ON THE DEFECTIVITY OF SEGRE-VERONESE VARIETIES VIA COLLAPSING POINTS

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Abstract. The study of dimensions of secant varieties is a very classical subject which regained a lot of interest in the last part of the last century due to its relation with the study of tensor decompositions. The celebrated Alexander-Hirschowitz Theorem of 1995 completed the classification of Veronese varieties whose secant varieties have dimension less than the expected. Since then, a great literature has been dedicated to similar classifications for Segre and Segre-Veronese varieties. In 2013, Abo and Brambilla conjectured that Segre-Veronese embeddings of $\mathbb{P}^m \times \mathbb{P}^n$ in bidegree (c, d) are never defective if both c and d are larger or equal than three. They also proved the inductive step of a possible proof, namely they showed that if they are non-defective in the cases $(3, 3), (3, 4)$ and $(4, 4)$, then they are non-defective for higher bidegrees. In this paper we solve the case $(3, 3)$. Following a classical approach, we turn our attention to the equivalent problem of computing the dimensions of linear systems of divisors of bi-degree $(3, 3)$ in $\mathbb{P}^m \times \mathbb{P}^n$ with general 2-fat base points. The novelty is to use a degeneration technique that allows some of the base points to collapse together. The latter technique proved its power in a recent work of the first author and Mella in the context of identifiability for Waring decompositions of general polynomials.

The present work has not yet been submitted to the arXiv nor to a journal since we are still trying to use these methods to approach the cases $(3, 4)$ *and* $(4, 4)$*. At the moment of the submission to participate at MEGA2021, this should be considered as a work-in-progress, but we might submit it to a journal during the next few months before the conference.*

1. Introduction

A classical problem in algebraic geometry that goes back to late XIX century concerns the classification of *defective varieties*, i.e., the classification of algebraic varieties whose *secant varieties* have dimension strictly smaller than the one expected by a direct parameter count. In the last decades, this problem gained a lot of attention due to its relation with *additive decompositions* of *tensors*. Indeed, *Segre varieties* parametrize *decomposable tensors*; similarly, *Veronese varieties* and *Segre-Veronese varieties* are the symmetric and partially-symmetric analogous. We refer to [CGO14] and [BCC+18] for an overview on the geometric problem and to [Lan12] for the relations between secant varieties and questions on tensors.

The main result in this area of research is the celebrated *Alexander-Hirschowitz Theorem*, proven in [AH95], which classifies defective Veronese varieties, completing the work started more than 100 years earlier. Denote by V_n^d be the Veronese variety given by the embedding of $\mathbb{P}^n$ via the linear system of degree d divisors. Several examples of defective Veronese varieties were known already at the time of Clebsch, Palatini and Terracini, but we had to wait until the work of Alexander and Hirschowitz to have a complete proof that those were the only exceptional cases among Veronese varieties.

Theorem 1.1 (Alexander-Hirschowitz)**.** *Let n, d and r be positive integers. The Veronese variety V_n^d is r-defective if and only if either*

1. $d = 2$ *and* $2 \leq r \leq n$*, or*
2. $(n, d, r) \in \{(2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)\}$.

We refer to [BO08, Section 7] for an historical overview on this theorem. After this result, the community tried to extend the classification of defective varieties to Segre and Segre-Veronese varieties. Here we focus on the Segre-Veronese variety with two factors, i.e., the image $\mathrm{SV}_{m\times n}^{c,d}$ of the embedding of $\mathbb{P}^m \times \mathbb{P}^n$ via the linear system of divisors of bidegree (c, d). The Segre variety corresponds to $c = d = 1$. There are several known defective Segre varieties, and a conjectural classification can be found in [AOP09]. Many Segre-Veronese varieties are defective as well. Defective cases were found by Catalisano, Geramita and Gimigliano [CGG05, CGG08], Abrescia [Abr08], Bocci [Boc05], Dionisi and Fontanari [DF01], Abo and Brambilla [AB09], Carlini and Chipalkatti [CC03] and Ottaviani [Ott07]. In [AB13, Conjecture 5.5], Abo and Brambilla conjectured that these are the only defective cases. In all examples in which $\mathrm{SV}^{c,d}_{m\times n}$ is known to be defective, either c or d is strictly smaller than three. This suggested a weaker conjecture, stated in [AB13, Conjecture 5.6].

Conjecture 1.2 (Abo-Brambilla)**.** If $c \geq 3$ and $d \geq 3$, then $\mathrm{SV}^{c,d}_{m\times n}$ is not defective for any m and n.

Abo and Brambilla themselves managed to greatly reduce the problem. Thanks to [AB13, Theorem 1.3], in order to prove Conjecture 1.2 it is enough to prove that $\mathrm{SV}^{3,3}_{m\times n}$, $\mathrm{SV}^{3,4}_{m\times n}$ and $\mathrm{SV}^{4,4}_{m\times n}$ are not defective for every m and n. This reminds what happened with Theorem 1.1, where the last one to be overcome was the case of cubics. Low degrees are difficult to handle because they are rich of defective cases, therefore they cannot be used as base cases for inductive arguments. The purpose of this paper is to solve one of the three remaining cases.

Theorem 1.3. *For any m and n, then $\mathrm{SV}^{3,3}_{m\times n}$ is not defective.*

A classical approach to compute the dimension of secant varieties consists in translating the problem to the computation of the dimension of certain linear system of divisors with multiple base points. The latter can be computed with degeneration techniques in which the base points are assumed to have support in some special configuration in order to start an inductive argument; see Section 2.2 for details. The idea is well known since the work by Castelnuovo and Terracini at the beginning of last century. In the 1980s, Alexander and Hirschowitz improved drastically this method by introducing a new degeneration technique, called *differential Horace method*, which allowed them to complete the classification of defective Veronese varieties. Despite its major success in the proof of Theorem 1.1, as well as most of the results about defectiveness of Segre and Segre-Veronese varieties, the differential Horace method might lead to linear systems whose base locus has a complicated structure, making this approach sometimes difficult to apply.

In this paper, we employ a different degeneration approach, introduced by Evain in [Eva97]: the base points are not only degenerated to a special position, but also allowed to collide together; see Section 2.3 for details. The degenerated linear system has a 0-dimensional base point with a very special, yet understood, non-reduced structure. Apparently a disadvantage, this new scheme can be very useful to find the right specialization and the right inductive approach. This technique proved to be efficient for instance in [Eva99] for linear systems of plane curves, in [Gal19] for linear systems in $\mathbb{P}^3$ and in [GM19] in the context of Waring decompositions of polynomials.

As far as we know, this is the first time this method is applied in the multigraded case and to approach problems regarding classification of defective varieties. We believe that it has great potential to be exploited towards a complete proof of Conjecture 1.2.

Structure of the paper. In Section 2 we recall the basic definitions for secant varieties and linear systems with multiple base points. We also illustrate the tools we use in our computation, such as Castelnuovo exact sequence and collisions of fat points. In Section 3 we prove a series of results that we apply in the proof of our main theorem, which is presented in Section 4. In Appendix A we describe the software computations we performed to check the initial cases of our inductive proofs. In Appendix B we collect long and tedious arithmetic computations needed in our proofs.

Acknowledgements. The project started while F.G. was a postdoc at MPI Leipzig (Germany) and A.O. was a postdoc at OVGU Magdeburg (Germany). A.O. thanks MPI Leipzig for providing a perfect environment during several visits at the beginning of the project. F.G. is supported by the fund FRA 2018 of University of Trieste - project DMG, and acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics and Geosciences, University of Trieste. A.O. acknowledges partial financial support from A. v. Humboldt Foundation/Stiftung through a fellowship for postdoctoral researchers.

2. Basics and background

Definition 2.1. Fixed $m, n, c, d \in \mathbb{N}$, the **Segre-Veronese variety** $\mathrm{SV}^{c,d}_{m\times n}$ is the image of the embedding of $\mathbb{P}^m \times \mathbb{P}^n$ via the linear system of divisors of bidegree (c, d).

The Segre-Veronese variety has a precise interpretation in terms of partially symmetric tensors. Let Sym^d_{n+1} be the $\mathbb{C}$-vector space of degree d homogeneous polynomials in $n+1$ variables with complex coefficients. The variety $\mathrm{SV}^{c,d}_{m\times n}$ is parametrized by partially symmetric tensors in $\mathrm{Sym}^c_{m+1}\otimes\mathrm{Sym}^d_{n+1}$ which are *decomposable*, or of *rank* 1, i.e.,

$$\mathrm{SV}^{c,d}_{m\times n} = \{f\otimes g \ : \ f\in \mathrm{Sym}^c_{m+1}, g\in\mathrm{Sym}^d_{n+1}\}\subset \mathbb{P}\left(\mathrm{Sym}^c_{m+1}\otimes\mathrm{Sym}^d_{n+1}\right)$$

Definition 2.2. Let $V\subset\mathbb{P}^N$ be a projective variety. The **r-th secant variety** of V is the Zariski-closure of the union of all linear spaces spanned by r points on V, i.e.,

$$\sigma_r(V) := \overline{\bigcup_{p_1,\ldots,p_r\in V}\langle p_1,\ldots,p_r\rangle}\subset\mathbb{P}^N.$$

In the case of Segre-Veronese varieties, the r-th secant variety consists of the Zariski-closure of the set of *rank-r* partially symmetric tensors, i.e.,

$$\sigma_r(\mathrm{SV}^{c,d}_{m\times n}) = \overline{\left\{\sum_{i=1}^r f_i\otimes g_i \ : \ f_i\in\mathrm{Sym}^c_{m+1}, g_i\in\mathrm{Sym}^d_{n+1}\right\}}\subset\mathbb{P}\left(\mathrm{Sym}^c_{m+1}\otimes\mathrm{Sym}^d_{n+1}\right).$$

Our goal is to compute the dimension of $\sigma_r(\mathrm{SV}^{c,d}_{m\times n})$. By a count of parameters, its **expected dimension** is

$$\begin{aligned}\operatorname{edim}\sigma_r\left(\mathrm{SV}^{c,d}_{m\times n}\right) &= \min\left\{\dim\mathbb{P}\left(\mathrm{Sym}^c_{m+1}\otimes\mathrm{Sym}^d_{n+1}\right), r(\dim\mathrm{SV}^{c,d}_{m\times n}+1)-1\right\} = \\ &= \min\left\{\binom{c+m}{m}\binom{d+n}{n}-1, r(m+n)+r-1\right\}.\end{aligned}$$

The actual dimension is always smaller than or equal to the expected one. If it is strictly smaller, then we say that the variety is **r-defective**. A variety is **defective** if it is r-defective for some r.

2.1. **Linear systems with non-reduced base locus.** A standard approach to study the dimension of secant varieties is to translate the problem to a question about dimensions of linear systems with multiple base points. For this purpose, we fix some notation we will use through the paper.

Definition 2.3. Let p be a point on a variety V defined by an ideal I_p. If $a\in\mathbb{N}$, then the **a-fat point** supported at p is the 0-dimensional scheme defined by I_p^a. If $p_1,\ldots,p_r\in V$, then the **fat points scheme of type** $(a_1,\ldots,a_r)$, denoted by $a_1p_1+\ldots+a_rp_r$, is the union of fat points defined by the ideal $I_{p_1}^{a_1}\cap\ldots\cap I_{p_r}^{a_r}$. We call it **general** when $p_1,\ldots,p_r$ are general points of V.

Definition 2.4. Let $p_1,\ldots,p_r\in\mathbb{P}^n$ and let $a_1,\ldots,a_r\in\mathbb{N}$. If $X=a_1p_1+\ldots+a_rp_r$ is a fat points scheme, then we denote by $\mathcal{L}^d_n(X)$ the vector space $I_X\cap\mathrm{Sym}^d_{n+1}$ of homogeneous polynomials of degree d vanishing with multiplicity a_i at p_i. We will write just $\mathcal{L}^d_n$ instead of $\mathcal{L}^d_n(\varnothing)$. The **virtual dimension** of $\mathcal{L}^d_n(X)$ is given by the count of parameters

$$\operatorname{vdim}\mathcal{L}^d_n(X) := \binom{n+d}{n}-\sum_{i=1}^r\binom{a_i+n-1}{n}.$$

In a similar way, if $p_1,\ldots,p_r\in\mathbb{P}^m\times\mathbb{P}^n$ and $X=a_1p_1+\ldots+a_rp_r$, then we denote by $\mathcal{L}^{c,d}_{m\times n}(X)$ the vector space $I_X\cap(\mathrm{Sym}^c_{n+1}\otimes\mathrm{Sym}^d_{n+1})$ of bihomogeneous polynomials of bidegree (c,d) vanishing with multiplicity a_i at p_i. Its **virtual dimension** is defined as

$$\operatorname{vdim}\mathcal{L}^{c,d}_{m\times n}(X) := \binom{m+c}{m}\binom{n+d}{n}-\sum_{i=1}^r\binom{a_i+m+n-1}{m+n}.$$

In both cases, the **expected dimension** is the maximum between 0 and the virtual dimension. Hence the actual dimension is larger or equal to the expected one. If the inequality is strict, then we say that the linear system is **special**. If the virtual dimension is non-negative and the linear system is not special, we say that it is **regular**.

It is important to recall what happens to the dimension of linear systems under specialization of the base locus. Consider a fat points scheme $X=a_1p_1+\ldots+a_rp_r\subset\mathbb{P}^m\times\mathbb{P}^n$. By semicontinuity, there exists a Zariski-open subset of $(\mathbb{P}^m\times\mathbb{P}^n)^{\times r}$ where the dimension of $\mathcal{L}^{c,d}_{m\times n}(X)$ is constant and takes the minimal value among all possible choices of $(p_1,\ldots,p_r)\in(\mathbb{P}^m\times\mathbb{P}^n)^{\times r}$. We denote by $\mathcal{L}^{c,d}_{m\times n}(a_1,\ldots,a_r)$ the linear system associated to a general choice of the support. In particular,

$$\dim\mathcal{L}^{c,d}_{m\times n}(X)\geq\dim\mathcal{L}^{c,d}_{m\times n}(a_1,\ldots,a_r).$$

In case of repetitions in the vector $(a_1,\dots,a_r)$, we use the notation a^s for the s-tuple $(a,\dots,a)$.

The expected dimension of the secant variety $\sigma_r(\mathrm{SV}^{c,d}_{m\times n})$ and the expected dimension of the linear system $\mathcal{L}^{c,d}_{m\times n}(X)$ are related. Indeed,

$$\operatorname{codim}\sigma_r(\mathrm{SV}^{c,d}_{m\times n}) = \dim\mathcal{L}^{c,d}_{m\times n}(2^r).$$

This is a consequence of the more general Terracini's Lemma which applies to any algebraic variety embedded via a linear system, as the Segre-Veronese varieties. See [BCC$^+$18, Corollary 1] for a recent reference. In particular, $\mathrm{SV}^{c,d}_{m\times n}$ is r-defective if and only if $\mathcal{L}^{c,d}_{m\times n}(2^r)$ is special. This allows us to translate Theorem 1.3 into the following statement.

Theorem 2.5. *If m,n and r are positive integers, then $\mathcal{L}^{3,3}_{m\times n}(2^r)$ is non-special.*

For fixed values of m and n, in order to prove this theorem it is enough to check few values of r, thanks to the following straightforward observation.

Remark 2.6. Let $X'\subset X$ be two fat points schemes of $\mathbb{P}^m\times\mathbb{P}^n$. Then:

- if $\mathcal{L}^{c,d}_{m\times n}(X)$ is regular, then $\mathcal{L}(X')$ is regular;
- if $\mathcal{L}^{c,d}_{m\times n}(X')=0$, then $\mathcal{L}(X)=0$.

In order to prove that $\mathcal{L}^{c,d}_{m\times n}(2^r)$ is non-special for every r, it is enough to consider

$$r_* := \max\left\{s\in\mathbb{N}\ :\ \operatorname{vdim}\mathcal{L}^{c,d}_{m\times n}(2^s)\geq 0\right\} \text{ and } r^* := \min\left\{s\in\mathbb{N}\ :\ \operatorname{vdim}\mathcal{L}^{c,d}_{m\times n}(2^s)< 0\right\}$$

and prove that $\mathcal{L}^{c,d}_{m\times n}(2^{r_*})$ is regular and $\mathcal{L}^{c,d}_{m\times n}(2^{r^*})$ is zero.

2.2. **Inductive methods.** Our proof of Theorem 2.5 relies on a classical inductive approach. Let V be a projective variety and let H be a subvariety of V. Consider a linear system $\mathcal{L}$ on V and let $\mathcal{L}_H$ be the linear system on H given by

$$\mathcal{L}_H := \{D\cap H\ :\ D\in\mathcal{L}\}.$$

Let $X\subset V$ be a fat points scheme. Then there is an exact sequence of vector spaces

$$0\to\mathcal{L}(X+H)\to\mathcal{L}(X)\to\mathcal{L}_H(X\cap H), \tag{2.1}$$

sometimes called *Castelnuovo exact sequence*, where $\mathcal{L}(X+H)$ denotes the subsystem of $\mathcal{L}$ of divisors containing $X\cup H$.

Definition 2.7. Let H be a divisor of V. Let I_X be the ideal defining a fat points scheme $X\subset V$.

- The **residue** of X with respect to H is the subscheme $\mathrm{Res}_H(X)\subset V$ defined by the saturation $(I_X:I_H)^{\mathrm{sat}}$.
- The **trace** of X on H is the scheme-theoretic intersection $\mathrm{Tr}_H(X)=H\cap X$, defined by I_X+I_H.

In this paper we are interested in the case $V=\mathbb{P}^m\times\mathbb{P}^n$ and $\mathcal{L}=\mathcal{L}^{c,d}_{m\times n}$. Let $H\cong\mathbb{P}^{m-1}\times\mathbb{P}^n$ be a divisor of bidegree $(1,0)$. An analogous statements holds if H has bidegree $(0,1)$. Then (2.1) becomes

$$0\to\mathcal{L}^{c-1,d}_{m\times n}(\mathrm{Res}_H(X))\to\mathcal{L}^{c,d}_{m\times n}(X)\to\mathcal{L}^{c,d}_{(m-1)\times n}(\mathrm{Tr}_H(X)), \tag{2.2}$$

where the left-most arrow corresponds to the multiplication by H. Hence,

$$\operatorname{vdim}\mathcal{L}^{c,d}_{m\times n}(X)\leq\dim\mathcal{L}^{c,d}_{m\times n}(X)\leq\dim\mathcal{L}^{c-1,d}_{m\times n}(\mathrm{Res}_H(X))+\dim\mathcal{L}^{c,d}_{(m-1)\times n}(\mathrm{Tr}_H(X)). \tag{2.3}$$

If $X=a_1p_1+\ldots+a_rp_r+b_1q_1+\ldots+b_sq_s$, where $p_1,\dots,p_r$ are general points and $q_1,\dots,q_s$ are general on H, then

$$\mathrm{Res}_H X = a_1p_1+\ldots+a_rp_r+(b_1-1)q_1+\ldots+(b_s-1)q_s\subset\mathbb{P}^m\times\mathbb{P}^n \text{ and}$$
$$\mathrm{Tr}_H X = b_1q_1+\ldots+b_sq_s\subset H\cong\mathbb{P}^{m-1}\times\mathbb{P}^n.$$

A straightforward computation gives

$$\operatorname{vdim}\mathcal{L}^{c-1,d}_{m\times n}(\mathrm{Res}_H(X))+\operatorname{vdim}\mathcal{L}^{c,d}_{(m-1)\times n}(\mathrm{Tr}_H(X))=\operatorname{vdim}\mathcal{L}^{c,d}_{m\times n}(X). \tag{2.4}$$

By employing (2.3), we use the Castelnuovo exact sequence in two ways.

(1) If we want to prove that $\mathcal{L}^{c,d}_{m\times n}(X)=0$, then it is enough to prove that

$$\mathcal{L}^{c-1,d}_{m\times n}(\mathrm{Res}_H(X))=\mathcal{L}^{c,d}_{(m-1)\times n}(\mathrm{Tr}_H(X))=0.$$

(2) If we want to prove that $\mathcal{L}^{c,d}_{m\times n}(X)$ is regular, by (2.4) it is enough to prove that both $\mathcal{L}^{c-1,d}_{m\times n}(\mathrm{Res}_H(X))$ and $\mathcal{L}^{c,d}_{(m-1)\times n}(\mathrm{Tr}_H(X))$ are regular.

Other classical tools are degeneration arguments. As we recalled, if $\tilde{X}$ is a specialization of the scheme X, then $\dim \mathcal{L}^{c,d}_{m\times n}(\tilde{X}) \geq \dim \mathcal{L}^{c,d}_{m\times n}(X)$. Here is how we will use this fact: if $H \cong \mathbb{P}^{m'} \times \mathbb{P}^{n'}$ is a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$, then

$$\begin{aligned} \operatorname{vdim} \mathcal{L}^{c,d}_{m\times n}(\tilde{X}) = \operatorname{vdim} \mathcal{L}^{c,d}_{m\times n}(X) \leq \dim \mathcal{L}^{c,d}_{m\times n}(X) \leq \dim \mathcal{L}^{c,d}_{m\times n}(\tilde{X}) \\ \leq \dim \mathcal{L}^{c,d}_{m\times n}(\tilde{X} + H) + \dim \mathcal{L}^{c,d}_{m'\times n'}(\tilde{X} \cap H). \end{aligned} \tag{2.5}$$

Therefore, in order to prove that $\mathcal{L}^{c,d}_{m\times n}(X)$ is non-special, the task is to find a suitable specialization $\tilde{X}$ for which we are able to compute the two summands on the right-hand-side of (2.5) and such that the upper bound coincides with the lower bound.

Note that sequence (2.2) allows a double induction: in one of the summands we have a lower degree while in the other we have a lower dimension. The classical method of specializing the support of X does not always work due to arithmetic constrains that does not allow to match the upper and the lower bound in (2.5). This was the case of the system $\mathcal{L}^3_n(2^r)$ of cubics in projective space with general 2-fat base points. After a series of papers, Alexander and Hirschowitz refined the classical method and managed to complete the proof of the classification of special linear systems $\mathcal{L}^d_n(2^r)$ and, as byproduct, of the celebrated classification of defective Veronese varieties. This method is called *differential Horace method* and, in the last decades, it has been used to prove the non-speciality of several linear systems in projective and multiprojective space. Despite its success, this method requires a deep understanding of the geometry of the problem and a clever choice of specialization. For this reason, in this paper we consider a different type of specialization in which the components of the base locus are allowed to collide together. We explain it in the next section.

Sometimes it is convenient to use the exact sequence (2.1) not only when H is an hyperplane, but also with H of higher codimension. In [BO08, Section 5], Brambilla and Ottaviani used this approach to obtain a different proof of the classification of the Alexander-Hirschowitz Theorem. We will borrow this technique to prove some of our results in Section 3.

2.3. **Collapsing points.** In this section we recall the specialization method we use to prove Theorem 2.5. We refer to [GM19, Remark 20 and Proposition 21] or [Gal19, Construction 10] for more details.

Remark 2.8. Let V be a smooth variety of dimension n. We consider a general scheme of type (2^{n+1}) on V and we let it *collapse* to one component, i.e., we let all the points of its support approach the same point $q \in V$ from general directions. The result of such a limit is a scheme supported at q, containing the triple point $3q$ with the following property: its restriction to a general line L containing q is a triple point of L, but there are $\binom{n+1}{2}$ lines through q such that the restriction is a 4-tuple point on each of these special lines is a 4-ple point. We call it a triple point with $\binom{n+1}{2}$ tangent directions. If we call E the exceptional divisor of $\mathrm{Bl}_q V$, then these tangent directions correspond to simple points $\{t_{ij} \mid 0 \leq i < j \leq n\}$ of E.

Example 2.9. Since the limit is a local construction, we work out an example on $\mathbb{A}^2$. Let Δ be a complex disk around the origin. Let $Y = \mathbb{A}^2 \times \Delta$ and $Y_t = \mathbb{A}^2 \times \{t\}$ for $t \in \Delta$. Fix a point $q \in Y_0$ and three general maps $\sigma_1, \sigma_2, \sigma_3 : \Delta \to Y$ such that $\sigma_1(t), \sigma_2(t), \sigma_3(t)$ are general points of Y_t for $t \neq 0$ and $\sigma_1(0) = \sigma_2(0) = \sigma_3(0) = q$. For every $t \neq 0$, let

$$X_t = 2\sigma_1(t) + 2\sigma_2(t) + 2\sigma_3(t) \subset Y_t$$

be a general scheme of fat points of type (2^3). We are interested in the limit $X_0 := \lim_{t\to 0} X_t$. For every $t \neq 0$ the ideal I_{X_t} contains a plane cubic C_t, consisting of the union of three lines, hence the limit C_0 belongs to $I_{X_0}(3)$. Actually, one can show that $I_{X_0}(3) = \langle C_0 \rangle$. By [Gal19, Proposition 13], X_0 strictly contains a 3-fat point but does not contain a 4-fat point. In order to completely understand its structure, we look at the blow-up $\mu : \tilde{Y} \to Y$ of Y at the point q with exceptional divisor W. Let $\tilde{\sigma}_i : \Delta \to \tilde{Y}$ be the map corresponding to σ_i. We want to stress that, since the sections σ_i are general, $\tilde{\sigma}_1(0), \tilde{\sigma}_2(0), \tilde{\sigma}_3(0)$ are three general points of W. See Figure 1. If we set $\tilde{Y}_t = \mu^{-1}(Y_t)$, then $\tilde{Y}_t \cong Y_t \cong \mathbb{A}^2$ for every $t \neq 0$, but the special fiber $\tilde{Y}_0$ has two irreducible components. We write $\tilde{Y}_0 \cong W \cup \mathrm{Bl}_q Y_0$. If we call $E = W \cap \mathrm{Bl}_q \mathbb{A}^2$, then $E \cong \mathbb{P}^1$ is the exceptional divisor of $\mathrm{Bl}_q \mathbb{A}^2$.

Let $\tilde{I}_{X_0}$ be the ideal consisting of all the strict transforms of elements of I_{X_0}. Since $X_0 \supsetneq 3q$, then $\tilde{I}_{X_0}(3)_{|E} \subsetneq H^0\mathcal{O}_E(3)$. Indeed, $\tilde{I}_{X_0}(3)_{|E}$ is the system of cubics of E containing the general scheme of fat points $2\tilde{\sigma}_1(0) + 2\tilde{\sigma}_2(0) + 2\tilde{\sigma}_3(0)$. There is exactly one such cubic, consisting of the union of the three lines $\langle \tilde{\sigma}_i(0), \tilde{\sigma}_j(0)\rangle$, which cut three simple points $t_{ij} = R \cap \langle \tilde{\sigma}_i(0), \tilde{\sigma}_j(0)\rangle$ on E. We regard X_0 as the fat point $3q$ together with three infinitely near simple points corresponding to t_{12}, t_{13}, t_{23}.

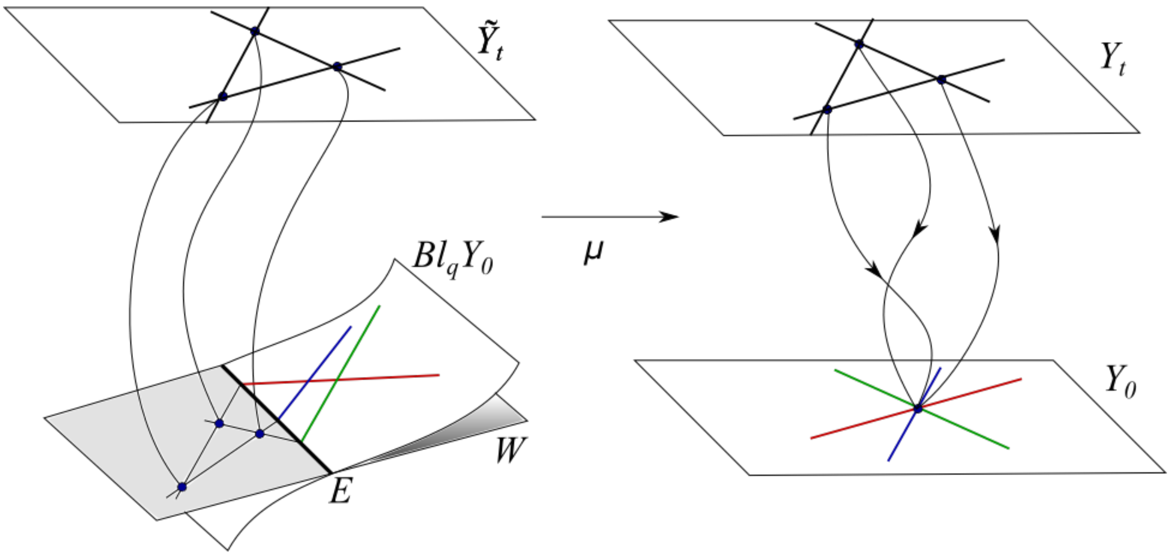

FIGURE 1. The collision of three 2-fat points as described in Example 2.9.

It is crucial to notice that the tangent directions described in Remark 2.8 are not in general position: for every choice of a set of indices $I \subset \{0, \ldots, n\}$ of cardinality $s \geq 3$, the points $\{t_{ij} \mid i, j \in I \text{ and } i < j\}$ are contained in a linear space $\mathbb{P}^{s-2} \subset E$.

Example 2.10. Consider the collision of four 2-fat points in $\mathbb{A}^3$. We proceed in the same way as in Example 2.9 and we see that $\tilde{I}_{X_0}(3)_{|W}$ is the system of cubics of $W \cong \mathbb{P}^3$ containing a general scheme of fat points of type (2^4) supported, say, at p_0, p_1, p_2, p_3. Its base locus consists of the $\binom{4}{2} = 6$ lines joining each pair of points. These six lines cut six simple points $t_{ij} = E \cap \langle p_i, p_j\rangle$ on $E \cong \mathbb{P}^2$, but they are not in general position. For instance, all the three points t_{12}, t_{13} and t_{23} belong to the line $E \cap \langle p_1, p_2, p_3\rangle$. The six points are in the configuration described in Figure 2.

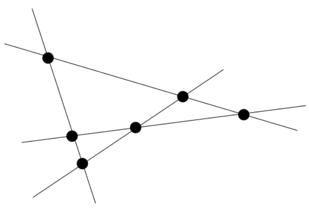

FIGURE 2. Six points in a *star configurations*, i.e., as intersections of a four general lines.

For later purpose we need to check that, even if the infinitely near points are not in general position, they are not too special. More precisely, we show that they impose independent conditions on low degree divisors of the exceptional divisor E.

Lemma 2.11. *Let $n \geq 2$. Let $p_0, \ldots, p_n \in \mathbb{P}^n$ be general points and let E be a hyperplane such that $\{p_0, \ldots, p_n\} \cap E = \varnothing$. Define $t_{ij} := \langle p_i, p_j\rangle \cap E$ and set*

$$X := \{t_{ij} \mid 1 \leq i < j \leq n\}.$$

Then the linear systems $\mathcal{L}^2_{n-1}(X)$, $\mathcal{L}^3_{n-1}(X)$ and $\mathcal{L}^3_{n-1}(2X)$ on E are non-special.

Proof. The system $\mathcal{L}^2_{n-1}(X)$ is non-special by [GM19, Lemma 25]. As a consequence, $\mathcal{L}^3_{n-1}(X)$ is non-special as well, so we focus on $\mathcal{L}^3_{n-1}(2X)$. We argue by induction on n.

- *Case* $n = 2$. It is enough to observe that every linear system on $\mathbb{P}^1$ is non-special.
- *Case* $n \geq 3$. For $i \in \{0, \ldots, n\}$, let $H_i := \langle p_j \mid j \neq i \rangle = \mathbb{P}^{n-1}$. By induction hypothesis, $\mathcal{L}^3_{n-1}(2X)_{|H_i \cap E} = \mathcal{L}^3_{n-2}(\mathrm{Tr}_H(2X)) = 0$, so $H_i \cap E$ is a fixed component of $\mathcal{L}^3_{n-1}(2X)$. That is impossible, because elements of $\mathcal{L}^3_{n-1}(2X)$ have degree 3. □

The proof of Theorem 2.5 will be presented in Section 4 and it depends on a series of lemmas. In order to make improve the readability of the proof and to better understand the strategy, we collect them in the following section.

3. Lemmata

In this section we prove a series of technical results that will lead to the proof of our main theorem. At a first glance, these lemmas might seem unrelated; therefore, the reader might want to come back to them after having read the proof of Theorem 2.5 in Section 4. We prove these intermediate results with the inductive approach and the degeneration technique illustrated in Sections 2.2 and 2.3. The specializations need to be chosen carefully to satisfy several arithmetic properties: in order to make our proofs easier to read, we moved some elementary but tedious computations to Appendix B.

Notation 3.1. For any $m, n \in \mathbb{N}$, we define

$$r^*(m,n) := \left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rceil \qquad r_*(m,n) := \left\lfloor \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rfloor$$

$$k^*(m,n) := r^*(m,n) - m - n - 1 \qquad k_*(m,n) := r_*(m,n) - m - n - 1$$

The first two results will help us dealing with linear systems whose base locus has simple points in special position. We denote by $\mathrm{Bs}(\mathcal{L})$ the base locus of a linear system $\mathcal{L}$.

Proposition 3.2. *Let V be a smooth variety and let $\mathcal{L}$ be a linear system on V. Let $\varphi : V \dashrightarrow \mathbb{P}(\mathcal{L}^\vee)$ be the rational map induced by $\mathcal{L}$. Fix a point $x \in V$ and suppose that there exists a nonempty Zariski-open subset $U \subset V$ such that*

(1) *$x \notin \mathrm{Bs}(\mathcal{L})$, and*
(2) *$x \in \mathrm{Bs}(\mathcal{L} \otimes I^2_{p,V})$ for every $p \in U$.*

Then every element of $\mathcal{L} \otimes I^2_{p,V}$ is singular at x.

Proof. Let $W := \overline{\varphi(V)}$. Let $p \in U$ and let $q := \varphi(p)$ be a general point of W. We can write its tangent space as an intersection of hyperplanes $\mathbb{T}_q W = H_1 \cap \ldots \cap H_s$. For any $i \in \{1, \ldots, s\}$, let $D_i := \varphi^*(H_i) \in \mathcal{L}$. By construction, each D_i is singular at p, i.e., $D_i \in \mathcal{L} \otimes I^2_{p,V}$. By hypothesis (2), $x \in D_i$ for every $i \in \{1, \ldots, s\}$. Therefore, by hypothesis (1), we deduce that $\varphi(x) \in \mathbb{T}_q(W)$. By [FOV99, Proposition 4.6.11], W is a cone and $\varphi(x)$ belongs to its vertex.

Now, take $B \in \mathcal{L} \otimes I^2_{p,V}$. Then $\varphi(B)$ is tangent to W at q and, therefore, it passes through $\varphi(x)$. Since the latter is in the vertex of W, $\varphi(B)$ is singular at $\varphi(x)$; hence B is singular at x. □

Observe that in the proof of Proposition 3.2 we proved something stronger: not only $\mathcal{L} \otimes I^2_{p,V}$ is singular at x, but W is a cone and $\varphi(x)$ belongs to its vertex.

Lemma 3.3 ([CGG07, Lemma 1.9])**.** *Let V be a projective variety and let $H \subset V$ be a positive dimensional subvariety. Let $X \subset V$ be a scheme of fat points and let $Y \subset H$ be a set of points. Let $\mathcal{L}$ be a linear system on V. Assume that*

(1) *Y imposes independent conditions on $\mathcal{L}(X)_{|H}$, and*
(2) *$\dim \mathcal{L}(X) - \dim(\mathcal{L}(X) \otimes \mathcal{I}_H) \geq \#Y$.*

Then Y imposes independent on $\mathcal{L}(X)$. In particular, if $\dim \mathcal{L}(X) \leq \#Y$ and $\mathcal{L}(X) \otimes \mathcal{I}_H = 0$, then $\mathcal{L}(X + Y) = 0$.

The rest of the section is devoted to proving the building blocks of Theorem 2.5. In order to help the reader to follow the structure of the proofs, we will add explicit numerical examples.

Lemma 3.4. *If $1 \leq m \leq n$, then $\mathcal{L}^{3,3}_{m \times n}(3, 2^{k^*(m,n)})$ is regular.*

Proof. We proceed by induction on m. The case $m = 1$ is Lemma 3.5. Assume that $m \geq 2$. Let $D \subset \mathbb{P}^m \times \mathbb{P}^n$ be a divisor of bidegree $(1, 0)$ and consider a scheme of fat points X of type $(3, 2^{k^*(m,n)})$ such that $X \cap D$ is of type $(3, 2^{k^*(m-1,n)})$. Note that

$$\begin{aligned}
&\mathrm{vdim}\mathcal{L}^{3,3}_{m\times n}(X)\\
&\quad= \binom{m+3}{3}\binom{n+3}{3} - \binom{m+n+2}{2} - (m+n+1)\left(\left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rceil - (m+n+1)\right)\\
&\quad\geq (m+n+1)^2 - \binom{m+n+2}{2} - (m+n) = \binom{m+n+1}{2} - (m+n) > 0. \qquad (3.1)
\end{aligned}$$

As explained in Section 2.2, it is enough to prove that residue and trace of $\mathcal{L}^{3,3}_{m\times n}(X)$ with respect to D are regular:

- *Trace.* The trace of X on $D \cong \mathbb{P}^{m-1}\times\mathbb{P}^n$ is a general scheme of fat points of type $(3, 2^{k^*(m-1,n)})$; hence, $\mathcal{L}^{3,3}_{(m-1)\times n}(\mathrm{Tr}_D(X))$ is regular by induction.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is of type $(2^{k^*(m,n)-k^*(m-1,n)+1}, 1^{k^*(m-1,n)})$ where $\mathrm{Res}_D(X)\cap D$ is general of type $(2, 1^{k^*(m-1,n)})$ on D. The system $\mathcal{L}^{2,3}_{m\times n}(\mathrm{Res}_D(X))$ has non-negative virtual dimension by Lemma B.3, and $\mathcal{L}^{2,3}_{m\times n}(2^{k^*(m,n)-k^*(m-1,n)+1})$ is regular by Lemma 3.6. In order to prove that $\mathcal{L}^{2,3}_{m\times n}(\mathrm{Res}_D(X))$ is regular, we need to show that the $k^*(m-1, n)$ simple points on D impose independent conditions on $\mathcal{L}^{2,3}_{m\times n}(2^{k^*(m,n)-k^*(m-1,n)+1})$. We apply Lemma 3.3 by showing that
$$k^*(m-1,n) \leq \dim \mathcal{L}^{2,3}_{m\times n}(2^{k^*(m,n)-k^*(m-1,n)+1}) - \dim \mathcal{L}^{1,3}_{m\times n}(1, 2^{k^*(m,n)-k^*(m-1,n)}). \qquad (3.2)$$
Note that $\mathcal{L}^{1,3}_{m\times n}(1, 2^{k^*(m,n)-k^*(m-1,n)}) = 0$ by Lemma B.2 and [BBC12, Theorem 3.1]. Hence inequality (3.2) follows by Lemma B.3. □

Lemma 3.5. *If n is a positive integer, then $\mathcal{L}^{3,3}_{1\times n}(3, 2^{k^*(1,n)})$ is regular.*

Proof. We proceed by induction on n. The case $n = 1$ is checked directly with the support of an algebra software; see Appendix A. Assume $n \geq 2$. Let $D \subset \mathbb{P}^1 \times \mathbb{P}^n$ be a divisor of bidegree $(0, 1)$ and consider the scheme of fat points X of type $(3, 2^{k^*(1,n)})$ such that $X\cap D$ is general of type $(3, 2^{k^*(1,n-1)})$ on D. Note that $\mathrm{vdim}\,\mathcal{L}^{3,3}_{1\times n}(X) \geq 0$ by (3.1). As explained in Section 2.2, it is enough to prove that residue and trace of $\mathcal{L}^{3,3}_{1\times n}(X)$ with resepct to D are regular.

- *Trace.* The trace of X on $D \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ is a general scheme of fat points of type $(3, 2^{k^*(1,n-1)})$ and $\mathcal{L}^{3,3}_{1\times(n-1)}(3, 2^{k^*(1,n-1)})$ is regular by induction.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is of type $(2^{1+k^*(1,n)-k^*(1,n-1)}, 1^{k^*(1,n-1)})$, where $\mathrm{Res}_D(X)\cap D$ is general of type $(2, 1^{k^*(1,n-1)})$ on D. By symmetry and by Lemma B.3, the virtual dimension of $\mathcal{L}^{3,2}_{1\times n}(2^{1+k^*(1,n)-k^*(1,n-1)}, 1^{k^*(1,n-1)})$ is non-negative. The system $\mathcal{L}^{3,2}_{1\times n}(2^{1+k^*(1,n)-k^*(1,n-1)})$ is regular by [BBC12, Theorem 3.1]. Hence, in order to prove that $\mathcal{L}^{3,2}_{1\times n}(\mathrm{Res}_D(X))$ is regular, we need to show that the additional $k^*(1, n-1)$ simple points lying on D impose independent conditions on $\mathcal{L}^{3,2}_{1\times n}(2^{1+k^*(1,n)-k^*(1,n-1)})$. We apply Lemma 3.3 by showing that
$$k^*(1, n-1) \leq \dim \mathcal{L}^{3,2}_{1\times n}(2^{1+k^*(1,n)-k^*(1,n-1)}) - \dim \mathcal{L}^{3,1}_{1\times n}(1, 2^{k^*(1,n)-k^*(1,n-1)}). \qquad (3.3)$$
Note that $\mathcal{L}^{3,1}_{1\times n}(1, 2^{k^*(1,n)-k^*(1,n-1)}) = 0$ by Lemma B.4 and [BBC12, Theorem 3.1]. Hence (3.3) follows by symmetry by Lemma B.3. □

Lemma 3.6. *If $1 \leq m \leq n$, then $\mathcal{L}^{2,3}_{m\times n}(2^{1+k^*(m,n)-k^*(m-1,n)})$ is regular.*

Proof. In order to simplify the notation, we set $f(m, n) := 1 + k^*(m, n) - k^*(m-1, n)$. We argue by induction on m. The case $m = 1$ follows by [Abr08, Theorem 4.2]. Assume that $m \geq 2$ and let $D \subset \mathbb{P}^m \times \mathbb{P}^n$ be a divisor of bidegree $(1, 0)$. Let X be a scheme of fat points of type $(2^{f(m,n)})$ such that $X \cap D$ is general of type $(2^{f(m-1,n)})$ on D. Note that we are allowed to do it because $f(m-1, n) \leq f(m, n)$ by Lemma B.5(3). Note that $\mathrm{vdim}\,\mathcal{L}^{2,3}_{m\times n}(X) \geq 0$ by Lemma B.6. Hence, it is enough to prove that residue and trace of $\mathcal{L}^{2,3}_{m\times n}(X)$ with respect to D are regular.

- *Trace.* The trace is a general scheme of fat points on $D \cong \mathbb{P}^{m-1} \times \mathbb{P}^n$ of type $(2^{f(m-1,n)})$ and $\mathcal{L}^{2,3}_{(m-1)\times n}(2^{f(m-1,n)})$ is regular by induction.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is a scheme of fat points of type $(2^{f(m,n)-f(m-1,n)}, 1^{f(m-1,n)})$, where $\mathrm{Res}_D(X) \cap D$ is a general scheme of type $(1^{f(m-1,n)})$ on D, which has positive virtual dimension by Lemma B.8. By Lemma B.7(1),
$$f(m,n) - f(m-1,n) \leq \left\lfloor \frac{(m+1)\binom{n+3}{3}}{m+n+1} \right\rfloor - m,$$
hence the linear system $\mathcal{L}^{1,3}_{m\times n}(2^{f(m,n)-f(m-1,n)})$ is regular by [BCC11, Theorem 2.3]. In order to prove that $\mathcal{L}^{1,3}_{m\times n}(\mathrm{Res}_D(X))$ is regular, we need to show that the $f(m-1,n)$ simple points on D impose independent conditions on $\mathcal{L}^{1,3}_{m\times n}(2^{f(m,n)-f(m-1,n)})$. We apply Lemma 3.3 by showing that
$$f(m-1,n) \leq \dim \mathcal{L}^{1,3}_{m\times n}(2^{f(m,n)-f(m-1,n)}) - \dim \mathcal{L}^{0,3}_{m\times n}(2^{f(m,n)-f(m-1,n)}). \tag{3.4}$$
Note that $\mathcal{L}^{0,3}_{m\times n}(2^{f(m,n)-f(m-1,n)}) \cong \mathcal{L}^{3}_{n}(2^{f(m,n)-f(m-1,n)}) = 0$ by Lemma B.5(4) and Theorem 1.1. Hence, (3.4) follows by Lemma B.8. □

Lemma 3.7. *If $n \geq m \geq 2$, then* $\dim \mathcal{L}^{3,3}_{m\times n}(4, 2^{k_*(m,n)}) = 0$.

Proof. We argue by induction on m. By Lemma 3.8, $\mathcal{L}^{3,3}_{2\times n}(4, 2^{k_*(2,n)}) = 0$. Assume that $m \geq 3$ and take a divisor $D \subset \mathbb{P}^m \times \mathbb{P}^n$ of bidegree $(1,0)$. Let X be a scheme of fat points of type $(4, 2^{k_*(m,n)})$ such that $X \cap D$ is general of type $(4, 2^{k_*(m-1,n)})$ on D. As explained in Section 2.2, it is enough to prove that residue and trace of $\mathcal{L}^{3,3}_{m\times n}(X)$ are zero.

- *Trace.* The trace of X on D is a general scheme of type $(4, 2^{k_*(m-1,n)})$ and we know that $\mathcal{L}^{3,3}_{(m-1)\times n}(4, 2^{k_*(m-1,n)}) = 0$ by induction hypothesis.
- *Residue.* The residue of X with respect to D is of type $(3, 2^{k_*(m,n)-k_*(m-1,n)}, 1^{k_*(m-1,n)})$ where $X \cap D$ is general of type $(3, 1^{k_*(m-1,n)})$ on D. The residue linear system is expected to be zero by Lemma B.9. The linear system $\mathcal{L}^{2,3}_{m\times n}(3, 2^{k_*(m,n)-k_*(m-1,n)})$ is regular by Lemma 3.9. Now, we need to prove that the extra $k_*(m-1,n)$ simple points on D impose enough conditions to make $\mathcal{L}^{2,3}_{m\times n}(\mathrm{Res}_D(X))$ to be zero. By Lemma 3.3, it is enough to prove that $\mathcal{L}^{1,3}_{m\times n}(2, 2^{k_*(m,n)-k_*(m-1,n)}) = 0$. Thanks to [BCC11, Theorem 2.3], we just need to show that
$$1 + k_*(m,n) - k_*(m-1,n) \geq \left\lceil \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rceil + m,$$
and this is done in Lemma B.10. □

Lemma 3.8. *If n is a positive integer, then* $\dim \mathcal{L}^{3,3}_{2\times n}(4, 2^{k_*(2,n)}) = 0$.

Proof. We argue by induction on n. By a software computation, we check $\dim \mathcal{L}^{3,3}_{2\times 1}(4, 2^{k_*(2,1)}) = 0$; see Appendix A. Assume $n \geq 2$ and consider a divisor $D \subset \mathbb{P}^2 \times \mathbb{P}^n$ of bidegree $(0,1)$. Let X be a scheme of fat points of type $(4, 2^{k_*(2,n)})$ such that $X \cap D$ is general of type $(4, 2^{k_*(2,n-1)})$ on D. As explained in Section 2.2, it is enough to prove that the residue and trace of $\mathcal{L}^{3,3}_{2\times n}(X)$ are zero.

- *Trace.* The trace of X on D is a general scheme of type $(4, 2^{k_*(2,n-1)})$ and the linear system $\mathcal{L}^{3,3}_{2\times(n-1)}(4, 2^{k_*(2,n-1)})$ is zero by induction hypothesis.
- *Residue.* The residue of X with respect to D is a scheme of type $(3, 2^{k_*(2,n)-k_*(2,n-1)}, 1^{k_*(2,n-1)})$ where $X \cap D$ is general of type $(3, 1^{k_*(2,n-1)})$. The residue linear system is expected to be zero by Lemma B.11. By Lemma 3.16, $\mathcal{L}^{3,2}_{2\times n}(3, 2^{k_*(2,n)-k_*(2,n-1)})$ is regular. Now, we need to prove that the $k_*(2,n-1)$ simple points on D impose enough conditions to make $\mathcal{L}^{3,2}_{2\times n}(\mathrm{Res}_D(X))$ zero. By Lemma 3.3 it is enough to show that $\dim \mathcal{L}^{3,1}_{2\times n}(2^{1+k_*(2,n)-k_*(2,n-1)}) = 0$. By [BCC11, Theorem 2.3], the latter is guaranteed by
$$1 + k_*(2,n) - k_*(2,n-1) \geq \left\lceil \frac{10(n+1)}{n+3} \right\rceil + n$$
which holds by Lemma B.12. □

Lemma 3.9. *Let $n \geq m \geq 2$. Then $\mathcal{L}^{2,3}_{m\times n}(3, 2^{k_*(m,n)-k_*(m-1,n)})$ is regular.*

Proof. In order to simplify the notation, set

$$\ell(m,n) := k_*(m,n) - k_*(m-1,n).$$

We proceed by induction on m. The case $m = 2$ is solved by Lemma 3.10. Assume $m \geq 3$ and consider a divisor $D \subset \mathbb{P}^m \times \mathbb{P}^n$ of bidegree $(1,0)$. Let X be a scheme of fat points of type $(3, 2^{\ell(m,n)})$ such that $X \cap D$ is general of type $(3, 2^{\ell(m-1,n)})$ on D. We are allowed to do it because $\ell(m-1,n) \leq \ell(m,n)$ by Lemma B.5(1). As explained in Section 2.2, it is enough to prove that the residue and the trace of $\mathcal{L}^{2,3}_{m\times n}(X)$ with respect to D are regular.

- *Trace.* The trace of X on D is a general scheme of fat points of type $(3, 2^{\ell(m-1,n)})$ and $\mathcal{L}^{2,3}_{(m-1)\times n}(3, 2^{\ell(m-1,n)})$ is regular by inductive hypothesis.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is a scheme of fat points of type $(2^{1+\ell(m,n)-\ell(m-1,n)}, 1^{\ell(m-1,n)})$, where $\mathrm{Res}_D(X) \cap D$ is a general scheme of type $(2, 1^{\ell(m-1,n)})$. The system $\mathcal{L}^{1,3}_{m\times n}(\mathrm{Res}_D(X))$ has non-negative virtual dimension by Lemma B.13. Lemma B.7(2) shows that
$$1 + \ell(m,n) - \ell(m-1,n) \leq \left\lfloor \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rfloor - m,$$
thus $\mathcal{L}^{1,3}_{m\times n}(2^{1+\ell(m,n)-\ell(m-1,n)})$ is regular by [BCC11, Theorem 2.3]. In order to prove that $\mathcal{L}^{1,3}_{m\times n}(\mathrm{Res}_D(X))$ is regular, we need to prove that the $\ell(m-1,n)$ general simple points impose independent conditions on $\mathcal{L}^{1,3}_{m\times n}(2^{1+\ell(m,n)-\ell(m-1,n)})$. We apply Lemma 3.3 by showing that
$$\ell(m-1,n) \leq \dim \mathcal{L}^{1,3}_{m\times n}(2^{1+\ell(m,n)-\ell(m-1,n)}) - \dim \mathcal{L}^{0,3}_{m\times n}(1, 2^{\ell(m,n)-\ell(m-1,n)}). \tag{3.5}$$
Note that $\mathcal{L}^{0,3}_{m\times n}(1, 2^{\ell(m,n)-\ell(m-1,n)}) \cong \mathcal{L}^{3}_{n}(1, 2^{\ell(m,n)-\ell(m-1,n)}) = 0$ by Lemma B.5(2) and Theorem 1.1. Hence, (3.5) follows by Lemma B.13. □

Lemma 3.10. *Let $n \geq 2$. Then $\mathcal{L}^{2,3}_{2\times n}(3, 2^{\ell(2,n)})$ is regular.*

Proof. We proceed by induction on n. We check the case $n = 2$ by a software computation; see Appendix A. Let $n \geq 3$ and set

$$s(n) := \frac{n(n+3)}{2} \in \mathbb{N}.$$

Let $D \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a bidegree $(1,0)$ divisor and let X be a scheme of fat points of type $(3, 2^{\ell(2,n)})$ such that $X \cap D$ is of type $(3, 2^{s(n)})$. Note that we are allowed to do it because $s(n) \leq \ell(2,n)$ by Lemma B.15(1). As explained in Section 2.2, it is enough to prove that the residue and trace of $\mathcal{L}^{2,3}_{2\times n}(X)$ with respect to D are regular.

- *Trace.* The trace on X is a general scheme of fat points of type $(3, 2^{s(n)})$ on D and the linear system $\mathcal{L}^{2,3}_{1\times n}(3, 2^{s(n)})$ is regular by Lemma 3.11.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is a scheme of fat points of type $(2^{1+\ell(2,n)-s(n)}, 1^{s(n)})$ where $X \cap D$ is general of type $(2, 1^{s(n)})$ on D. The system $\mathcal{L}^{1,3}_{m\times n}(\mathrm{Res}_D(X))$ has non-negative virtual dimension by Lemma B.16. By Lemma B.14, we have
$$1 + \ell(2,n) - s(n) \leq \left\lfloor \frac{3}{n+3}\binom{n+3}{3} \right\rfloor - 2,$$
hence $\mathcal{L}^{1,3}_{2\times n}(2^{1+\ell(2,n)-s(n)})$ is regular by [BCC11, Theorem 2.3]. Now we need to show that the simple points on D impose independent conditions on $\mathcal{L}^{1,3}_{2\times n}(2^{1+\ell(2,n)-s(n)})$. By Lemma 3.3, it is enough to show that
$$s(n) \leq \dim \mathcal{L}^{1,3}_{2\times n}(2^{1+\ell(2,n)-s(n)}) - \dim \mathcal{L}^{0,3}_{2\times n}(1, 2^{\ell(2,n)-s(n)}). \tag{3.6}$$
Note that $\mathcal{L}^{0,3}_{2\times n}(1, 2^{\ell(2,n)-s(n)}) \cong \mathcal{L}^{3}_{n}(1, 2^{\ell(2,n)-s(n)}) = 0$ by Lemma B.15(2) and Theorem 1.1. Hence, (3.6) follows by Lemma B.16. □

Lemma 3.11. *If $n \geq 2$, then $\mathcal{L}^{2,3}_{1\times n}(3, 2^{s(n)})$ is regular. In particular, it is zero.*

Proof. Note that
$$\operatorname{vdim}\mathcal{L}^{2,3}_{1\times n}(3,2^{s(n)}) = 3\binom{n+3}{3} - \binom{n+3}{2} - \frac{n(n+2)(n+3)}{2} = 0.$$
We have to prove that it is indeed zero. We proceed by induction on n. A software computation shows that $\dim\mathcal{L}^{2,3}_{1\times 2}(3,2^{s(2)}) = \dim\mathcal{L}^{2,3}_{1\times 3}(3,2^{s(3)}) = 0$; see Appendix A . Assume that $n \geq 4$. Let $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be a subvariety defined by two general forms of bidegree $(0,1)$. Let $X = X_A + X_\circ$ be a scheme of fat points of type $(3,2^{s(n)})$, where

X_A	is a scheme of type $(3,2^{s(n-2)})$ with general support on $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$;
$X_\circ$	is a general scheme of type (2^{2n+1}) with support outside A.

As explained in Section 2.2, we consider the exact sequence
$$0 \to \mathcal{L}^{2,3}_{1\times n}(A+X) \to \mathcal{L}^{2,3}_{1\times n}(X) \to \mathcal{L}^{2,3}_{1\times(n-2)}(\operatorname{Tr}_A X).$$
Then it is enough to prove that both the left-most and the right-most linear systems are zero. By induction hypothesis, $\mathcal{L}^{2,3}_{1\times(n-2)}(\operatorname{Tr}_A X) = 0$, while $\mathcal{L}^{2,3}_{1\times n}(A+X) = 0$ by Lemma 3.12 □

Lemma 3.12. *Let $n \geq 4$. Let $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be a subvariety of $\mathbb{P}^1 \times \mathbb{P}^n$ defined by two general forms of bidegree $(0,1)$. Let $X = X_A + X_\circ$ be a scheme of fat points as in the proof of Lemma 3.11. Then* $\dim\mathcal{L}^{2,3}_{1\times n}(A+X) = 0$.

Proof. We proceed by induction on n. A software computation shows that the statement holds for $n = 4$ and $n = 5$; see Appendix A. Assume that $n \geq 6$. Let $B \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be another subvariety defined by two general forms of bidegree $(0,1)$. Consider a specialization $Y = Y_{A\cap B} + Y_A + Y_B + Y_\circ$ of X, where

$Y_{A\cap B}$	is a scheme of type $(3,2^{s(n-4)})$ with general support on $A\cap B \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$
Y_A	is a scheme of type (2^{2n-3}) with general support on A, outside B;
Y_B	is a scheme of type (2^{2n-3}) with general support on B, outside A;
$Y_\circ$	is a scheme of type (2^4) with general support outside $A\cup B$.

Now it is enough to prove that $\mathcal{L}^{2,3}_{1\times n}(A+Y) = 0$. Consider the exact sequence
$$0 \to \mathcal{L}^{2,3}_{1\times n}(A+B+Y) \to \mathcal{L}^{2,3}_{1\times n}(A+Y) \to \mathcal{L}^{2,3}_{1\times(n-2)}(A\cap B + \operatorname{Tr}_B Y).$$

- *Trace.* By induction hypothesis, $\mathcal{L}^{2,3}_{1\times(n-2)}(A\cap B + \operatorname{Tr}_B Y) = 0$.
- *Residue.* By Lemma 3.13, $\mathcal{L}^{2,3}_{1\times n}(A+B+Y) = 0$. □

Lemma 3.13. *Let $n \geq 6$. Let $A, B \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be subvarieties of $\mathbb{P}^1 \times \mathbb{P}^n$, each defined by two general forms of bidegree $(0,1)$. Let $Y = Y_{A\cap B} + Y_A + Y_B + Y_\circ$ be a scheme of fat points as in the proof of Lemma 3.12. Then $\mathcal{L}^{2,3}_{1\times n}(A+B+Y) = 0$.*

Proof. We proceed by induction on n. A software computation shows that our statement holds for $n = 6$ and $n = 7$; see Appendix A. Assume that $n \geq 8$. Let $C \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be another subvariety defined by two general forms of bidegree $(0,1)$. We consider the specialization $Z = Z_{A\cap B\cap C} + Z_{A\cap B} + Z_{A\cap C} + Z_{B\cap C} + Z_A + Z_B + Z_C$ of Y, where

$Z_{A\cap B\cap C}$	is a scheme of type $(3,2^{s(n-6)})$ with general support on $A\cap B\cap C \cong \mathbb{P}^1 \times \mathbb{P}^{n-6}$;
$Z_{A\cap B}$	is a scheme of type (2^{2n-7}) with general support on $A\cap B \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$, outside C;
$Z_{A\cap C}$	is a scheme of type (2^{2n-7}) with general support on $A\cap C \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$, outside B;
$Z_{B\cap C}$	is a scheme of type (2^{2n-7}) with general support on $B\cap C \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$, outside A;
Z_A	is a scheme of type (2^4) with general support on $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, outside $B\cup C$;
Z_B	is a scheme of type (2^4) with general support on $B \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, outside $A\cup C$;
Z_C	is a scheme of type (2^4) with general support on $C \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, outside $A\cup B$.

Then it is enough to prove that $\mathcal{L}^{2,3}_{1\times n}(A+B+Z) = 0$. Consider the exact sequence
$$0 \to \mathcal{L}^{2,3}_{1\times n}(A+B+C+Z) \to \mathcal{L}^{2,3}_{1\times n}(A+B+Z) \to \mathcal{L}^{2,3}_{1\times(n-2)}(A\cap C + B\cap C + \operatorname{Tr}_C Z).$$

- *Trace.* By induction hypothesis, $\mathcal{L}^{2,3}_{1\times(n-2)}(A\cap C + B\cap C + \operatorname{Tr}_C Z) = 0$.
- *Residue.* The system $\mathcal{L}^{2,3}_{1\times n}(A+B+C+Z)$ is zero by Lemma 3.14. □

Lemma 3.14. *Let $n \geq 8$. Let $A, B, C \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be subvarieties of $\mathbb{P}^1 \times \mathbb{P}^n$, each defined by two general forms of bidegree $(0,1)$. Let $Z = Z_{A\cap B\cap C} + Z_{A\cap B} + Z_{A\cap C} + Z_{B\cap C} + Z_A + Z_B + Z_C$ be a scheme of fat points as in the proof of Lemma 3.13. Then $\mathcal{L}^{2,3}_{1\times n}(A+B+C+Z) = 0$.*

Proof. We proceed by induction on n. A software computation shows that the statement holds for $n = 8$ and $n = 9$; see Appendix A. Assume that $n \geq 10$. Let $E \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$ be another subvariety defined by two general forms of bidegree $(0,1)$. Let $W = W_{A\cap B\cap C\cap E} + W_{A\cap B\cap E} + W_{A\cap C\cap E} + W_{B\cap C\cap E} + W_{A\cap E} + W_{B\cap E} + W_{C\cap E}$ be a specialization of Z such that

$W_{A\cap B\cap C\cap E}$	is a scheme of type $(3, 2^{s(n-8)})$ with general support on $A\cap B\cap C\cap E \cong \mathbb{P}^1 \times \mathbb{P}^{n-8}$;
$W_{A\cap B\cap E}$	is a scheme of type (2^{2n-11}) with general support on $A\cap B\cap E \cong \mathbb{P}^1 \times \mathbb{P}^{n-6}$;
$W_{A\cap C\cap E}$	is a scheme of type (2^{2n-11}) with general support on $A\cap C\cap E \cong \mathbb{P}^1 \times \mathbb{P}^{n-6}$;
$W_{B\cap C\cap E}$	is a scheme of type (2^{2n-11}) with general support on $B\cap C\cap E \cong \mathbb{P}^1 \times \mathbb{P}^{n-6}$;
$W_{A\cap E}$	is a scheme of type (2^4) with general support on $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$;
$W_{B\cap E}$	is a scheme of type (2^4) with general support on $B \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$;
$W_{C\cap E}$	is a scheme of type (2^4) with general support on $C \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$;
$W_{A\cap B\cap C}$	is a scheme of type (2^{2n-11}) with general support on $A\cap B\cap C$;
$W_{A\cap B}$	is a scheme of type (2^4) with general support on $A\cap B \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$;
$W_{A\cap C}$	is a scheme of type (2^4) with general support on $A\cap C \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$;
$W_{B\cap C}$	is a scheme of type (2^4) with general support on $B\cap C \cong \mathbb{P}^1 \times \mathbb{P}^{n-4}$.

Then it is enough to prove that $\mathcal{L}^{2,3}_{1\times n}(A+B+C+W) = 0$. Consider the exact sequence

$$0 \to \mathcal{L}^{2,3}_{1\times n}(A+B+C+E+W) \to \mathcal{L}^{2,3}_{1\times n}(A+B+C+W) \to \mathcal{L}^{2,3}_{1\times (n-2)}((A+B+C)\cap E + \mathrm{Tr}_E\, W).$$

- *Trace.* By induction hypothesis, $\mathcal{L}^{2,3}_{1\times (n-2)}((A+B+C)\cap E + \mathrm{Tr}_E\, W) = 0$.
- *Residue.* The system $\mathcal{L}^{2,3}_{1\times n}(A+B+C+E+W)$ is a subsystem of $\mathcal{L}^{2,3}_{1\times n}(A+B+C+E)$ which is zero because the ideal of $A+B+C+E$ is generated in bidegree $(0,4)$. ☐

Example 3.15 (Lemma 3.11, Lemma 3.12, Lemma 3.13, Lemma 3.14 for $n = 10$)**.** We show that $\mathcal{L}^{2,3}_{1\times 10}(3, 2^{65}) = 0$. Let $A \cong \mathbb{P}^1 \times \mathbb{P}^8$ be a subvariety defined by two general forms of bidegree $(0,1)$. Let $X = X_A + X_\circ$ be a scheme of type $(3, 2^{65})$ where

X_A	is of type $(3, 2^{44})$ with general support on A, indeed $s(8) = 44$;
$X_\circ$	is general of type (2^{21}).

Let B, C be other two general subvarieties of $\mathbb{P}^1 \times \mathbb{P}^{10}$, each one defined by two general forms of bidegree $(0,1)$. We consider a series of specializations of the scheme X: we describe them as union of distinct components, each one with general support in the space indicated by the diagrams in Figure 3.

By using a series of Castelnuovo exact sequence, we obtain the following chain of inequalities

$$\begin{aligned}
\dim\mathcal{L}^{2,3}_{1\times 10}(X) &\leq \dim\mathcal{L}^{2,3}_{1\times 10}(A+X) + \dim\mathcal{L}^{2,3}_{1\times 8}(\mathrm{Tr}_A(X)) \\
&\leq \Big(\dim\mathcal{L}^{2,3}_{1\times 10}(A+B+Y) + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap B + \mathrm{Tr}_B(Y))\Big) \\
&\qquad + \dim\mathcal{L}^{2,3}_{1\times 8}(\mathrm{Tr}_A(X)) \\
&\leq \Big(\dim\mathcal{L}^{2,3}_{1\times 10}(A+B+C+Z) + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap C + B\cap C + \mathrm{Tr}_C(Z))\Big) \\
&\qquad + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap B + \mathrm{Tr}_B(Y)) + \dim\mathcal{L}^{2,3}_{1\times 8}(\mathrm{Tr}_A(X)) \\
&\leq \Big(\dim\mathcal{L}^{2,3}_{1\times 10}(A+B+C+E+W) + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap E + B\cap E + C\cap E + \mathrm{Tr}_E(W))\Big) \\
&\qquad + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap C + B\cap C + \mathrm{Tr}_C(Z)) + \dim\mathcal{L}^{2,3}_{1\times 8}(A\cap B + \mathrm{Tr}_B(Y)) + \dim\mathcal{L}^{2,3}_{1\times 8}(\mathrm{Tr}_A(X)).
\end{aligned}$$

In each step of the latter chain of inequalities, we may assume that the linear systems obtained from the traces on $\mathbb{P}^1 \times \mathbb{P}^8$ are known to be equal to zero by induction. Hence, we are left with proving that $\mathcal{L}^{2,3}_{1\times 10}(A+B+C+E+W) = 0$. This follows for the straightfoward observation that the ideal of $A\cup B\cup C\cup E$ is generated by forms in bidegree $(0,4)$ and, therefore, $\mathcal{L}^{2,3}_{1\times 10}(A+B+C+E) = 0$.

Lemma 3.16. *If $n \geq 2$, then $\mathcal{L}^{3,2}_{2\times n}(3, 2^{k_*(2,n)-k_*(2,n-1)})$ is regular.*

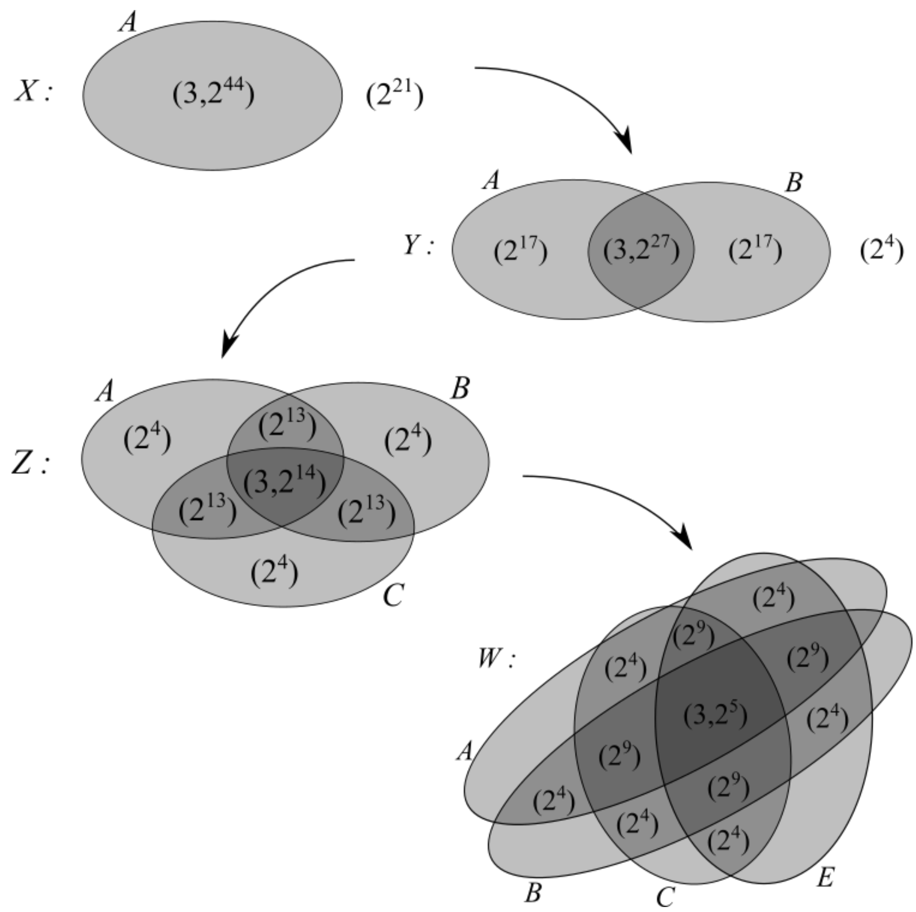

FIGURE 3. The specialization used in Example 3.15

Proof. In order to shorten the notation, we set

$$b(n) := k_*(2,n) - k_*(2,n-1).$$

We proceed by induction on n. A software computation shows that $\mathcal{L}^{3,2}_{2\times 2}(3, 2^{b(2)})$ is regular; see Appendix A. Assume that $n \geq 3$. Let $D \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a general divisor of bidegree $(0,1)$ and let $X = X_D + X_\circ$ be a scheme of fat points of type $(3, 2^{b(n)})$ such that

$$\begin{array}{ll} X_D & \text{is a scheme of type } (3, 2^{b(n-1)}) \text{ with general support on } D \cong \mathbb{P}^2 \times \mathbb{P}^{n-1}; \\ X_\circ & \text{is a scheme of type } (2^{b(n)-b(n-1)}) \text{ with general support outside } D. \end{array}$$

As explained in Section 2.2, it is enough to prove that residue and trace of $\mathcal{L}^{3,2}_{2\times n}(Y)$ with respect to D are regular.

- *Trace.* The trace $\mathrm{Tr}_D(X)$ is a general scheme of type $(3, 2^{b(n-1)})$ on $D \cong \mathbb{P}^2 \times \mathbb{P}^{n-1}$ and $\mathcal{L}^{3,2}_{2\times(n-1)}(3, 2^{b(n-1)})$ is regular by induction.
- *Residue.* The residue $\mathrm{Res}_D(X)$ is a scheme of fat points of type $(2^{1+b(n)-b(n-1)}, 1^{b(n-1)})$ such that $\mathrm{Res}_D(X) \cap D$ is general of type $(2, 1^{b(n-1)})$ on D and $\mathcal{L}^{3,1}_{2\times n}(\mathrm{Res}_D(X))$ is regular by Lemma 3.17. □

Lemma 3.17. *Let $n \geq 3$. Let $D \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a general divisor of bidegree $(0,1)$ and let Y be a scheme of fat points of type $(2^{1+b(n)-b(n-1)}, 1^{b(n-1)})$ such that $Y \cap D$ is a general scheme of type $(2, 1^{b(n-1)})$ on D. Then $\mathcal{L}^{3,1}_{2\times n}(Y)$ is regular.*

Proof. Set $v(n) := \mathrm{vdim}\, \mathcal{L}^{3,1}_{2\times n}(Y)$ and let $P \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a set of $v(n)$ general simple points. In order to prove the statement, it suffices to show that $\mathcal{L}^{3,1}_{2\times n}(Y + P) = 0$. We argue by induction on n. The cases $n \in \{3,4,5\}$ are checked by an explicit software computation; see Appendix A. Now assume that $n \geq 6$. By Lemma B.17,

$$b(n) - b(n-1) = b(n-3) - b(n-4),$$

so it makes sense to specialize some of the base points to a subvariety of codimension 3. Let $A \subset \mathbb{P}^2 \times \mathbb{P}^n$ be the subvariety defined by the vanishing of 3 general bidegree $(0,1)$ forms. We consider the specializations $Z = Z_{D\cap A} + Z_A + Z_D$ of Y and and $Q = Q_A + Q_\circ$ of P, where

$$\begin{array}{rl} Z_A & \text{is a scheme of type } (2^{b(n-3)-b(n-4)}) \text{ on } A \cong \mathbb{P}^2 \times \mathbb{P}^{n-3}, \text{ outside } D; \\ Z_{D\cap A} & \text{is a scheme of type } (2, 1^{b(n-4)}) \text{ general points on } A \cap D; \\ Z_D & \text{is a set of } b(n-1) - b(n-4) \text{ general points on } D, \text{ outside } A; \\ Q_A & \text{is a set of } v(n-3) \text{ general points on } A, \text{ outside } D; \\ Q_\circ & \text{is a set of } v(n) - v(n-3) \text{ general points outside } A. \end{array}$$

Such specialization is possible by Lemma B.17 and Lemma B.18. Now, it is enough to prove that $\mathcal{L}^{3,1}_{2\times n}(Z+Q) = 0$. Consider the exact sequence

$$0 \to \mathcal{L}^{3,1}_{2\times n}(A+Z+Q) \to \mathcal{L}^{3,1}_{2\times n}(Z+Q) \to \mathcal{L}^{3,1}_{2\times(n-3)}(\mathrm{Tr}_A(Z+Q)).$$

Then it is enough to prove the left-most and the right-most linear systems are zero.

- The trace $\mathrm{Tr}_A(Z+Q) = Z_A + Z_{D\cap A} + Q_A$: the linear system $\mathcal{L}^{3,1}_{2\times(n-3)}(Z_A + Z_{D\cap A})$ is regular by induction and its dimension is exactly the cardinality of Q_A; hence, $\mathcal{L}^{3,1}_{2\times(n-3)}(\mathrm{Tr}_A(Z+Q)) = 0$.
- By Lemma 3.18, $\mathcal{L}^{3,1}_{2\times n}(A+Z+Q) = 0$.

□

Lemma 3.18. *Let $n \geq 6$. In the notation of the proof of Lemma 3.17, let $D \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a general divisor of bidegree $(0,1)$ and let $A \subset \mathbb{P}^2 \times \mathbb{P}^n$ be the subvariety defined by 3 general bidegree $(0,1)$ forms. Let Z and Q be schemes of fat points as in the proof of Lemma 3.17. Then $\dim \mathcal{L}^{3,1}_{2\times n}(A+Z+Q) = 0$.*

Proof. We proceed by a triple-step induction on n. The cases $n \in \{6,7,8\}$ are checked by an explicit software computation; see Appendix A. Assume that $n \geq 9$. Let $B \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a subvariety defined by 3 general bidegree $(0,1)$ forms. We consider a specialization W of Z and R of Q such that $W = W_{A\cap B} + W_{A\cap B\cap D} + W_{A\cap D} + W_D$ and $R = R_{A\cap B} + R_A + R_\circ$ where

$$\begin{array}{rl} W_{A\cap B} & \text{is a scheme of type } (2^{b(n-6)-b(n-7)}) \text{ on } A\cap B \cong \mathbb{P}^2 \times \mathbb{P}^{n-6}, \text{ outside } D; \\ W_{A\cap B\cap D} & \text{is a scheme of type } (2, 1^{b(n-7)}) \text{ on } A\cap B\cap D \cong \mathbb{P}^2 \times \mathbb{P}^{n-7}; \\ W_{A\cap D} & \text{is a set of 10 points on } A\cap D, \text{ outside } B; \\ W_D & \text{is a set of 10 points on } D, \text{ outside } A\cup B; \\ R_{A\cap B} & \text{is a set of } v(n-6) \text{ points on } A\cap B, \text{ outside } D; \\ R_A & \text{is a set of } v(n-3) - v(n-6) \text{ points on } A, \text{ outside } B\cup D; \\ R_\circ & \text{is a set of } v(n) - v(n-3) \text{ points outside } A\cup B\cup D. \end{array}$$

Consider

$$0 \to \mathcal{L}^{3,1}_{2\times n}(A+B+W+R) \to \mathcal{L}^{3,1}_{2\times n}(A+W+R) \to \mathcal{L}^{3,1}_{2\times(n-3)}((A\cap B) + \mathrm{Tr}_B(W+R)).$$

It is enough to show that the left-most and right-most linear system are zero.

- By Lemma B.17 and Lemma B.18, the linear system $\mathcal{L}^{3,1}_{2\times(n-3)}((A\cap B) + \mathrm{Tr}_B(W+R))$ is zero by induction.
- The linear system $\mathcal{L}^{3,1}_{2\times n}(A+B+W+R)$ is contained in the linear system $\mathcal{L}^{3,1}_{2\times n}(A+B)$ which is zero because the ideal of $A\cup B$ is generated in bi-degree $(0,2)$. □

Example 3.19 (Lemma 3.16, Lemma 3.17, Lemma 3.18 for $n=8$)**.** We show that $\mathcal{L}^{3,2}_{2\times 8}(3, 2^{29})$ is regular, i.e., $\dim \mathcal{L}^{3,2}_{2\times 8}(3, 2^{29}) = 65$. Let D be a general divisor of bidegree $(0,1)$ and let X be a scheme of type $(3, 2^{29})$ such that $X \cap D$ is general of type $(3, 2^{26})$. By Castelnuovo exact sequence

$$65 \leq \dim \mathcal{L}^{3,2}_{2\times 8}(X) \leq \dim \mathcal{L}^{3,1}_{2\times 8}(\mathrm{Res}_D(X)) + \mathcal{L}^{3,2}_{2\times 7}(\mathrm{Tr}_D(X)). \tag{3.7}$$

We assume to know that $\mathcal{L}^{3,2}_{2\times 7}(\mathrm{Tr}_D(X))$ is regular, i.e., $\dim \mathcal{L}^{3,2}_{2\times 7}(\mathrm{Tr}_D(X)) = 45$. The residue $Y := \mathrm{Res}_D(X)$ is a scheme of type $(2^4, 1^{26})$ such that $Y \cap D$ is general of type $(2, 1^{26})$. Since $\mathrm{vdim}\, \mathcal{L}^{3,1}_{2\times 8}(Y) = 20$, we consider a set of twenty general points P. We want to prove that $\dim \mathcal{L}^{3,1}_{2\times 8}(Y+P) = 0$. Let A, B be two subvarieties of codimension 3 defined by general forms of bidegree $(0,1)$. We consider a series of specializations of the scheme $Y+P$: we describe them as union of distinct components, each one with general support in the space indicated by the diagrams in Figure 4.

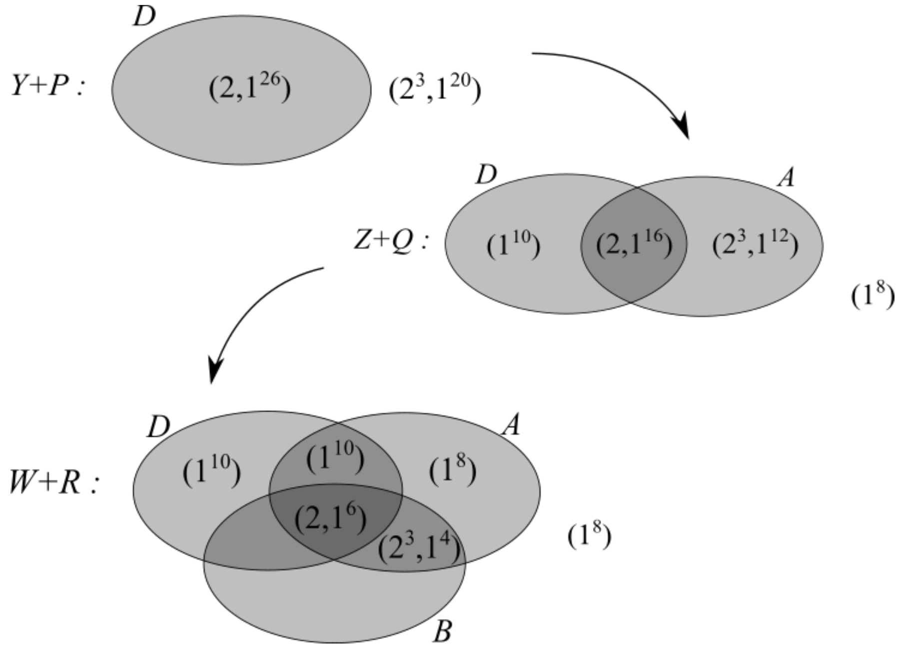

FIGURE 4. The specialization used in Example 3.19

By using a series of Castelnuovo exact sequences, we obtain the following chain of inequalities

$$\begin{aligned}\dim \mathcal{L}^{3,1}_{2\times 8}(Y+P) &\leq \dim \mathcal{L}^{3,1}_{2\times 8}(Z+Q) \leq \dim \mathcal{L}^{3,1}_{2\times 8}(A+Z+Q) + \dim \mathcal{L}^{3,1}_{2\times 5}(\mathrm{Tr}_A(Z+Q))\\ &\leq \Big(\dim \mathcal{L}^{3,1}_{2\times 8}(A+B+W+R) + \dim \mathcal{L}^{3,1}_{2\times 5}((A\cap B)+\mathrm{Tr}_B(W+R)\Big)\\ &\qquad + \dim \mathcal{L}^{3,1}_{2\times 5}(\mathrm{Tr}_A(Z+Q)).\end{aligned}$$

We assume that at each step of the chain of inequalities the traces on $\mathbb{P}^2\times\mathbb{P}^5$ are known to be zero. Hence, we are left with checking that $\dim \mathcal{L}^{3,1}_{2\times 8}(A+B+W+R)=0$ which holds by the straightforward observation that the ideal of $A\cup B$ is generate in bidegree $(0,2)$ and therefore $\dim \mathcal{L}^{3,1}_{2\times 8}(A+B)=0$.

4. Proof of theorem 2.5

We are now ready to prove our main theorem.

Notation 4.1. Let p be a point in $\mathbb{P}^m\times\mathbb{P}^n$ and let $T=\{t_1,\dots,t_r\}$ be a set of points in the exceptional divisor of $\mathrm{Bl}_p(\mathbb{P}^m\times\mathbb{P}^n)$. We denote by $(m[T])$ the type of a 0-dimensional scheme $Z\subset\mathbb{P}^m\times\mathbb{P}^n$ supported at p, such that:

- $mp\subset Z\subsetneq (m+1)p$;
- if $D\in I_Z$ is a divisor of $\mathbb{P}^m\times\mathbb{P}^n$ then the strict transform of D in $\mathrm{Bl}_p(\mathbb{P}^m\times\mathbb{P}^n)$ cuts the exceptional divisor E in a divisor of degree m in E passing through the points $t_1,\dots,t_r$.

Without loss of generality, we may assume that $n\geq m$. Moreover, the case $m=1$ is proved by [AB13, Theorem 1.2], so we may assume that $n\geq m\geq 2$. As we recalled in Remark 2.6, it is enough to prove our statement for $r\in\{r_*(m,n),r^*(m,n)\}$.

Let $X\subset\mathbb{P}^m\times\mathbb{P}^n$ be a general scheme of fat points of type (2^r) and let X_0 be the specialization of X when $m+n+1$ of the r double points collide together as explained in Remark 2.8. Let $N:=\binom{m+n+1}{2}$ and let $T=\{t_1,\dots,t_N\}$ be the set of the infinitely near points. If S is a subset of T, we will denote by $\mathcal{L}^{3,3}_{m\times n}(3[S])$ the linear system of bidegree $(3,3)$ divisors containing the triple point and whose strict transform contains S. In order to prove that $\mathcal{L}^{3,3}_{m\times n}(X)$ is nonspecial it is enough to prove that $\mathcal{L}^{3,3}_{m\times n}(X_0)$ is nonspecial. By generality, we write

$$\mathcal{L}^{3,3}_{m\times n}(X_0)=\mathcal{L}^{3,3}_{m\times n}(3[T],2^{r-m-n-1}),$$

where T is a set of $N:=\binom{m+n+1}{2}$ tangent directions to the triple point, in the special position described by Remark 2.8. Observe that $\mathcal{L}^{3,3}_{m\times n}(3,2^{r-m-n-1})$ is regular by Lemma 3.4, so we only need to prove that $T=\{t_1,\dots,t_N\}$ impose independent conditions to $\mathcal{L}^{3,3}_{m\times n}(3,2^{r-m-n-1})$. Assume by contradiction that $\mathcal{L}^{3,3}_{m\times n}(3[T], 2^{r-m-n-1})$ is special, i.e., there exists $i \in \{1, \dots, N-1\}$ such that $t_{i+1} \in \mathrm{Bs}(\mathcal{L}^{3,3}_{m\times n}(3[t_1, \dots, t_i], 2^{r-m-n-1})$.

Define
$$h := \min\left\{a \in \mathbb{N} \mid \mathcal{L}^{3,3}_{m\times n}(3[t_1, \dots, t_{i+1}], 2^a) \text{ is special}\right\}.$$
Let q be the point of support of the component of X_0 of multiplicity 3 and let $E \cong \mathbb{P}^{m+n-1}$ be the exceptional divisor of the blow-up $\mathrm{Bl}_q(\mathbb{P}^m \times \mathbb{P}^n)$.

Claim 1. $h \geq 1$.

Proof of Claim 1. It is enough to show that T imposes independent conditions on the strict transform $\mathcal{L}$ on X of the linear system $\mathcal{L}^{3,3}_{m\times n}(3)$. In order to do so, we apply Lemma 3.3:

- The hypothesis Lemma 3.3(1) is satisfied because T imposes independent conditions on $\mathcal{L}_{|E}$, by Lemma 2.11.
- Note that the sublinear space of divisors of $\mathcal{L}$ which contain the exceptional divisor E is the strict transform of $\mathcal{L}^{3,3}_{m\times n}(4)$ on $\mathrm{Bl}_q(\mathbb{P}^m \times \mathbb{P}^n)$. Hence, in order to check that Lemma 3.3(2) is verified we need to prove that
$$\dim \mathcal{L}^{3,3}_{m\times n}(3) - \dim \mathcal{L}^{3,3}_{m\times n}(4) \geq \binom{m+n+1}{2}.$$
This is checked in Lemma B.1. □

By definition of h, we have that $\mathcal{L}^{2,d}_{2\times n}(3[t_1, \dots, t_{i+1}], 2^{h-1})$ is non-special. In particular,
$$t_{i+1} \notin \mathrm{Bs}(\mathcal{L}^{3,3}_{m\times n}(3[t_1, \dots, t_i], 2^{h-1})) \text{ and } t_{i+1} \in \mathrm{Bs}(\mathcal{L}^{3,3}_{m\times n}(3[t_1, \dots, t_i], 2^{h})).$$
By Proposition 3.2, every element of $\mathcal{L}^{3,3}_{m\times n}(3[t_1, \dots, t_i], 2^h)$ is singular at t_{i+1}. In particular, the general element of $\mathcal{L}^{3,3}_{m\times n}(3[T], 2^h)$ is singular at t_{i+1} and, by monodromy, it is singular at each point of T. In other words, the restriction of the strict transform of $\mathcal{L}^{3,3}_{m\times n}(3[T], 2^h)$ to E is a subsystem of the linear system of cubics on E singular at T, which is zero by Lemma 2.11. We deduce that
$$\mathcal{L}^{3,3}_{m\times n}(3[T], 2^h) \subset \mathcal{L}^{3,3}_{m\times n}(4, 2^h).$$
By Lemma 3.7, we conclude that
$$\mathcal{L}^{3,3}_{m\times n}(3[T], 2^h) = 0. \tag{4.1}$$
Now, in both cases $r \in \{r_*(m,n), r^*(m,n)\}$ we get a contradiction:

- if $r = r_*$ and $\mathrm{vdim}\, \mathcal{L}^{3,3}_{m\times n}(3[T], 2^{r-m-n-1}) > 0$, then (4.1) contradicts the fact that the actual dimension is at least the virtual dimension;
- if $r = r^*$ and $\mathrm{vdim}\, \mathcal{L}^{3,3}_{m\times n}(3[T], 2^{r-m-n-1}) \leq 0$, then (4.1) contradicts the assumption that $\mathcal{L}^{3,3}_{m\times n}(3[T], 2^{r-m-n-1})$ is special.

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Appendix A. Computations with algebraic software

We used induction to prove several results in Section 3. In this appendix we show how to check the base cases with a straightforward interpolation, using the Algebra software Macaulay2. The complete code can be found in the ancillary file of the arXiv submission or on the webpage of the second author.

Fixed positive integers a, b, m, n, we consider the monomial basis `B` of the vector space of forms of bidegree (a, b) in the bigraded coordinate ring `S` of $\mathbb{P}^m \times \mathbb{P}^n$, i.e.,

```
S = QQ[x_0..x_m] ** QQ[y_0..y_n];
B = first entries super basis({a,b},S).
```

Sometimes we deal with linear subspaces of $\mathcal{L}^{a,b}_{m\times n}$ whose base locus contains a union of general subspaces defined by bidegree $(0, 1)$ forms. In this case we assume that such forms are chosen among the coordinates of $\mathbb{P}^m \times \mathbb{P}^n$ and we use as `B` the monomial basis of the homogeneous part in bidegree (a, b) of the ideal defining the subspace. For example, in the case of Lemma 3.14, the base locus contains the union of three general codimension-2 subspaces defined by forms of bidegree $(0, 1)$; hence,

```
A1 = sub(ideal(y_0,y_1),S);
A2 = sub(ideal(y_2,y_3),S);
A3 = sub(ideal(y_4,y_5),S);
B = first entries super basis({a,b},intersect({A1,A2,A3}));
```

Now we consider the element in the span of `B` with generic coefficients, i.e.,

```
C = QQ[c_0..c_(#B-1)];
R = C[x_0..x_m]**C[y_0..y_n];
F = sum for i to #B-1 list c_i*sub(B_i,R).
```

At this point, we impose the conditions given by the scheme of fat points in the base locus. The scheme of fat points is defined by two attributes: a matrix `P` whose columns are the coordinates of the points supporting the scheme and a list of integers `M` giving the type of the scheme. Hence, we obtain a system of linear equations in the $\mathtt{c}_i$'s whose solution is exactly the vector space we want to compute. Therefore, we just need to compute the rank of the matrix `V` of the coefficients.

```
V = sub(sub(diff(matrix {for j to #B-1 list C_j},
        transpose diff(symmetricPower(M_0-1,vars(R)),F)),
      for i to m+n+1 list R_i => P_0_i),QQ);
for j from 1 to #M-1 do (
    V = V || sub(sub(diff(matrix {for j to #B-1 list C_j},
```

```
            transpose diff(symmetricPower(M_j-1,vars(R)),F)),
        for i to m+n+1 list R_i => P_j_i),QQ);
);
```

Appendix B. Arithmetic computations

Here we collect some arithmetic properties that we use in the paper. For the convenience of the reader, we recall here the definition of the constants that we used.

$$r^*(m,n) := \left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rceil \qquad r_*(m,n) := \left\lfloor \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rfloor$$

$$k^*(m,n) := r^*(m,n) - m - n - 1 \qquad k^*(m,n) := r^*(m,n) - m - n - 1$$

$$f(m,n) := 1 + k^*(m,n) - k^*(m-1,n) \qquad \ell(m,n) := k_*(m,n) - k_*(m-1,n)$$

$$s(n) := \frac{n(n+3)}{2} \qquad b(n) := k_*(2,n) - k_*(2,n-1)$$

$$v(n) := 10(n+1) - (n+3)(1 + b(n) - b(n-1)) - b(n-1)$$

Lemma B.1. *If* $m, n \in \mathbb{N}$, *then* $\dim \mathcal{L}^{3,3}_{m\times n}(3) - \dim \mathcal{L}^{3,3}_{m\times n}(4) \geq \binom{m+n+1}{2}$.

Proof. A direct computation shows that a j-fat point imposes independent conditions on $\mathcal{L}^{a,b}_{m\times n}$ for any $a, b \geq j-1$. Therefore, we only need to check that

$$\binom{m+3}{3}\binom{n+3}{3} - \binom{m+n+2}{2} - \binom{m+3}{3}\binom{n+3}{3} + \binom{m+n+3}{3} \geq \binom{m+n+1}{2}.$$

Such inequality is equivalent to

$$(m+n+2)(m+n+1)\left(\frac{m+n+3}{6} - \frac{1}{2}\right) \geq \frac{(m+n+1)(m+n)}{2}$$
$$\Leftrightarrow \frac{m+n+2}{6} \geq \frac{1}{2} \Leftrightarrow m+n \geq 1. \qquad \square$$

Lemma B.2. *Let* $2 \leq m \leq n$. *Then*

$$k^*(m,n) - k^*(m-1,n) \geq \left\lceil \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rceil + m.$$

Proof. First we bound

$$\begin{aligned}
&k^*(m,n) - k^*(m-1,n) - \left\lceil \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rceil - m \\
&= \left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rceil - (m+n+1) - \left\lceil \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n} \right\rceil + (m+n) - \left\lceil \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rceil - m \\
&\geq \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} - (m+n+1) - \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n} - 1 + (m+n) - \frac{m+1}{m+n+1}\binom{n+3}{3} - 1 - m \\
&= \binom{n+3}{3}\frac{\binom{m+3}{3}(m+n) - \binom{m+2}{3}(m+n+1) - (m+1)(m+n)}{(m+n)(m+n+1)} - m - 3.
\end{aligned}$$

Let

$$\begin{aligned}
A(m,n) := \binom{n+3}{3}&\left(\binom{m+3}{3}(m+n) - \binom{m+2}{3}(m+n+1) - (m+1)(m+n)\right) \\
&- (m+3)(m+n)(m+n+1).
\end{aligned}$$

We need to show that $A(m,n) \geq 0$. We distinguish two cases.

If $m \geq 9$, then we consider $A(m,n) - A(m,n-1) \in \mathbb{C}[m][n]$, which equals

$$\left(\frac{1}{3}m^2 + \frac{1}{3}m\right)n^3 + \left(\frac{1}{6}m^3 + m^2 + \frac{5}{6}m\right)n^2 + \left(\frac{1}{2}m^3 + \frac{2}{3}m^2 - \frac{11}{6}m - 6\right)n + \frac{1}{3}m^3 - 2m^2 - \frac{19}{3}m.$$

Since $m \geq 9$, all the coefficients are positive and so $A(m,n) \geq A(m,n-1)$. In particular, since $m \leq n$, we have that

$$A(m,n) \geq A(m,m) = \frac{5}{36}m^6 + \frac{11}{12}m^5 + \frac{71}{36}m^4 - \frac{31}{12}m^3 - \frac{127}{9}m^2 - \frac{19}{3}m \geq 0.$$

Assume now that $m \in \{2,\dots,8\}$. The polynomial $A(m,n) \in \mathbb{C}[m][n]$ has positive leading coefficient. It is easy to check that $A(m,n) \geq 0$ for $n \geq N(m)$, where

$$N(m) = \begin{cases} 3 & \text{if } m = 2 \\ 2 & \text{if } m \in \{3,\dots,8\}. \end{cases}$$

Since $m \leq n$, the only remaining case is $(m,n) = (2,2)$, which is checked directly. □

Lemma B.3. *If $m \geq 2$, then* $\operatorname{vdim} \mathcal{L}^{2,3}_{m\times n}(2^{k^*(m,n)-k^*(m-1,n)+1}) \geq k^*(m-1,n)$.

Proof. We can bound

$$\begin{aligned}
&\operatorname{vdim} \mathcal{L}^{2,3}_{m\times n}(2^{k^*(m,n)-k^*(m-1,n)+1}) - k^*(m-1,n)\\
&= \binom{m+2}{2}\binom{n+3}{3} - (m+n+1)\big(k^*(m,n)+1\big) + (m+n)k^*(m-1,n)\\
&= \binom{m+2}{2}\binom{n+3}{3} - (m+n+1)\left(\left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rceil - m - n\right) + (m+n)\left(\left\lceil \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rceil - m - n\right)\\
&= \binom{m+2}{2}\binom{n+3}{3} - (m+n+1)\left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rceil + (m+n)\left\lceil \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rceil + m + n\\
&\geq \binom{m+2}{2}\binom{n+3}{3} - \binom{m+3}{3}\binom{n+3}{3} - m - n + \binom{m+2}{3}\binom{n+3}{3} + m + n = 0.
\end{aligned}$$
□

Lemma B.4. *If $n \geq 2$, then* $\operatorname{vdim} \mathcal{L}^{3,1}_{1\times n}(1, 2^{k^*(1,n)-k^*(1,n-1)}) \leq 0$.

Proof.

$$\begin{aligned}
&\operatorname{vdim}\ \mathcal{L}^{3,1}_{1\times n}(1, 2^{k^*(1,n)-k^*(1,n-1)})\\
&\quad = 4(n+1) - 1 - (n+2)\left(\left\lceil \frac{4\binom{n+3}{3}}{n+2}\right\rceil - (n+2) - \left\lceil \frac{4\binom{n+2}{3}}{n+1}\right\rceil + (n+1)\right)\\
&\quad \leq 4n+3 - (n+2)\left(\frac{4\binom{n+3}{3}}{n+2} - (n+2) - \frac{4\binom{n+2}{3}}{n+1} - 1 + (n+1)\right)\\
&\quad = 4n+3 - 4\binom{n+3}{3} + 4\frac{(n+2)^2 n}{6} + 2(n+2) = -\frac{4}{3}n^2 + \frac{4}{3}n + 3 < 1.
\end{aligned}$$
□

Lemma B.5. *If $n \geq m \geq 2$, then*

$$\ell(m,n) - \ell(m-1,n) \geq \frac{(n+3)(n+2)}{6} + 2. \tag{B.1}$$

In particular:

1. $\ell(m-1,n) \leq \ell(m,n)$,
2. $\operatorname{vdim} \mathcal{L}^3_n(1, 2^{\ell(m,n)-\ell(m-1,n)}) \leq 0$,
3. $f(m-1,n) \leq f(m,n)$ *and*
4. $\operatorname{vdim} \mathcal{L}^3_n(2^{f(m,n)-f(m-1,n)}) \leq 0$.

Proof. We need to prove that

$$\left\lfloor \frac{\binom{m+3}{3}\binom{n+3}{3}}{(m+n+1)}\right\rfloor - (m+n+1) - \left\lfloor \frac{\binom{m+2}{3}\binom{n+3}{3}}{(m+n+1)}\right\rfloor + (m+n) - \frac{(n+3)(n+2)}{6} - 2 \geq 0.$$

The left-hand-side is bigger or equal than

$$\frac{\binom{m+3}{3}\binom{n+3}{3}}{(m+n+1)} - 1 - (m+n+1) - \frac{\binom{m+2}{3}\binom{n+3}{3}}{(m+n+1)} + (m+n) - \frac{(n+3)(n+2)}{6} - 2$$
$$= \frac{A(m,n)}{12(\,m+\,n+1)},$$

where

$$A(m,n) = \left(m^2+3\,m\right)n^3 + \left(6\,m^2+16\,m\right)n^2 + \left(11\,m^2+23\,m-48\right)n + 6\,m^2 - 42\,m - 48.$$

For $m \geq 8$, by Descartes' rule of signs, the univariate polynomial $A(m,n) \in \mathbb{C}[m][n]$ has no positive root and we deduce that $A(m,n) \geq 0$. For $2 \leq m \leq 7$, we can directly check that the polynomial $A(m,n)$ is positive for $n \geq 2$. This completes the proof of (B.1).

From (B.1), we clearly get that $\ell(m-1,n) \leq \ell(m,n)$. Moreover,

$$\begin{aligned} \operatorname{vdim} \mathcal{L}^3_n(1, 2^{\ell(m,n)-\ell(m-1,n)}) &= \binom{n+3}{3} - 1 - (n+1)(\ell(m,n) - \ell(m-1,n)) \\ &\leq \binom{n+3}{3} - 1 - \binom{n+3}{3} - 2(n+1) \leq 0. \end{aligned}$$

Note that

$$\begin{aligned} f(m,n) &= k^*(m,n) - k^*(m-1,n) \geq k_*(m,n) - k_*(m-1,n) - 1 = \ell(m,n) - 1. \\ f(m,n) &= k^*(m,n) - k^*(m-1,n) \leq k_*(m,n) + 1 - k_*(m-1,n) = l(m,n) + 1. \end{aligned}$$

Therefore, from (B.1), we obtain

$$f(m,n) - f(m,n) \geq l(m,n) - 1 - \ell(m-1,n) - 1 \geq \frac{(n+3)(n+2)}{6} \tag{B.2}$$

and consequently

$$\begin{aligned} \operatorname{vdim} \mathcal{L}^3_n(2^{f(m,n)-f(m-1,n)}) &= \binom{n+3}{3} - (n+1)(f(m,n) - f(m-1,n)) \\ &\leq \binom{n+3}{3} - \binom{n+3}{3} = 0. \end{aligned}$$ □

Lemma B.6. *If $m \geq 2$, then* $\operatorname{vdim} \mathcal{L}^{2,3}_{m\times n}(2^{f(m,n)}) \geq 0$.

Proof.

$$\begin{aligned} &\operatorname{vdim}\ \mathcal{L}^{2,3}_{m\times n}(2^{f(m,n)}) \\ &= \binom{m+2}{2}\binom{n+3}{3} - (m+n+1)\left(\left\lceil \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rceil - \left\lceil \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n} \right\rceil\right) \\ &\geq \binom{m+2}{2}\binom{n+3}{3} - \binom{m+3}{3}\binom{n+3}{3} - (m+n) + \frac{m+n+1}{m+n}\binom{m+2}{3}\binom{n+3}{3} \\ &= -\binom{m+2}{3}\binom{n+3}{3} - m - n + \frac{m+n+1}{m+n}\binom{m+2}{3}\binom{n+3}{3} \\ &= \frac{1}{m+n}\binom{m+2}{3}\binom{n+3}{3} - m - n = \frac{A(m,n)}{36(m+n)} \end{aligned}$$

where

$$\begin{aligned} A(m,n) = \left(m^3+3\,m^2+2\,m\right)n^3 + \left(6\,m^3+18\,m^2+12\,m-36\right)n^2 + \\ \left(11\,m^3+33\,m^2-50\,m\right)n + 6\,m^3 - 18\,m^2 + 12\,m. \end{aligned}$$

By Descartes' rule of sign, the univariate polynomial $A(m,n)$ has no positive roots for $n \geq m \geq 2$ and, in particular, $A(m,n) > 0$. □

Lemma B.7. *If $n \geq m \geq 2$, then*

$$2 + \ell(m,n) - \ell(m-1,n) \leq \left\lfloor \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rfloor - m. \tag{B.3}$$

As a consequence, we obtain that:

(1) $f(m,n) - f(m-1,n) \leq \left\lfloor \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rfloor - m$;

(2) $1 + \ell(m,n) - \ell(m-1,n) \leq \left\lfloor \frac{m+1}{m+n+1}\binom{n+3}{3} \right\rfloor - m$.

Proof. Note that (1) follows from (B.3) because $f(m,n) - f(m-1,n) \leq 2 + \ell(m,n) - \ell(m-1,n)$ by (B.2), while (2) is trivial. Hence, we only have to prove (B.3), i.e.,

$$2 + k_*(m,n) - 2k_*(m-1,n) + k_*(m-2,n) \leq \left\lfloor \frac{(m+1)\binom{n+3}{3}}{m+n+1} \right\rfloor - m.$$

I.e.,

$$2 + \left\lfloor \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} \right\rfloor - (m+n+1) - 2\left\lfloor \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n} \right\rfloor + 2(m+n)$$
$$+ \left\lfloor \frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1} \right\rfloor - (m+n-1) - \left\lfloor \frac{(m+1)\binom{n+3}{3}}{m+n+1} \right\rfloor - m \leq 0.$$

The left-hand-side is smaller or equal to

$$\begin{aligned} 2 + \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1} - (m+n+1) - 2\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n} + 2 + 2(m+n)& \\ + \frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1} - (m+n-1) - \frac{(m+1)\binom{n+3}{3}}{m+n+1} + 1& \\ = \frac{A(m,n)}{18(m+n)(m+n-1)(m+n+1)}& \end{aligned}$$

with

$$\begin{aligned} A(m,n) &= \left(-3\,m^2 - 3\,m\right) n^4 + \left(-2\,m^3 - 18\,m^2 - 16\,m + 90\right) n^3 \\ &\quad + \left(-12\,m^3 - 33\,m^2 + 249\,m\right) n^2 + \left(-22\,m^3 + 252\,m^2 + 4\,m - 90\right) n + 78\,m^3 - 78\,m. \end{aligned}$$

We regard at $A(m,n)$ as a polynomial in $\mathbb{C}[m][n]$.

- For $m \geq 12$, by Descartes' rule of signs, the univariate polynomial $A(m,n)$ has at most one positive root. It is immediate to check that $A(m,0) > 0$ and $A(m,m) < 0$, i.e., such unique positive root is in the interval $(0,m)$. Since $m \leq n$, we deduce that $A(m,n) \leq 0$.
- For $2 \leq m \leq 11$, since the leading term of the univariate polynomial $A(m,n)$ is negative, we have that $A(m,n) \leq 0$ for $n \geq N(m)$ where

$$N(m) = \begin{cases} 5 & \text{if } m = 2; \\ 4 & \text{if } m = 3; \\ 3 & \text{if } m \in \{4,5,6,7,8\}; \\ 2 & \text{if } m \in \{9,10,11\}. \end{cases}$$

  Since $n \geq m$, the only remaining cases are $(m,n) \in \{(2,2),(2,3),(2,4),(3,3)\}$ where the statement is checked directly. $\square$

Lemma B.8. *If $n \geq m \geq 2$, then* $\operatorname{vdim} \mathcal{L}^{1,3}_{m\times n}(2^{f(m,n)-f(m-1,n)}) \geq f(m-1,n)$.

Proof. We have to prove that

$$(m+1)\binom{n+3}{3} - (m+n+1)(k^*(m,n) - 2k^*(m-1,n) + k^*(m-2,n)) - k^*(m-1,n) + k^*(m-2,n) \geq 0.$$

The left-hand-side is

$$\begin{aligned}
&(m+1)\binom{n+3}{3}-(m+n+1)\left(\left\lceil\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rceil-(m+n+1)-2\left\lceil\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rceil+2(m+n)\right.\\
&\qquad\left.+\left\lceil\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}\right\rceil-(m+n-1)\right)-\left\lceil\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rceil+(m+n)+\left\lceil\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}\right\rceil-(m+n-1)\\
&\geq(m+1)\binom{n+3}{3}-(m+n+1)\left(\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}+1-(m+n+1)-2\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}+2(m+n)\right.\\
&\qquad\left.+\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}+1-(m+n-1)\right)-\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}-1+(m+n)+\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}-(m+n-1)\\
&\qquad=\frac{A(m,n)}{36(m+n)(m+n-1)},
\end{aligned}$$

where

$$\begin{aligned}
A(m,n)&=\left(3\,m^2+3\,m\right)n^4+\left(2\,m^3+18\,m^2+16\,m-72\right)n^3\\
&\quad+\left(12\,m^3+33\,m^2-195\,m\right)n^2+\left(22\,m^3-198\,m^2-4\,m+72\right)n-60\,m^3+60\,m.
\end{aligned}$$

We consider $A(m,n)$ as a polynomial in $\mathbb{C}[m][n]$.

- for $m\geq 9$, by Descartes' rule of signs, the univariate polynomial $A(m,n)$ has a unique positive root; moreover, it is immediate to check that $A(m,0)<0$ while $A(m,m)>0$ and, therefore, such unique positive root is in the interval $(0,m)$. Since $n\geq m$, we deduce that $A(m,n)\geq 0$;
- for $2\leq m\leq 8$, the univariate polynomial $A(m,n)$ is positive for $n\geq N(m)$ where

$$N(m)=\begin{cases}4 & \text{if } m=2;\\ 3 & \text{if } m\in\{3,4\};\\ 2 & \text{if } m\in\{5,6,7,8,9\}.\end{cases}$$

  Therefore, the only cases left are $(m,n)\in\{(2,2),(2,3)\}$ where the statement is checked directly. ☐

Lemma B.9. *Let $n\geq m\geq 2$. Then* $\operatorname{vdim}\mathcal{L}^{2,3}_{m\times n}(3,2^{k_*(m,n)-k_*(m-1,n)})\leq k_*(m-1,n)$.

Proof. We need to show that

$$\begin{aligned}
&\binom{m+2}{2}\binom{n+3}{3}-\binom{m+n+2}{2}-(m+n+1)\left(\left\lfloor\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rfloor-(m+n+1)\right.\\
&\qquad\left.-\left\lfloor\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rfloor+(m+n)\right)-\left\lfloor\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rfloor+(m+n)\leq 0.
\end{aligned}$$

The left-hand-side is smaller or equal to

$$\begin{aligned}
&\binom{m+2}{2}\binom{n+3}{3}-\binom{m+n+2}{2}-(m+n+1)\left(\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}-1-(m+n+1)\right.\\
&\qquad\left.-\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}+(m+n)\right)-\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}+1+(m+n)\\
&\qquad=\frac{-m^2-2\,m\,n-n^2+3\,m+3\,n+4}{2}.
\end{aligned}$$

Let $A(m,n)=-m^2+(-2\,n+3)\,m-n^2+3\,n+4\in\mathbb{C}[n][m]$. By Descartes' rule of signs, for $n\geq 4$, the univariate polynomial $A(m,n)$ has no positive roots and we deduce that $A(m,n)\leq 0$. In the remaining cases $(m,n)\in\{(2,2),(2,3),(3,3)\}$, the statement is checked directly. ☐

Lemma B.10. *Let $n\geq m\geq 2$. Then*

$$1+k_*(m,n)-k_*(m-1,n)\geq\left\lceil\frac{m+1}{m+n+1}\binom{n+3}{3}\right\rceil+m.$$

Proof. We need to show that

$$1+\left\lfloor \frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rfloor-(m+n+1)-\left\lfloor \frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rfloor+(m+n)-\left\lceil \frac{m+1}{m+n+1}\binom{n+3}{3}\right\rceil-m\geq 0.$$

The left-hand-side is bigger or equal to

$$\begin{aligned}&1+\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}-1-(m+n+1)-\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}+(m+n)-\frac{m+1}{m+n+1}\binom{n+3}{3}-1-m\\ &=\frac{A(m,n)}{36(m+n)(m+n+1)}\end{aligned}$$

where

$$\begin{aligned}A(m,n)&=\left(3\,m^2+3\,m\right)n^4+\left(2\,m^3+18\,m^2+16\,m\right)n^3\\ &+\left(12\,m^3+33\,m^2-15\,m-72\right)n^2+\left(22\,m^3-54\,m^2-184\,m-72\right)n-24\,m^3-108\,m^2-84\,m.\end{aligned}$$

- For $m\geq 5$, by Descartes' rule of signs, the univariate polynomial $A(m,n)\in\mathbb{C}[m][n]$ has at most one positive root. It is immediate to check that $A(m,0)<0$ while $A(m,n)>0$; hence, such unique positive root is in the interval $(0,m)$. Since $n\geq m$, we deduce that $A(m,n)\geq 0$.
- For $2\leq m\leq 4$, the univariate polynomial $A(m,n)$ is positive for $n\geq 2$. □

Lemma B.11. *If $n\geq 2$, then* $\operatorname{vdim}\mathcal{L}^{3,2}_{2\times n}(3,2^{k_*(2,n)-k_*(2,n-1)})\leq k_*(2,n-1)$.

Proof. We need to prove that

$$\begin{aligned}10\binom{n+2}{2}-\binom{n+4}{4}-(n+3)\left(\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{10\binom{n+2}{3}}{n+2}\right\rfloor+(n+2)\right)\\ -\left\lfloor\frac{10\binom{n+2}{3}}{n+2}\right\rfloor+(n+2)\leq 0.\end{aligned}$$

The left-hand-side is smaller or equal to

$$\begin{aligned}&10\binom{n+2}{2}-\binom{n+4}{4}-(n+3)\left(\frac{10\binom{n+3}{3}}{n+3}-1-(n+3)-\frac{10\binom{n+2}{3}}{n+2}+(n+2)\right)\\ &-\frac{10\binom{n+2}{3}}{n+2}+1+(n+2)\\ &=\frac{-n^4-10\,n^3-35\,n^2+22\,n+192}{24}.\end{aligned}$$

The latter is negative for $n\geq 2$. □

Lemma B.12. *Let $n\geq 2$. Then* $1+k_*(2,n)-k_*(2,n-1)\geq\left\lceil\frac{10(n+1)}{n+3}\right\rceil+n$.

Proof. We need to prove that

$$1+\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{10\binom{n+2}{3}}{n+2}\right\rfloor+(n+2)-\left\lceil\frac{10(n+1)}{n+3}\right\rceil-n\geq 0.$$

The left-hand-side is bigger or equal to

$$1+\frac{10\binom{n+3}{3}}{n+3}-1-(n+3)-\frac{10\binom{n+2}{3}}{n+2}+(n+2)-\frac{10(n+1)}{n+3}-1-n=\frac{7\,n^2-5\,n-18}{3(n+3)}.$$

The latter is positive for $n\geq 2$. □

Lemma B.13. *If $n\geq m\geq 2$, then* $\operatorname{vdim}\mathcal{L}^{1,3}_{m\times n}(2^{1+\ell(m,n)-\ell(m-1,n)})\geq\ell(m-1,n))$.

Proof. We need to prove that

$$\begin{aligned}(m+1)\binom{n+3}{3}-(m+n+1)\left(1+k_*(m,n)-2k_*(m-1,n)+k_*(m-2,n)\right)\\ -k_*(m-1,n)+k_*(m-2,n)\geq 0\end{aligned}$$

The left-hand-side is

$$\begin{aligned}
&(m+1)\binom{n+3}{3}-(m+n+1)\left(1+\left\lfloor\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}\right\rfloor-2\left\lfloor\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rfloor+\left\lfloor\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}\right\rfloor\right)\\
&\qquad\qquad\qquad\qquad-\left\lfloor\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}\right\rfloor+\left\lfloor\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}\right\rfloor\\
\geq\ &(m+1)\binom{n+3}{3}-(m+n+1)\left(1+\frac{\binom{m+3}{3}\binom{n+3}{3}}{m+n+1}-2\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}-2+\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}\right)\\
&\qquad\qquad\qquad\qquad-\frac{\binom{m+2}{3}\binom{n+3}{3}}{m+n}+\frac{\binom{m+1}{3}\binom{n+3}{3}}{m+n-1}-1\\
=\ &\frac{A(m,n)}{36(m+n+1)(m+n-1)}
\end{aligned}$$

where

$$\begin{aligned}
A(m,n)=&\left(3\,m^2+3\,m\right)n^4+\left(2\,m^3+18\,m^2+16\,m+36\right)n^3\\
&+\left(12\,m^3+33\,m^2+129\,m-36\right)n^2+\left(22\,m^3+126\,m^2-76\,m\right)n\\
&+48\,m^3-36\,m^2-12\,m
\end{aligned}$$

By Descartes' rule of signs, the univariate polynomial $A(m,n)\in\mathbb{C}[m][n]$ has no positive roots for $m\geq 2$. Since $A(m,0)\geq 0$, we deduce that $A(m,n)\geq 0$ and the statement follows. □

Lemma B.14. *Let $n\geq 3$. In the notation of Lemma 3.10,*

$$1+\ell(2,n)-s(n)\leq\left\lfloor\frac{3}{n+3}\binom{n+3}{3}\right\rfloor-2.$$

Proof. A software computation shows that the claim holds for $n\in\{3,4\}$. We assume that $n\geq 5$ and we bound

$$\begin{aligned}
1+&\ell(2,n)-s(n)-\left\lfloor\frac{3}{n+3}\binom{n+3}{3}\right\rfloor+2=3+k_*(2,n)-k_*(1,n)-\frac{(n+3)n}{2}-\frac{(n+2)(n+1)}{2}\\
&=3+\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{4\binom{n+3}{3}}{n+2}\right\rfloor+(n+2)-\frac{(n+3)n}{2}-\frac{(n+2)(n+1)}{2}\\
&\leq 3+\frac{5(n+2)(n+1)}{3}-(n+3)-\frac{2(n+3)(n+1)}{3}+1+(n+2)-\frac{(n+3)n}{2}-\frac{(n+2)(n+1)}{2}\\
&=\frac{2}{3}(5-n)\leq 0.
\end{aligned}$$

□

Lemma B.15. *Let $n\geq 3$. Then*

$$\ell(2,n)-s(n)\geq\left\lceil\frac{\binom{n+3}{3}-1}{n+1}\right\rceil. \tag{B.4}$$

In particular:

1. *$s(n)\leq\ell(2,n)$ and*
2. *$\operatorname{vdim}\mathcal{L}^3_n(1,2^{\ell(2,n)-s(n)})\leq 0$.*

Proof. We directly check that

$$\begin{aligned}\ell(2,n)-s(n)-\left\lceil\frac{\binom{n+3}{3}-1}{n+1}\right\rceil &= k_*(2,n)-k_*(1,n)-s(n)-\left\lceil\frac{\binom{n+3}{3}-1}{n+1}\right\rceil\\ &=\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{4\binom{n+3}{3}}{n+2}\right\rfloor+(n+2)-s(n)-\left\lceil\frac{n(n^2+6n+11)}{6(n+1)}\right\rceil\\ &\geq\frac{5(n+2)(n+1)}{3}-1-(n+3)-\frac{2(n+3)(n+1)}{3}+(n+2)-\frac{n(n^2+6n+11)}{6(n+1)}-1\\ &=\frac{5\,n^3+14\,n^2-n-4}{6(n+1)},\end{aligned}$$

where the latter is positive for $n\geq 1$. □

Lemma B.16. *Let $n\geq 3$. Then*

$$\operatorname{vdim}\mathcal{L}^{1,3}_{2\times n}(2^{1+\ell(2,n)-s(n)})\geq s(n). \tag{B.5}$$

Proof. We need to prove that

$$3\binom{n+3}{3}-(n+3)\left(1+\ell(2,n)-s(n)\right)-s(n)\geq 0.$$

The left-hand-side is

$$\begin{aligned}&3\binom{n+3}{3}-(n+3)\left(1+\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{4\binom{n+3}{3}}{n+2}\right\rfloor+(n+2)\right)+(n+2)\frac{n(n+3)}{2}\\ \geq\ &3\binom{n+3}{3}-(n+3)\left(1+\frac{10\binom{n+3}{3}}{n+3}-(n+3)-\frac{4\binom{n+3}{3}}{n+2}+1+(n+2)\right)+(n+2)\frac{n(n+3)}{2}\\ =\ &\frac{n^2-5\,n-24}{6},\end{aligned}$$

where the latter is positive for $n\geq 8$. In the cases $3\leq n\leq 7$, (B.5) can be checked directly. □

Lemma B.17. *Let $n\geq 3$. In the notation of the proof of Lemma 3.16,*

$$b(n)-b(n-1)=\begin{cases}4 & n\equiv 1 \mod 3;\\ 3 & \textit{otherwise}\ .\end{cases}$$

In particular, we deduce that $b(n)-b(n-3)=10>0$.

Proof. By definition

$$b(n)=k_*(2,n)-k_*(2,n-1)=\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-(n+3)-\left\lfloor\frac{10\binom{n+2}{3}}{n+2}\right\rfloor+(n+2).$$

Then

$$\begin{aligned}b(n)-b(n-1)&=\left\lfloor\frac{10\binom{n+3}{3}}{n+3}\right\rfloor-2\left\lfloor\frac{10\binom{n+2}{3}}{n+2}\right\rfloor+\left\lfloor\frac{10\binom{n+1}{3}}{n+1}\right\rfloor\\ &=\left\lfloor\frac{5(n+2)(n+1)}{3}\right\rfloor-2\left\lfloor\frac{5(n+1)n}{3}\right\rfloor+\left\lfloor\frac{5n(n-1)}{3}\right\rfloor\end{aligned}$$

Let $n=3m+1$ for $m\in\mathbb{Z}$. Then

$$\begin{aligned}b(n)-b(n-1)&=5(m+1)(3m+2)-2\left\lfloor\frac{5(3m+2)(3m+1)}{3}\right\rfloor+5(3m+1)m=\\ &=5(m+1)(3m+2)-2(15m^2+15m+3)+5(3m+1)m=4.\end{aligned}$$

Let $n=3m+2$ for $m\in\mathbb{Z}$. Then

$$\begin{aligned}b(n)-b(n-1)&=5(3m+4)(m+1)-10(3m+2)(m+1)+\left\lfloor\frac{5(3m+2)(3m+1)}{3}\right\rfloor=\\ &=5(3m+4)(m+1)-10(3m+2)(m+1)+(15m^2+15m+3)=3.\end{aligned}$$

Let $n = 3m$ for $m \in \mathbb{Z}$. Then

$$\begin{aligned} b(n) - b(n-1) &= \left\lfloor \frac{5(3m+2)(3m+1)}{3} \right\rfloor - 10(3m+1)m + 5m(3m-1) = \\ &= (15m^2 + 15m + 3) - 10(3m+1)m + 5m(3m-1) = 3. \end{aligned}$$

□

Lemma B.18. *Let $n \geq 4$. In the notation of the proof of Lemma 3.16,*

$$v(n) - v(n-3) = \begin{cases} 5 & \text{for } n \equiv 1 \mod 3; \\ 8 & \text{otherwise.} \end{cases}$$

Proof. By definition,

$$\begin{aligned} v(n) - v(n-3) = 10(n+1) &- (n+3)(1 + b(n) - b(n-1)) - b(n-1) \\ &- 10(n-2) + n(1 + b(n-3) - b(n-4)) + b(n-4) = \\ &= 30 - 3(1 + b(n) - b(n-1)) - (b(n-1) - b(n-4)). \end{aligned}$$

By Lemma B.17, we deduce that, for any $n \in \mathbb{Z}$,

$$b(n) - b(n-3) = \sum_{i=n-2}^{n} (b(i) - b(i-1)) = 10.$$

Hence,

$$v(n) - v(n-3) = \begin{cases} 30 - 3 \cdot 5 - 10 = 5 & \text{for } n \equiv 1 \mod 3; \\ 30 - 3 \cdot 4 - 10 = 8 & \text{otherwise.} \end{cases}$$

□

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