(p-2)}{2}+((3n+1)m+n)(p-1)+(n+1)(m+1)=(np+1)(mp+1)$. Each of these monomials belongs to the monomials appearing in the expression of $(x^p)^{(k,\ell)}$ for some $0\leq k\leq mp$ and $0\leq \ell\leq np$. We conclude that every $(x^p)^{(k,\ell)}$ has exactly one monomial that is supported on a triangle of $T_{m,n}$ and this monomial is its leading monomial. □

We formulate the following conjectural generalization of Zobnin's result to the case of two independent variables.

Conjecture 1.14. For all $m,p\in\mathbb{N}$, $\{(x^p)^{(k,\ell)}\}_{0\leq k\leq mp,0\leq\ell\leq 2p}$ is a Gröbner basis of the differential ideal generated by x^p in the truncated partial differential ring $\mathbb{C}[x^{(\leq m,\leq 2)}]$ w.r.t. any $T_{m,2}$-ordering.

As indicated in the following proposition, we have computational evidence that this conjecture holds true.

Proposition 1.15. *For $m\leq 12$ and $p\leq 5$, the set of differential polynomials $\{(x^p)^{(k,\ell)}\}_{0\leq k\leq mp,0\leq\ell\leq 2p}$ is a Gröbner basis of the differential ideal generated by x^p in the truncated partial differential ring $\mathbb{C}[x^{(\leq m,\leq 2)}]$ w.r.t. any $T_{m,2}$-ordering.*

Proof. Computations in `Singular` for the degrevlex ordering prove the claim for m and p as indicated in the following table:

m	1	2	3	4	5	6	7	8	9	10	11	12
p ≤	62	21	12	9	8	7	6	6	6	5	5	5

□

Theorem 1.16. *If $m\geq 1$, $n\geq 3$, and $p\geq 2$, then the family $\{(x^p)^{(k,\ell)}\}_{0\leq k\leq mp,0\leq\ell\leq np}$ is* not *a Gröbner basis of the differential ideal generated by x^p in the ring $\mathbb{C}[x^{(\leq m,\leq n)}]$ w.r.t any $T_{m,n}$-ordering.*

Proof. If $\{(x^p)^{(k,\ell)}\}_{0\leq k\leq mp,0\leq\ell\leq np}$ is a Gröbner basis of the differential ideal generated by x^p for the $T_{m,n}$-ordering $\prec$, then the same statement holds for the $T_{m-1,n}$-ordering $\prec$. Therefore, we restrict our proof to the case $m=1$. Let us consider the differential polynomials $(x^p)^{(p-1,3)}$ and $(x^p)^{(p-1,0)}$. We will show that their S-polynomial does not have an LCM-representation. By [4, Theorem 2.9.6], the $(x^p)^{(k,\ell)}$ then are not a Gröbner basis. Note that $\operatorname{lm}((x^p)^{(p-1,3)})=x_{0,3}x_{1,0}^{p-1}$ and $\operatorname{lm}((x^p)^{(p-1,0)})=x_{0,0}x_{1,0}^{p-1}$. Their least common multiple is

$$\operatorname{LCM}(\operatorname{lm}((x^p)^{(p-1,3)}),\operatorname{lm}((x^p)^{(p-1,0)}))=x_{0,0}x_{0,3}x_{1,0}^{p-1}.$$

We proceed by proof by contradiction. Suppose that

$$S((x^p)^{(p-1,3)},(x^p)^{(p-1,0)})=\sum_{a,b}(x^p)^{(a,b)}g_{a,b},$$

where $\operatorname{lm}((x^p)^{(a,b)}g_{a,b})\prec x_{0,0}x_{0,3}x_{1,0}^{p-1}$. Since all monomials in $S((x^p)^{(p-1,3)},(x^p)^{(p-1,0)})$ are of degree $p+1$ and homogeneous with respect to both derivatives ∂_s and ∂_t, we can write

$$S((x^p)^{(p-1,3)},(x^p)^{(p-1,0)})=\sum_{p-2\leq a\leq p-1,0\leq b\leq 3}c_{a,b}(x^p)^{(a,b)}x_{p-1-a,3-b},$$

where $c_{a,b}$ are constants and $(x^p)^{(a,b)}x_{p-1-a,3-b} \prec x_{0,1}x_{0,3}x_{1,0}^{p-1}$. We now list the polynomials that can show up in the previous equality with their leading monomials:

$$\begin{aligned}
&(x^p)^{(p-2,0)}x_{1,3}, \quad \mathrm{lm}((x^p)^{(p-2,0)}x_{1,3}) = x_{0,0}^2 x_{1,0}^{p-2}x_{1,3},\\
&(x^p)^{(p-2,1)}x_{1,2}, \quad \mathrm{lm}((x^p)^{(p-2,1)}x_{1,2}) = x_{0,0}x_{0,1}x_{1,0}^{p-2}x_{1,2},\\
&(x^p)^{(p-2,2)}x_{1,1}, \quad \mathrm{lm}((x^p)^{(p-2,2)}x_{1,1}) = x_{0,1}^2 x_{1,0}^{p-2}x_{1,1},\\
&(x^p)^{(p-2,3)}x_{1,0}, \quad \mathrm{lm}((x^p)^{(p-2,3)}x_{1,0}) = x_{0,1}x_{0,2}x_{1,0}^{p-2}x_{1,1},\\
&(x^p)^{(p-1,0)}x_{0,3}, \quad \mathrm{lm}((x^p)^{(p-1,0)}x_{0,3}) = x_{0,0}x_{1,0}^{p-1}x_{0,3},\\
&(x^p)^{(p-1,1)}x_{0,2}, \quad \mathrm{lm}((x^p)^{(p-1,1)}x_{0,2}) = x_{0,1}x_{1,0}^{p-1}x_{0,2},\\
&(x^p)^{(p-1,2)}x_{0,1}, \quad \mathrm{lm}((x^p)^{(p-1,2)}x_{0,1}) = x_{0,2}x_{1,0}^{p-1}x_{0,1},\\
&(x^p)^{(p-1,3)}x_{0,0}, \quad \mathrm{lm}((x^p)^{(p-1,3)}x_{0,0}) = x_{0,3}x_{1,0}^{p-1}x_{0,0}.
\end{aligned}$$

Within all these polynomials, only $(x^p)^{(p-2,0)}x_{1,3}$ and $(x^p)^{(p-2,1)}x_{1,2}$ have leading monomials $\prec x_{0,0}x_{0,3}x_{1,0}^{p-1}$. Thus,

$$S((x^p)^{(p-1,3)}, (x^p)^{(p-1,0)}) = c_{p-2,0}(x^p)^{(p-2,0)}x_{1,3} + c_{p-2,1}(x^p)^{(p-2,1)}x_{1,2}. \tag{4}$$

Note that $x_{0,2}x_{1,0}^{p-2}x_{1,1}$ is a monomial summand of the polynomial $(x^p)^{(p-1,3)}$. Then $x_{0,0}x_{0,2}x_{1,0}^{p-2}x_{1,1}$ shows up in $S((x^p)^{(p-1,3)}, (x^p)^{(p-1,0)})$ but not in $c_{p-2,0}(x^p)^{(p-2,0)}x_{1,3}+ c_{p-2,1}(x^p)^{(p-2,1)}x_{1,2}$, which is in contradiction to Equation (4). □

2. Linking $\dim_{\mathbb{C}}(R_n/C_{p,n})$ and $\dim_{\mathbb{C}}(R_{m,n}/C_{p,(m,n)})$ to lattice polytopes

We now investigate the sequences $\dim_{\mathbb{C}}(R_n/C_{p,n})$ and $\dim_{\mathbb{C}}(R_{m,n}/C_{p,(m,n)})$, both considered as sequence in p. We link them to lattice polytopes.

2.1. Polynomiality of $\dim_{\mathbb{C}}(R_n/C_{p,n})$.

We investigate the following question.

Question 2.1. *Fix $n \in \mathbb{N}$. As p varies, is $(\dim_{\mathbb{C}}(R_n/C_{p,n}))_{p\in\mathbb{N}}$ a polynomial in p of degree $n+1$?*

Before turning to the proof that this question has a positive answer, we present an explicit example.

Example 2.2 ($\dim_{\mathbb{C}}(R_6/C_{p,n})_{p\in\mathbb{N}}$)**.** Computations in `Singular` [6] reveal the first 13 entries of the sequence $\dim_{\mathbb{C}}(R_6/C_{p,n})_{p\in\mathbb{N}}$ to be

$$0, 1, 34, 353, 2037, 8272, 26585, 72302, 173502, 377739, 760804, 1437799, 2576795,$$

coinciding with the sequence www.oeis.org/A244881. With `Mathematica`, we compute the interpolating polynomial on the values for $p = 1, \dots, 20$ to be

$$\frac{17}{315}p^7 + \frac{17}{90}p^6 + \frac{53}{180}p^5 + \frac{19}{72}p^4 + \frac{13}{90}p^3 + \frac{17}{360}p^2 + \frac{1}{140}p,$$

which is indeed of degree $7 = 6 + 1$. △

Let $\prec$ denote the reverse lexicographical ordering on $R_n = \mathbb{C}[x_0, \dots, x_n]$. In the following lemma, we determine the initial ideal of $C_{p,n}$ w.r.t. $\prec$. The main ingredients for the proof are the main result in [24] and Proposition 1.2.

Lemma 2.3. *The initial ideal of $C_{p,n}$ with respect to $\prec$ is generated by the family $\left\{x_i^{u_i}x_{i+1}^{u_{i+1}} \mid u_i + u_{i+1} = p,\ 0 \le i \le n-1\right\}$.*

Proof. Let us first prove that the leading monomials of our family of generators are $x_i^{u_i}x_{i+1}^{u_{i+1}}$. Let $0 \leq k < np$ be of the form $k = mp + (p-a)$, where $1 \leq a \leq p$ and $0 \leq m \leq n-1$. For $k = np$, the leading term of f_k is x_n^p, where f_k denotes the coefficient of t^k in the polynomial $f_{p,n}$. We claim that the leading monomial of the polynomial f_k is $x_m^a x_{m+1}^{p-a}$. Suppose that $x_0^{i_0}\cdots x_n^{i_n} \succ x_m^a x_{m+1}^{p-a}$ for some monomial summand $x_0^{i_0}\cdots x_n^{i_n}$ in f_k. This implies that $i_0 = \cdots = i_{m-1} = 0$. Then $i_m + \cdots + i_n = p$ and $mi_m + \cdots + ni_n = mp + p - a = k$. Since $i_m \leq a$, from

$$(a - i_m) + (m+1)(i_m + \cdots + i_n - p) + (i_{m+2} + \cdots + (n-m-1)i_n) = 0$$

we conclude that $i_m = a$, $i_{m+1} = p - a$, and $i_{m+2} = \cdots = i_n = 0$. Therefore, $x_m^a x_{m+1}^{p-a}$ is indeed the leading monomial of f_k. We now consider the truncated differential ring ring $\mathbb{C}[x^{(\leq n)}]$. As rings, $\mathbb{C}[x^{\leq n}] \cong \mathbb{C}[x_0, \ldots, x_n] = R_n$. Then the following holds:

$$\mathrm{in}_{\prec}\langle\{(x^p)^{(k)} \mid 0 \leq k \leq np\}\rangle = \langle\{\mathrm{lm}\left(\sum_{k=0}^{np}[r_k](x^p)^{(k)}\right) \big| r_k \in \mathbb{C}[x^{(\infty)}]\rangle,$$

where $[r_k]$ denotes the equivalence class of r_k in $\mathbb{C}[x^{\leq n}]$. By Zobnin's result, $\mathrm{lm}\left(\sum_{k=0}^{np}(x^p)^{(k)}r_k\right)$ is contained in the ideal generated by the family of elements $\{\mathrm{lm}(x^p)^{(k)}\}_{k\in\mathbb{N}}$. Therefore, the initial ideal of $\langle\{(x^p)^{(k)}\}_{0\leq k\leq np}\rangle$ is generated by $\{\mathrm{lm}((x^p)^{(k)})\}_{0\leq k\leq np}$, concluding the proof. $\square$

Lemma 2.4. *The convex polytope*

$$P_n \coloneqq \left\{(u_0, \ldots, u_n) \in (\mathbb{R}_{\geq 0})^{n+1} \big| u_i + u_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n-1\right\}$$

is a lattice polytope whose vertices are binary vectors with no consecutive 1*s.*

Before proving the lemma, we recall some definitions from graph theory. Let $G = (V, E)$ be an undirected graph, where V denotes the set of vertices and E the set of edges. A *clique* of G is a complete subgraph of G. A graph is *perfect* if for every subgraph, the chromatic number equals the clique number of that subgraph. A subset $S \subseteq V$ of vertices is called *stable* if no two elements of S are adjacent. Borrowing the notation from [10], the *stable set polytope of* G is the $|V|$-dimensional polytope

$$\mathrm{Stab}(G) \coloneqq \mathrm{conv}\left\{\chi^S \in \mathbb{R}^V \mid S \subseteq V \text{ stable}\right\},$$

where the *incidence vectors* $\chi^S = (\chi_v^S)_{v\in V} \in \mathbb{R}^V$ are defined as

$$\chi_v^S \coloneqq \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{else.} \end{cases}$$

The *fractional stable set polytope of* G is defined as

$$\mathrm{QStab}(G) \coloneqq \left\{ x \in \mathbb{R}^V \mid 0 \leq x(v)\, \forall v \in V,\ \sum_{v\in Q} x(v) \leq 1 \text{ for all cliques } Q \text{ of } G \right\}.$$

Hence $\mathrm{Stab}(G) = \mathrm{conv}\{\{0,1\}^V \cap \mathrm{QStab}(G)\}$. Chvátal [3, Theorem 3.1] proved that a graph G is perfect if and only if $\mathrm{Stab}(G) = \mathrm{QStab}(G)$. If follows from Fulkerson's theory of anti-blocking polyhedra [7] that this result is equivalent to the perfect graph theorem. The latter was conjectured by Berge [1] and proven by Lovász [15].

Proof of Lemma 2.4. Consider the graph $G = (\{0, 1, \ldots, n\}, \{[i, i+1]\}_{i=0,\ldots,n-1})$. Observe that P_n is precisely the fractional stable set polytope of G. Since G is a perfect graph, $\mathrm{QStab}(G) = \mathrm{Stab}(G)$ and P_n has binary vertices as claimed. $\square$

For an n-dimensional polytope $P \subseteq \mathbb{R}^n$ with integer vertices and $t \in \mathbb{N}$, denote by $L_P(t) \coloneqq |tP \cap \mathbb{Z}^n|$ the number of lattice points of the dilated polytope tP. E. Erhart proved that this number is a rational polynomial in t of degree n, i.e., there exist rational numbers $l_{P,0}, \ldots, l_{P,n}$, s.t.

$$L_P(t) = l_{P,n}t^n + \cdots + l_{P,1}t + l_{P,0}.$$

The polynomial $L_P \in \mathbb{Q}[t]$ is called the *Erhart polynomial* of P.

Proposition 2.5. *The number* $\dim_{\mathbb{C}}(R_n/C_{p,n})$ *is the Ehrhart polynomial of the polytope* P_n *defined in Lemma 2.4 evaluated at* $p-1$.

Proof. From Lemma 2.3 we read that $x_0^{u_0}\cdots x_n^{u_n}$ is a standard monomial if and only if $u_i + u_{i+1} < p$ for all $0 \leq i \leq n-1$. The claim then follows from Lemma 2.4. $\square$

2.2. **Investigation of** $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$. In this section, we generalize the results found for $R_n/C_{p,n}$ to two independent variables, i.e., to $R_{m,n}/C_{p,(m,n)}$.

Proposition 2.6. *For* $m \leq 12$ *and* $p \leq 5$, $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$ *is the Erhart polynomial of the* $3(m+1)$*-dimensional lattice polytope*

$$\begin{aligned}P_{(m,2)} \coloneqq \big\{ (u_{00}, u_{01}, u_{02} \ldots, u_{m0}, u_{m1}, u_{m2}) \in (\mathbb{R}_{\geq 0})^{3(m+1)} \big| u_{k_1,l_1} + u_{k_2,l_2} + u_{k_3,l_3} \leq 1 \\ \textit{for all indices s.t. } \{(k_1,l_1),(k_2,l_2),(k_3,l_3)\} \textit{ is a triangle of } T_{m,2} \big\}\end{aligned}$$

evaluated at $p-1$.

Proof. Let G be the edge graph of the regular triangulation from Figure 1 for $m=2$ with $3(m+1)$ vertices and $2+7m$ edges. Since this graph is perfect and the maximal cliques are precisely the triangles of $T_{m,2}$, $\mathrm{Stab}(G) = \mathrm{QStab}(G) = P_{(m,2)}$. By Theorem 1.15, $x_{00}^{u_{00}}\cdots x_{m2}^{u_{m2}}$ is a standard monomial if and only if for all triples of indices $\{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$ that are a triangle of $T_{m,2}$, $u_{i_1,j_1} + u_{i_2,j_2} + u_{i_3,j_3} \leq p-1$. $\square$

In terms of integer programming, Proposition 2.6 translates as follows.

Corollary 2.7. *For* $m \leq 12$ *and* $p \leq 5$, $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$ *is polynomial in* p *of degree* $3(m+1)$. *It is the number of non-negative integer solutions of the* $2m^2$ *linear inequalities* $x_{i_1,j_1} + x_{i_2,j_2} + x_{i_3,j_3} \leq p-1$, *where* $\{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$ *runs over the* $2m^2$ *many triangles of* $T_{m,2}$.

3. Regular unimodular triangulations of the $m \times n$-rectangle

We now outline how regular unimodular triangulations of the $m \times n$-rectangle give rise to T-orderings on the truncated partial differential ring $\mathbb{C}[x^{(\leq m, \leq n)}]$—or, equivalently, on the polynomial ring $\mathbb{C}[\{x_{k,\ell}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}]$.

Example 3.1 ($m = n = 2$)**.** Again, denote by $C_{p,(2,2)}$ the ideal in

$$R_{2,2} = \mathbb{C}[x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}]$$

generated by the $(2p+1)^2$ many coefficients $f_{k,\ell}$ of $s^k t^\ell$ in

$$f_{p,(2,2)} = (x_{00} + x_{01}t + x_{02}t^2 + x_{10}s + x_{11}st + x_{12}st^2 + x_{20}s^2 + x_{21}s^2t + x_{22}s^2t^2)^p.$$

Let $\prec$ denote the weighted degrevlex ordering on $R_{2,2}$ for the weight vector

$$w_{2,2} \coloneqq \big(2^8+1, \ldots, 2^8+1\big) - \big(1, 2, 2^2, \ldots, 2^8\big) \in \mathbb{N}^9,$$

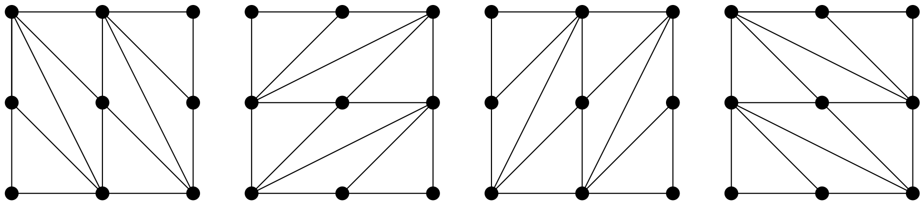

FIGURE 3. The four regular unimodular triangular regulations of the 2×2-square giving rise to a Gröbner basis, the first of which is $T_{2,2}$. Note that they all arise from $T_{2,2}$ by rotating and flipping.

i.e., putting weight 128 to x_{00}, weight 127 to x_{01}, and so on. Note that $w_{2,2}$ induces the triangulation $T_{2,2}$ in the *upper* hull convention. For $p=3$, we find that within the monomials of the $f_{k,\ell}$, the following 8 triples of pairwise different variables show up:

$$\{x_{00},x_{01},x_{10}\},\{x_{01},x_{02},x_{10}\},\{x_{02},x_{10},x_{11}\},\{x_{02},x_{11},x_{12}\},$$
$$\{x_{10},x_{11},x_{20}\},\{x_{11},x_{12},x_{20}\},\{x_{12},x_{20},x_{21}\},\{x_{12},x_{21},x_{22}\},$$

the indices of each of which define a triangle of $T_{2,2}$. Computations in `Singular` prove that the coefficients of $f_{3,(2,2)}$ are a Gröbner basis of $C_{3,(2,2)}\triangleleft R_{2,2}$ w.r.t. the weighted degrevlex ordering for $w_{2,2}$. We checked that the same statement holds true for $p\leq 25$ and hence the weighted degrevlex ordering is a $T_{2,2}$-ordering for $p\leq 25$. There are 64 regular unimodular triangulations of the 2×2-square in total, four of which give rise to a Gröbner basis in the sense above. Those are depicted in Figure 3. △

For $m=5,n=2$, we validated with `Singular` that the coefficients of $f_{p,(5,2)}$ are a Gröbner basis w.r.t the weighted degrevlex ordering $\prec$ for a vector inducing $T_{5,2}$ in the upper hull convention up to $p=9$, approving that $\prec$ is a $T_{5,2}$-ordering for $p\leq 9$. For the 8×2-rectangle, we validated this for $p\leq 6$. For greater values, even though computing over finite characteristics, the computations are expensive and did not terminate within several days.

Remark 3.2 (Truncated 2-wedges)**.** Let us consider the truncated 2-wedges of x^p as studied in [22]. The placing triangulation of the point configuration $[(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)]$ is induced by the vector $(1,2,4,8,16,32)$. Computations in `Singular` reveal that the coefficients of $(\sum_{k+\ell\leq 2}x_{k,\ell}s^kt^\ell)^3$ are a Gröbner basis w.r.t. the weighted degrevlex ordering for $(32,31,29,25,17,1)$. Mimicking this setup for the triangle $\{(0,0),(3,0),(0,3)\}$ does *not* give rise to a Gröbner basis. △

Example 3.3 ($m=3,n=2$)**.** Again, denote by $C_{p,(3,2)}$ the ideal in

$$R_{3,2}=\mathbb{C}\left[x_{00},x_{01},x_{02},x_{10},x_{11},x_{12},x_{20},x_{21},x_{22},x_{30},x_{31},x_{32}\right]$$

generated by the $(3p+1)(2p+1)$ coefficients of $f_{p,(3,2)}$. Let $\prec$ denote the weighted degrevlex ordering on $R_{3,2}$ for the weight vector

$$w_{3,2}\coloneqq(2^{11}+1,\dots,2^{11}+1)-(2^0,2^1,\dots,2^{11})\in\mathbb{N}^{12}\,.$$

For $p=3$, the following 12 triples of pairwise different variables show up within the leading monomials of the 70 coefficients:

$$\{x_{00},x_{01},x_{10}\},\{x_{01},x_{02},x_{10}\},\{x_{02},x_{10},x_{11}\},\{x_{02},x_{11},x_{12}\},$$
$$\{x_{10},x_{11},x_{20}\},\{x_{11},x_{12},x_{20}\},\{x_{12},x_{20},x_{21}\},\{x_{12},x_{21},x_{22}\},$$
$$\{x_{20},x_{21},x_{30}\},\{x_{21},x_{22},x_{30}\},\{x_{22},x_{30},x_{31}\},\{x_{22},x_{31},x_{32}\},$$

the indices of each of which define a triangle of $T_{3,2}$. Computations in `Singular` prove that the 70 coefficients $f_{k,\ell}$ of $s^k t^\ell$ in $f_{3,(3,2)}$ are indeed a Gröbner basis of $C_{3,(3,2)} \lhd R_{3,2}$ w.r.t. $\prec$, turning $\prec$ into a $T_{3,2}$-ordering for $p=3$. Note that there are 852 regular unimodular triangulations of the 3×2-rectangle in total, four of which give rise to a Gröbner basis in the sense above. Those four are depicted in Figure 4. △

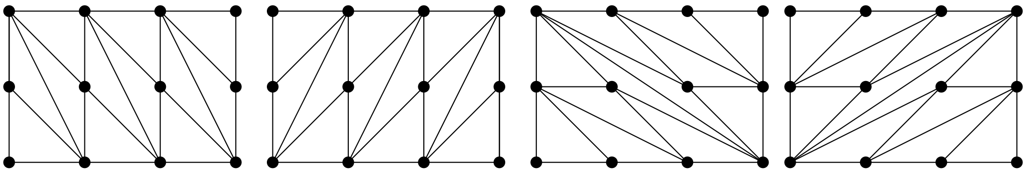

FIGURE 4. The four regular unimodular triangular regulations of the 3×2-rectangle giving rise to a Gröbner basis, the first of which is $T_{3,2}$

Question 3.4. *For which $m,n,p\in\mathbb{N}$ does there exist a regular unimodular triangulation T of the $m\times n$-rectangle such that the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of $C_{p,(m,n)}$ w.r.t. the weighted degree reverse lexicographical ordering for a vector inducing that triangulation in the* upper *hull convention?*

One natural continuation of the triangulation $T_{m,2}$ of the $m\times 2$-rectangle to $m\times n$ consists of m copies of the triangulation of the $1\times n$-rectangle that is depicted in Figure 2, namely the placing triangulation of the point configuration

$$[(0,0),(0,1),\ldots,(0,n),(1,0),\ldots,(1,n),\ldots,(m,0),\ldots,(m,n)]\,.$$

We point out that this triangulation does *not* lead to a positive answer of Question 3.4 in general. For instance, $T_{1,3}$ does not give rise to a Gröbner basis. Only the four triangulations depicted in Figure 5 do.

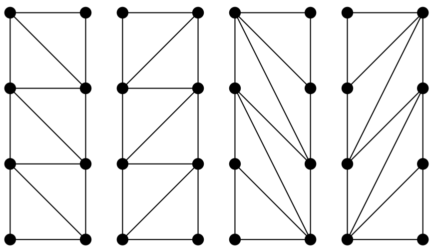

FIGURE 5. The four regular unimodular triangular regulations of the 1×3-rectangle giving rise to a Gröbner basis.

For $m=n=3$, the question has a negative answer. There are in total 46.452 regular unimodular triangulations of the 3×3-square. For none of the vectors in the relative interior of the secondary cone of those triangulations, the coefficients of $f_{3,(3,3)}$ are a Gröbner basis of $C_{3,(3,3)}$ w.r.t. the weighted degrevlex ordering.

Remark 3.5. As pointed out in [2], there are—up to symmetries—5941 regular unimodular triangulations of the 3×3-square. It would actually be sufficient to check the Gröbner basis property for each of those. △

It would be intriguing to find the reason for this failure and to determine all $m, n \in \mathbb{N}$ for which Question 3.4 has a positive answer. Let us point out that this problem gets computationally expensive quickly: for the 4×2-rectangle, there are 12.170 regular unimodular triangulations, whereas for the 4×3-rectangle, there are already 2.822.146.

Now let T be a triangulation of the $m \times n$-rectangle as asked for in Question 3.4. We end this article with two questions, for both of which we have computational evidence.

Question 3.6. *Are the four triangulations depicted in Figure 4, continued to the $m \times 2$-triangle,* all *regular unimodular triangulations that give rise to a Gröbner basis?*

Question 3.7. *As p varies, is $\dim_{\mathbb{C}}(R_{m,n}/C_{p,(m,n)})$ the Ehrhart polynomial of the fractional stable set polytope of the edge graph of T and is this graph perfect?*

References

[1] C. Berge. Sur une conjecture relative au problème des codes optimaux. *Comm. 13 e assemblée générale de l'URSI, Tokyo*, 1962.

[2] S. Brodsky, M. Joswig, R. Morrison, and B. Sturmfels. Moduli of tropical plane curves. *Research in the Mathematical Sciences*, 2(4), 2015.

[3] V. Chvátal. On certain polytopes associated with graphs. *J. Combin. Theory Ser. B*, 18:138–154, 1975.

[4] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer New York, 2007.

[5] J. A. De Loera, J. Rambau, and F. Santos. *Triangulations*, volume 25 of *Algorithms and computations in Mathematics*. Springer-Verlag Berlin Heidelberg, 2010.

[6] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 4-1-3—A computer algebra system for polynomial computations. http://www.singular.uni-kl.de, 2020.

[7] D. R. Fulkerson. Anti-blocking polyhedra. *J. Combin. Theory Ser. B*, 12:50–71, 1972.

[8] E. Gawrilow and M. Joswig. `polymake`: A framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

[9] R. A. Goward Jr. and K. E. Smith. The jet scheme of a monomial scheme. *Comm. Algebra*, 34:1591–1598, 2006.

[10] M. Grötschel. My favorite theorem. *OPTIMA*, 62:2–5, 1999.

[11] R. Gustavson, M. Kondratieva, and A. Ovchinnikov. New effective differential Nullstellensatz. *Advances in Mathematics*, 290:1138–1158, 2016.

[12] E. R. Kolchin. *Differential Algebra and Algebraic Groups*. Pure and applied mathematics. Academic Press, 1973.

[13] H. Levi. On the structure of differential polynomials and on their theory of ideals. *Trans. Amer. Math. Soc.*, 51(3):532–568, 1942.

[14] W. Li, A. Ovchinnikov, G. Pogudin, and T. Scanlon. Algorithms yield upper bounds in differential algebra. arxiv:2005.01608, 2020.

[15] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, 2, 1971.

[16] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[17] R. Moosa and T. Scanlon. Jet and prolongation spaces. *J. Inst. Math. Jussieu*, 9:391–430, 2010.

[18] F. Ollivier. Standard bases of differential ideals. *Lecture Notes in Comput. Sci.*, 508:304–321, 1990.

[19] J. Rambau. `TOPCOM`: Triangulations of point configurations and oriented matroids. In A. M. Cohen, X.-S. Gao, and N. Takayama, editors, *Mathematical Software—ICMS 2002*, pages 330–340. World Scientific, 2002.

[20] J. F. Ritt. *Differential Algebra*, volume 14 of *American Mathematical Society: Colloquium publications*. American Mathematical Society, 1950.

[21] B. Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University lecture series*. American Mathematical Society, Providence, R.I., 1996.

[22] C. Yuen. One higher dimensional analog of jet schemes. arXiv:math/0608633, 2006.

[23] C. Yuen. The multiplicity of jet schemes of a simple normal crossing divisor. *Comm. Algebra*, 35, 2007.

[24] A. I. Zobnin. One-element differential standard bases with respect to inverse lexicographical orderings. *J. Math. Sci. (N. Y.)*, 163(5):523–533, 2009.

Rida Ait El Manssour, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstrasse 22, 04103 Leipzig, Germany

Email address: rida.manssour@mis.mpg.de

Anna-Laura Sattelberger, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstrasse 22, 04103 Leipzig, Germany

Email address: anna-laura.sattelberger@mis.mpg.de