DIMENSION OF TENSOR NETWORK VARIETIES

ALESSANDRA BERNARDI, CLAUDIA DE LAZZARI, FULVIO GESMUNDO

Abstract. The tensor network variety is a variety of tensors associated to a graph and a set of positive integer weights on its edges, called bond dimensions. We determine an upper bound on the dimension of the tensor network variety. A refined upper bound is given in cases relevant for applications such as varieties of matrix product states and projected entangled pairs states. We provide a range (the "supercritical range") of the parameters where the upper bound is sharp.

1. Introduction

Tensor network varieties are varieties of tensors described by the combinatorial structure of a graph. They play a major role in quantum many-body physics, where they are used as a variational ansatz class to describe strongly correlated quantum systems whose entanglement structure is given by the underlying graph.

The original motivation in quantum physics is the description of quantum spin chains [AKLT88, FNW92, ÖR95]. In this setting, it is known that ground states of a local gapped Hamiltonian on 1-dimensional spin chains are well approximated by matrix product states, which are tensor network states associated to a circular graph [PGVWC07, VMC08]. We refer to [Orú14, STG$^+$19] for a full description of the subject from the point of view of quantum physics. Methods from differential and complex geometry were introduced in the study of these objects in [HMOV14] and more recently some important developments were achieved using methods from algebraic geometry and representation theory [BBM15, MSV19, GLW18, CLVW20, CGFW20, HGS$^+$20].

Moreover, tensor networks have a role in other areas of applied mathematics. In algebraic complexity theory, the model of computation of algebraic branching program is a "symmetrized version" of a tensor network [BC92, DMPY12]. In algebraic statistics, probabilistic graphical models are described as a joint probability distribution of a set of random variables whose correlations factor through the structure of the graph [Lau96, RS19]; these models find application in phylogenetics [ERSS05, AR08]. In machine learning, a linear network is essentially a tensor network where the contraction maps are usually precomposed with a nonlinear *activation* function [Ben09].

In this work, we approach the problem of determining the dimension of tensor network varieties, that is the closure of the set of tensors allowing a tensor network representation for a given graph. This provides a measure of how large the set of tensors allowing a certain tensor network representation is, which in turn gives a measure of the expressiveness of the tensor network class. We provide a completely general upper bound in Theorem 1.1 and we

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illustrate how to refine it in cases relevant for applications in Corollary 1.2 and Corollary 1.3. In Corollary 1.4, we give the exact value of the dimension of the tensor network variety in a particular range, where it can be realized as the closure of the orbit of the action of an algebraic group. In Section 2, we give a complete description of the objects we are going to study. Section 3 and Section 4 are devoted to the proof of the main results. Finally, in Section 5, we further analyze some cases arising from small values of the parameters, and we provide a more precise calculation of their dimension.

1.1. **Main results.** Let $V_1, \ldots, V_d$ be complex vector spaces with $\dim V_i = n_i$ and let Γ be a simple graph with vertex set $\mathbf{v}(\Gamma)$ of cardinality d and edge set $\mathbf{e}(\Gamma)$. The tensor network varieties associated to Γ in $V_1 \otimes \cdots \otimes V_d$ are irreducible algebraic varieties in $V_1 \otimes \cdots \otimes V_d$ depending on a collection of integer weights $\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$ on the edges of Γ, called *bond dimensions*. Write $\mathbf{n} = (n_1, \ldots, n_d)$ for the local dimensions of the tensor product, and let $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ be the tensor network variety associated to Γ with bond dimensions $\mathbf{m}$ in $V_1 \otimes \cdots \otimes V_d$; see Definition 2.2.

It will be clear from the definitions that if $\mathbf{m}$ and $\mathbf{m}'$ are two collections of weights such that $m'_e \leq m_e$ for every edge $e \in \mathbf{e}(\Gamma)$ then $\mathcal{TNS}^{\Gamma}_{\mathbf{m}',\mathbf{n}} \subseteq \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$.

Our main result is the following

Theorem 1.1. *Let $(\Gamma, \mathbf{m}, \mathbf{n})$ be a tensor network and let $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ be the corresponding tensor network variety. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} \leq \min \left\{ \sum_{v \in \mathbf{v}(\Gamma)} (n_v \cdot \textstyle\prod_{e \ni v} m_e) - d + 1 - \displaystyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X), \prod_{v \in \mathbf{v}(\Gamma)} n_v \right\}.$$

In the statement of Theorem 1.1, $\mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$ is the stabilizer under the action of the *gauge subgroup* of a generic d-tuple of linear maps, whose role will be made clear in Section 4.

The term $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$ in Theorem 1.1 makes the statement not immediate to apply in full generality, as it describes the dimension of the tensor network variety in terms of the dimension of another object which is not immediate to compute. However, as it will be explained in Section 4.4, the value $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$ can be bounded from above by the dimension of a potentially larger stabilizer which can be computed from the *local structure* of the graph, rather than from its global combinatorics. In fact, a consequence of Proposition 4.10 will be that the term $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$ is trivial in a wide range of cases.

The term $\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1)$ is the dimension of the gauge subgroup associated to the tensor network, see Section 3.2. The role of this group in the theory of tensor network was known and it is expected that it entirely controls the value of $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$. In fact, it is expected that in "most" cases the exact value of the dimension is

$$\min \left\{ \sum_{v \in \mathbf{v}(\Gamma)} (n_v \cdot \textstyle\prod_{e \ni v} m_e) - d + 1 - \displaystyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1), \prod_{v \in \mathbf{v}(\Gamma)} n_v \right\}. \tag{1}$$

However, in Section 5, we will observe that there are at least some cases where the inequality is strict.

Particularly relevant in the study of quantum many-body systems are *lattice graphs* for which we provide some examples in Figure 1. In the physics literature, elements of the tensor network

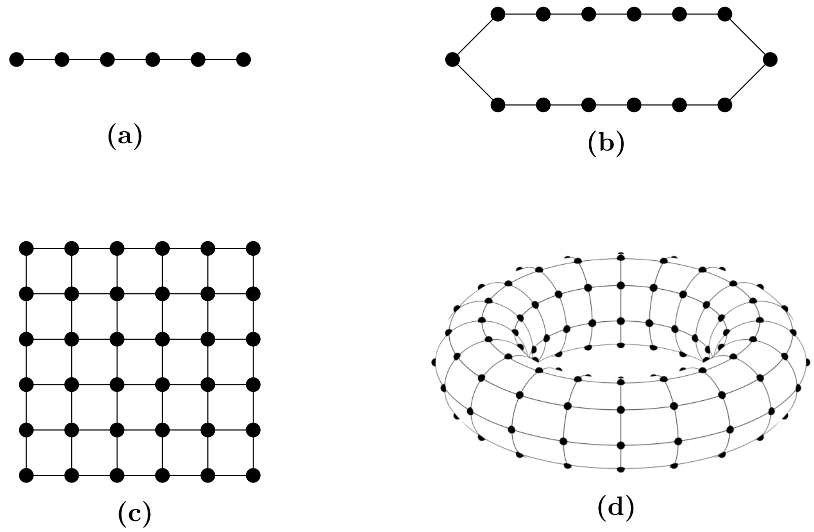


Figure 1. Examples of lattice graphs: MPS with open (a) and periodic (b) boundary conditions; PEPS with open (c) and periodic (d) boundary conditions

variety associated to a path or a cycle are called matrix product states (MPS), respectively with open or periodic boundary condition; elements of the tensor network variety associated to a grid (or more generally a two-dimensional lattice), either placed on a plane or on a torus, are called projected entangled pair states (PEPS), with open or periodic boundary conditions respectively.

In the case of matrix product states with open boundary conditions, a complete result regarding the dimension of the tensor network variety is given in [HMOV14, Thm 14]. In the language of Theorem 1.1, setting $\mathbf{m} = (m_1, \ldots, m_{d-1})$ and $\mathbf{n} = (n_1, \ldots, n_d)$ to be the collections of bond dimensions and of local dimensions on the path P_d, the result of [HMOV14] is (formally setting $m_0 = m_d = 1$)

$$\dim \mathcal{TNS}^{P_d}_{\mathbf{m},\mathbf{n}} = \min \left\{ \textstyle\sum_{i=1}^{d} n_i m_{i-1} m_i - \sum_{j=1}^{d-1} m_i^2, \prod_1^d n_i \right\};$$

this coincides with the expected value for the dimension in (1).

We refer to [PGVWC07, CLVW20] for the details on the construction of MPS, PEPS and other related entanglement structures and for their physical interpretation.

We state the following corollaries of Theorem 1.1 in the case of constant bond dimension and constant local dimension. It will be clear from the discussion of Section 4 that these hypotheses can be relaxed, but we state them in this restricted range for the sake of presentation.

Corollary 1.2. *Let (C_d, m, n) be the tensor network on the cycle graph on d vertices with constant bond dimension m and constant local dimension n. Then*

$$\dim \mathcal{TNS}^{C_d}_{m,n} \leq \min\{d(n-1)m^2 + 1, n^d\}.$$

Corollary 1.3. *Let Γ be a graph on d vertices such that all vertices of Γ have degree at least 3. Let (Γ, m, n) be the tensor network on Γ with constant bond dimension m and constant* *local dimension n. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{m,n} \leq \min \left\{ \sum_{v \in \mathbf{v}(\Gamma)} nm^{\deg(v)} - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m^2-1), n^d \right\}.$$

The equality in (1) is attained in the *supercritical range*, defined in Section 4.

Corollary 1.4. *Let $(\Gamma, \mathbf{m}, \mathbf{n})$ be a supercritical tensor network. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \min \left\{ \sum_{v \in \mathbf{v}(\Gamma)} (n_v \cdot \textstyle\prod_{e \ni v} m_e) - d + 1 - \displaystyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2-1), \prod_{v \in \mathbf{v}(\Gamma)} n_v \right\}.$$

1.2. **State of the art and related work.** Tensor network varieties appeared in [LQY12], where a number of basic geometric questions were answered, providing several insights. In [YL18], a comparison between tensor network varieties corresponding to different underlying graphs is proposed. The problem of dimension is also addressed: in particular, Theorem 7.4 and Theorem 9.1 in [YL18] correspond to the particular cases of Corollary 1.4 where the underlying graph is a path or a cycle respectively; in this case, the result follows also from Proposition 2.9 in [Ges16].

It would be interesting to have a general understanding of lower bounds for the dimension of tensor network varieties, but this is a challenging problem. In Section 2, we provide a parametrization of an open subset of the tensor network variety: hence determining lower bounds on the dimension can be reduced to determining lower bounds on the rank of the differential of the parametrization at a point; however, determining a suitable point, and computing such rank is non-trivial.

An indirect method to determine lower bounds on the dimension of tensor network varieties consists in determining subvarieties of known dimension contained in it. The result of Corollary 4.2 in [YL18] would give a lower bound of this form, whenever the dimension of the r-th secant variety of the Segre variety of rank one tensors is known. The result of Corollary 4.2 of [YL18] can be improved using lower bounds on the *border subrank* of the graph tensors introduced in Section 2 below. We only mention a result in this direction which follows from Theorem 6.6 in [Str87]. For $\mathbf{n} = (n_1, n_2, n_3)$, write $\sigma_{r,\mathbf{n}}$ for the r-th secant variety of the variety of rank one tensors in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$. Then, for $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1 \leq m_2 \leq m_3$, we have

$$\sigma_{r,\mathbf{n}} \subseteq \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}}$$

for every $r \leq m_1 m_2 - \left\lfloor \frac{(m_1+m_2-m_3)^2}{4} \right\rfloor$. In particular, if $m := m_1 = m_2 = m_3$, we get $\sigma_{r,\mathbf{n}} \subseteq \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}}$ for $r \leq \lceil 3/4 m^2 \rceil$; moreover, if $n = n_1 = n_2 = n_3$, [Lic85] provides $\dim \sigma_{r,\mathbf{n}} = \min\{r(3n-2), n^3\}$, with the only exception $(r,n) = (4,3)$ where $\dim \sigma_{4,3} = 26$; we deduce

$$\dim \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} \geq \min\{r(3n-2), n^3\}$$

for $r = \lceil 3/4 m^2 \rceil$. We do not expect this method to give a sharp bound except in trivial cases.

Indeed, we expect the upper bound of Theorem 1.1 to give the exact value of the dimension in "most" cases, in a way similar to the Alexander-Hirschowitz Theorem for secant varieties of Veronese varieties [AH95].

2. Definitions and preliminaries

We introduce tensor network varieties via the language of graph tensors, following [VC17, CVZ19]. In this work, we restrict to simple graphs; the theory generalizes to more general notions of graphs and we refer to [CLVW20, CGMZ20] for the definitions and the basics in the general setting.

Given tensors $T \in V_1 \otimes \cdots \otimes V_d$ and $S \in V_1' \otimes \cdots \otimes V_d'$, the *Kronecker product* of T and S, denoted $T \boxtimes S$, is the element $T \otimes S$ regarded as a tensor on d factors

$$T \boxtimes S \in (V_1 \otimes V_1') \otimes \cdots \otimes (V_d \otimes V_d').$$

Given a tensor $T \in V_1 \otimes \cdots \otimes V_d$, for every subset $I \subseteq \{1, \dots, d\}$, T defines a linear map $T_I : \bigotimes_{i \in I} V_i^* \to \bigotimes_{i' \notin I} V_{i'}$ called *flattening map* associated to I. We say that T is *concise* if all the flattening maps $T_i : V_i^* \to \bigotimes_{i' \neq i} V_{i'}$ are injective.

A *simple graph* is an undirected graph with no loops and no multiple edges. Let $\Gamma = (\mathbf{v}(\Gamma), \mathbf{e}(\Gamma))$ be a simple graph, with vertex set $\mathbf{v}(\Gamma) = \{1, \dots, d\}$ and edge set $\mathbf{e}(\Gamma) = \{e_1, \dots, e_R\}$. A collection of *bond dimensions* is a set of weights $\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$ on the edges of Γ. Given a collection of bond dimensions $\mathbf{m}$, define the graph tensor associated to Γ as follows. For an edge $e = \{i_1, i_2\}$, let

$$\mathbf{u}_{(e)}(m_e) = \sum_{j=1}^{m_e} v_j^{(i_1)} \otimes v_j^{(i_2)} \otimes \bigotimes_{i \neq i_1, i_2} v_0^{(i)} \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e} \otimes \mathbb{C}^1 \otimes \cdots \otimes \mathbb{C}^1$$

where for $p = 1, 2$, $\{v_j^{(i_p)} : j = 1, \dots, m_e\}$ are bases of a copy of $\mathbb{C}^{m_e}$ and $v_0^{(i)}$ is a generator of $\mathbb{C}^1$ for $i \neq i_1, i_2$. The superscripts indicate the ordering of the tensor factors.

The graph tensor associated to a graph Γ with bond dimensions $\mathbf{m}$ is

$$T(\Gamma, \mathbf{m}) = \boxtimes_{e \in \mathbf{e}(\Gamma)} \mathbf{u}_{(e)}(m_e); \tag{2}$$

this is a tensor of order d whose i-th factor has a local structure $W_i = \bigotimes_{e \ni i} \mathbb{C}^{m_e}$. In coordinates, we pictorially imagine the graph tensor $T(\Gamma, \mathbf{m})$ as the tensor product of identity matrices $\mathrm{Id}_{m_e} \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$ for $e \in \mathbf{e}(\Gamma)$ laying on the edges of the graph; this product is regarded as a tensor of order d where the i-th factor is the product of the spaces $\mathbb{C}^{m_e}$ incident to vertex i. Note that from this point of view one of the two copies of $\mathbb{C}^{m_e}$ is identified with its dual space $\mathbb{C}^{m_e *}$, see Figure 2.

Remark 2.1. Let Γ and Γ' be two graphs on the same set of vertices and with $\mathbf{e}(\Gamma) = \mathbf{e}(\Gamma') \cup \{e\}$. In other words, Γ' is the graph obtained from Γ after removing the edge e. Let $\mathbf{m}$ be a collection of bond dimensions on Γ and let $\mathbf{m}'$ be the collection $\mathbf{m}$ restricted to Γ'. It is clear from the definitions that if $m_e = 1$ then $T(\Gamma, \mathbf{m}) = T(\Gamma', \mathbf{m}')$ because in this case $\mathbf{u}_{(e)}(m_e)$ is a decomposable tensor hence $T \boxtimes \mathbf{u}_{(e)}(m_e) = T$ for every tensor T.

Remark 2.1 guarantees that up to modifying the underlying graph, one can always assume $m_e \geq 2$.

Let $n_i \in \mathbb{N}$ be integers associated to the vertices of Γ and let $V_i = \mathbb{C}^{n_i}$. Write $\mathbf{n} = (n_i)_{i=1,\dots,d}$ for the d-uple of dimensions of the vector spaces V_i; we say that $\mathbf{n}$ are the *local dimensions* associated to Γ. A triple $(\Gamma, \mathbf{m}, \mathbf{n})$ consisting of a simple graph, a collection of bond dimensions and a collection of local dimensions is a *tensor network*. A tensor network naturally provides the following algebraic variety.

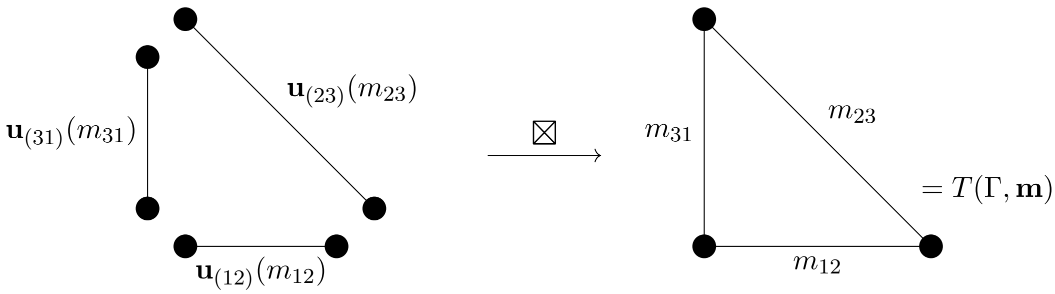


Figure 2. Pictorial representation of the construction of the graph tensor $T(\Gamma, \mathbf{m})$ on the triangular graph: $T(\Gamma, \mathbf{m})$ is the tensor product of the three identity matrices $\mathbf{u}_e(m_e)$, regarded as a tensor on three factors.

Definition 2.2 (Tensor Network Variety)**.** The *tensor network variety* in $V_1 \otimes \cdots \otimes V_d$ associated to the tensor network $(\Gamma, \mathbf{m}, \mathbf{n})$ is

$$\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \overline{\left\{ T \in V_1 \otimes \cdots \otimes V_d : T = (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathbf{m}), X_j \in \mathrm{Hom}(W_j, V_j) \right\}},$$

where the closure can be taken equivalently in the Euclidean or the Zariski topology.

The set $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ is an irreducible algebraic variety. Moreover, if $\mathbf{m}$ and $\mathbf{m}'$ are two collections of bond dimensions on Γ such that $m'_e \leq m_e$ for every edge $e \in \mathbf{e}(\Gamma)$, we have $\mathcal{TNS}^{\Gamma}_{\mathbf{m}',\mathbf{n}} \subseteq \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$.

It is known that if the graph Γ is a tree, then the closure in the definition of $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ is not needed, but if Γ contains cycles then there are examples where it is necessary [LQY12, CLVW20].

The definition of $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ provides a natural parametrization of a dense subset given by

$$\begin{aligned} \overline{\Phi} : \mathrm{Hom}(W_1, V_1) \oplus \cdots \oplus \mathrm{Hom}(W_d, V_d) &\to V_1 \otimes \cdots \otimes V_d, \\ (X_1, \ldots, X_d) &\mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathbf{m}). \end{aligned}$$

Let $\mathcal{TNS}^{\Gamma\circ}_{\mathbf{m},\mathbf{n}}$ be the image of the map $\overline{\Phi}$. The set $\mathcal{TNS}^{\Gamma\circ}_{\mathbf{m},\mathbf{n}}$ is often the object of interest in applications, as it coincides exactly with the set of tensors which have a tensor network representation with the given parameters. Since we are interested in the dimension of these objects, as we employ methods from algebraic geometry, we consider the algebraic variety obtained taking the closure. The map $\overline{\Phi}$ factors as follows:

$$\begin{tikzcd} \bigoplus_{i=1}^d \mathrm{Hom}(W_i, V_i) \arrow[r, "\mu"] \arrow[dr, "\overline{\Phi}"'] & \mathrm{Hom}(W_1 \otimes \cdots \otimes W_d, V_1 \otimes \cdots \otimes V_d) \arrow[d, "\Phi"] \\ & V_1 \otimes \cdots \otimes V_d \end{tikzcd}$$

where μ is the d-linear map defined as $\mu(X_1, \ldots, X_d) = X_1 \otimes \cdots \otimes X_d$. Denote the image of the map μ by $\mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$. It is the cone over the Segre embedding of $\mathbb{P}(\mathrm{Hom}(W_1, V_1)) \times \cdots \times \mathbb{P}(\mathrm{Hom}(W_d, V_d))$ in $\bigotimes_1^d \mathrm{Hom}(W_i, V_i) = \mathrm{Hom}(W_1 \otimes \cdots \otimes W_d, V_1 \otimes \cdots \otimes V_d)$ and its affine dimension is

$$\dim \mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) = \sum_{1=1}^{d} \dim(\mathrm{Hom}(W_i, V_i)) - d + 1.$$

The map Φ is simply the evaluation at the graph tensor; therefore the restriction of Φ to the subvariety $\mathrm{Hom}(W_1, \ldots, W_d, V_1, \ldots, V_d)$ provides a parametrization of $\mathcal{TNS}^{\Gamma\circ}_{\mathbf{m},\mathbf{n}}$.

The *dimension* of an irreducible algebraic variety is defined as the dimension of its tangent space at a smooth point. We refer to [Sha94, Ch. 3] for the basic properties of dimension. The Theorem of Dimension of the Fibers [Sha94, Thm. 1.25] provides

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \dim \left[\mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)\right] - \dim \Phi^{-1}(T) \tag{3}$$

where T is a generic tensor in the image of Φ.

The goal of the rest of the paper is to determine the value $\dim \Phi^{-1}(T)$ which, via (3), gives the value of $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$. Determining the exact value $\dim \Phi^{-1}(T)$ is hard in general. We focus on lower bounds, which via (3) provide upper bounds for $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$. This is done by determining the dimension of stabilizer of the graph tensor under the action of $GL(W_1) \times \cdots \times GL(W_d)$ and then showing that $\Phi^{-1}(T)$ contains orbits under the action of such stabilizer; lower bounds on the dimension of such orbit gives a lower bound on $\dim \Phi^{-1}(T)$.

3. Isotropy of tensors: the gauge subgroup

In this section, we determine the dimension of the isotropy group of graph tensors. First, we provide some preliminary results on the stabilizer of a tensor under the action of the general linear groups acting on the tensor factors; we then introduce the gauge subgroup of a tensor network and we prove that it coincides with the connected component of the identity of the isotropy group of the corresponding graph tensor.

3.1. **Isotropy groups of tensors.** Given vector spaces $V_1, \ldots, V_d$, consider the natural action of the group $GL(V_1) \times \cdots \times GL(V_d)$ on $V_1 \otimes \cdots \otimes V_d$. This defines a group homomorphism

$$\begin{aligned} GL(V_1) \times \cdots \times GL(V_d) &\to GL(V_1 \otimes \cdots \otimes V_d) \\ (g_1, \ldots, g_d) &\mapsto g_1 \otimes \cdots \otimes g_d \end{aligned}$$

whose kernel is the central subgroup $Z_{V_1 \otimes \cdots \otimes V_d} = \{(\lambda_1 \mathrm{Id}_{V_1}, \ldots, \lambda_d \mathrm{Id}_{V_d}) : \lambda_1 \cdots \lambda_d = 1\}$. Therefore, the group $G(V_1, \ldots, V_d) := GL(V_1) \times \cdots \times GL(V_d)/Z_{V_1 \otimes \cdots \otimes V_d}$ can be identified naturally with a subgroup of $GL(V_1 \otimes \cdots \otimes V_d)$ acting faithfully on $V_1 \otimes \cdots \otimes V_d$. The elements of $G(V_1, \ldots, V_d)$ will be denoted as tensor products $g_1 \otimes \cdots \otimes g_d$ for $g_j \in GL(V_j)$.

The corresponding Lie algebra action defines a Lie algebra homomorphism

$$\begin{aligned} \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_d) &\to \mathfrak{gl}(V_1 \otimes \cdots \otimes V_d) \\ (X_1, \ldots, X_d) &\mapsto X_1 \otimes \mathrm{Id}_{V_2} \otimes \cdots \otimes \mathrm{Id}_{V_d} + \cdots + \mathrm{Id}_{V_1} \otimes \cdots \otimes \mathrm{Id}_{V_{d-1}} \otimes X_d, \end{aligned}$$

whose kernel is the central algebra $\mathfrak{z}_{V_1 \otimes \cdots \otimes V_d} = \{(x_1 \mathrm{Id}_{V_1}, \ldots, x_d \mathrm{Id}_{V_d}) : x_1 + \cdots + x_d = 0\}$. Hence, the Lie algebra $\mathfrak{g}(V_1, \ldots, V_d) := \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_d)/\mathfrak{z}_{V_1 \otimes \cdots \otimes V_d}$ is a subalgebra of $\mathfrak{gl}(V_1 \otimes \cdots \otimes V_d)$ and coincides with the Lie algebra of $G(V_1, \ldots, V_d)$. With abuse of notation, denote the elements of $\mathfrak{g}(V_1, \dots, V_d)$ as d-tuples $\mathbf{X} = (X_1, \dots, X_d)$ with $X_j \in \mathfrak{gl}(V_j)$ with the understanding that $\mathbf{X}$ is identified with its image in $\mathfrak{g}(V_1, \dots, V_d)$.

Definition 3.1. Let $T \in V_1 \otimes \cdots \otimes V_d$ be a tensor. The isotropy group of T, denoted G_T, is the stabilizer of T under the action of $G(V_1, \dots, V_d)$:

$$G_T = \{g_1 \otimes \cdots \otimes g_d \in G(V_1, \dots, V_d) : g_1 \otimes \cdots \otimes g_d(T) = T\}.$$

The group G_T is algebraic and in general it is union of finitely many connected (irreducible) components. Let G_T° denote the connected component containing the identity: G_T° is normal in G_T (see, e.g., [Ges16, Lemma 2.1]) and $\dim G_T = \dim G_T^\circ$.

The isotropy Lie algebra of T, denoted $\mathfrak{g}_T$, is the Lie algebra of the group G_T, or equivalently the one of G_T°; it is the subalgebra of $\mathfrak{g}(V_1, \dots, V_d)$ which annihilates T under the Lie algebra action induced by $\mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_d)$ [Pro07, Sec. 1.2]

$$\mathfrak{g}_T = \{\mathbf{X} = (X_1, \dots, X_d) \in \mathfrak{g}(V_1, \dots, V_d) : \mathbf{X}.T = 0\},$$

where $\mathbf{X}.T = \sum_1^d \mathrm{Id}_{V_1} \otimes \cdots \otimes X_k \otimes \cdots \otimes \mathrm{Id}_{V_d}(T)$ denotes the image via the Lie algebra action. We have $\dim \mathfrak{g}_T = \dim G_T^\circ = \dim G_T$.

The dimension of the orbit-closure of a tensor T is therefore given by

$$\dim(G(V_1, \dots, V_d) \cdot T) = \left[\textstyle\sum_1^d (\dim V_i)^2 - d + 1\right] - \dim \mathfrak{g}_T.$$

We prove preliminary results on isotropy Lie algebras of tensors of higher order. Lemma 3.3 is classical and we record it here for the reader's convenience. Lemma 3.4 concerns the intersection of $\mathfrak{g}_T$ with the subalgebra of $\mathfrak{g}(V_1, \dots, V_d)$ consisting of elements acting only on a subset of the tensor factors; this will be used in a reduction in the proof of Theorem 3.6.

We first record an immediate linear algebra fact.

Lemma 3.2. *Let V be a vector space and let $A, B_1, \dots, B_N$ be subspaces of V such that there exists a subspace B such that $A \cap B = \{0\}$ and $B_j \subseteq B$ for every $j = 1, \dots, N$. Then $\bigcap_j (A \oplus B_j) = A \oplus \bigcap_j B_j$.*

The following result is classical and follows for instance from [Bri03, Section 1.1].

Lemma 3.3. *Let $T \in V_1 \otimes \cdots \otimes V_d$ be a non-concise tensor. Let $V_i' \subseteq V_i$ be subspaces such that $T \in V_1' \otimes \cdots \otimes V_d'$ is concise. Write $\mathfrak{h}_T$ for the isotropy Lie algebra of T in $\mathfrak{g}(V_1', \dots, V_d')$ (regarded as a subalgebra of $\mathfrak{g}(V_1, \dots, V_d)$) and $\mathfrak{g}_T$ for the isotropy Lie algebra of T in $\mathfrak{g}(V_1, \dots, V_d)$. Then*

$$\mathfrak{g}_T = \mathfrak{h}_T \oplus \mathfrak{p}$$

where $\mathfrak{p} \subseteq \mathfrak{g}(V_1, \dots, V_d)$ is the Lie algebra which annihilates the subspace $V_1' \otimes \cdots \otimes V_d'$, that is the algebra generated by $\bigoplus_1^d ({V_i'}^\perp \otimes V_i) \subseteq \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_d)$.

Lemma 3.4. *Let $T \in V_1 \otimes \cdots \otimes V_d$. For $I \subseteq \{1, \dots, d\}$, let $F_T := T_{I^c} : \bigotimes_{j \in I^c} V_j^* \to \bigotimes_{i \in I} V_i$ be the flattening map of T corresponding to the subset I. Then*

$$\mathfrak{g}_T \cap \mathfrak{g}(V_i : i \in I) = \bigcap_{S \in \operatorname{Im} F_T} \mathfrak{g}_S. \tag{4}$$

In particular, if T is concise, $\mathfrak{g}_T \cap \mathfrak{gl}(V_j) = 0$ for every j.

Proof. Let $k = |I|$; up to reordering the factors, assume $I = \{1, \dots, k\}$.

Given $\mathbf{X} \in \mathfrak{g}(V_1, \dots, V_d)$, write $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ with $\mathbf{X}_1 = (X_1, \dots, X_k)$ and $\mathbf{X}_2 = (X_{k+1}, \dots, X_d)$. Let $\mathbf{X}.T$ be the image of T via the action of $\mathbf{X}$ and let $F_{\mathbf{X}.T}$ be the corresponding flattening map. By Leibniz's rule, given an element $S' \in V_{k+1}^* \otimes \cdots \otimes V_d^*$, $F_{\mathbf{X}.T}$ is characterized by the expression

$$F_{\mathbf{X}.T}(S') = F_T(\mathbf{X}_2.S') + \mathbf{X}_1.F_T(S'),$$

where $\mathbf{X}_2$ acts on $V_{k+1}^* \otimes \cdots \otimes V_d^*$, $\mathbf{X}_1$ acts on $V_1 \otimes \cdots \otimes V_k$.

Now, let $\mathbf{X} \in \mathfrak{g}_T \cap (\mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_k))$. Hence, $\mathbf{X} = (\mathbf{X}_1, \mathbf{0})$ and $\mathbf{X}.T = 0$. Therefore $0 = F_{\mathbf{X}.T}(S') = \mathbf{X}_1.F_T(S')$, showing $\mathbf{X}_1 \in \mathfrak{g}_S$ for every $S \in \mathrm{Im}\, F_T$.

Conversely, let $\mathbf{X}_1 \in \bigcap_{S \in \mathrm{Im}\, F_T} \mathfrak{g}_S$. Let $S_1, \dots, S_N \in \mathrm{Im}\, F_T$ be a set of generators and write $T = \sum_{i=1}^N S_i \otimes P_i$ for some $P_i \in V_{k+1} \otimes \cdots \otimes V_d$. Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{0})$. Then

$$\mathbf{X}.T = \sum_{i=1}^N (\mathbf{X}.S_i) \otimes P_i + \sum_{i=1}^N S_i \otimes \mathbf{X}.P_i = \sum_{i=1}^N (\mathbf{X}_1.S_i) \otimes P_i = 0$$

showing $\mathbf{X} \in \mathfrak{g}_T$.

This concludes the proof of (4).

The last claim follows by taking $I = \{j\}$: if T is concise, then F_T is surjective and therefore $\bigcap_{S \in \mathrm{Im}\, F_T} \mathfrak{g}_S = \bigcap_{v \in V_j} \mathfrak{g}_v = 0$. ☐

By linearity the intersection in Lemma 3.4 can be restricted to a basis of the image of the flattening map $\mathrm{Im}\, F_T$, as it is clear from the proof.

3.2. **Gauge subgroup.** Let Γ be a graph and $\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$ a collection of bond dimensions. Let $T = T(\Gamma, \mathbf{m}) \in W_1 \otimes \cdots \otimes W_d$ be the associated graph tensor. Fix an edge $e = \{i_1, i_2\} \in \mathbf{e}(\Gamma)$: by definition of $T(\Gamma, \mathbf{m})$ there exist vector spaces U_e, W'_{i_1}, W'_{i_2} such that $W_{i_1} = U_e \otimes W'_{i_1}$, and $W_{i_2} = U_e^* \otimes W'_{i_2}$ where $\dim U_e = m_e$ and the tensor product structure depends on the local structure at the vertices i_1 and i_2. The group $GL(U_e) \times GL(U_e^*)$ acts on the factor $U_e \otimes U_e^*$ of $W_{i_1} \otimes W_{i_2}$ with kernel the central subgroup $Z_e = \{(\lambda \mathrm{Id}_{U_e}, \lambda^{-1} \mathrm{Id}_{U_e^*}) : \lambda \in \mathbb{C}^*\}$.

This defines a homomorphism

$$\Psi_e : (GL(U_e) \times GL(U_e^*))/Z_e \to GL(W_{k_1} \otimes W_{k_2}) \to G(W_k : k \in \mathbf{v}(\Gamma)).$$

As e varies among the edges of Γ, the images of the different Ψ_e's commute and therefore they induce a homomorphism

$$\Psi : \bigtimes_{e \in \mathbf{e}(\Gamma)} (GL(U_e) \times GL(U_e^*))/Z_e \to G(W_k : k \in \mathbf{v}(\Gamma)),$$

which turns out to be injective. Regrouping the factors, we can write

$$\mathrm{Im}\, (\Psi) = \left[\bigtimes_{v \in \mathbf{v}(\Gamma)} H_v \right] / \left[\bigtimes_{e \in \mathbf{e}(\Gamma)} Z_e \right]$$

where $H_v = \bigtimes_{v \ni e} GL_{m_e}$; here GL_{m_e} is $GL(U_e)$ or $GL(U_e^*)$ depending on whether U_e or U_e^* is the tensor factor appearing in W_v. With abuse of notation, we will denote by H_v the quotient

$\left\langle H_v, \left[\bigtimes_{e\in\mathbf{e}(\Gamma)} Z_e\right]\right\rangle / \left[\bigtimes_{e\in\mathbf{e}(\Gamma)} Z_e\right] \subseteq \mathrm{Im}\ (\Psi)$ as well, where for subgroups H, K, one denotes by $\langle H, K\rangle$ the subgroup generated by H and K.

Let $GL^{\Delta}_{m_e} \subseteq GL(U_e)\times GL(U_e^*)$ be the subgroup lying "diagonally", that is $GL^{\Delta}_{m_e} = \{(A, A^{-1^T}) \in GL(U_e) \times GL(U_e^*) : A \in GL(U_e)\}$; its image under the homomorphism Ψ_e is a copy of $PGL_{m_e} \subseteq G(W_1,\dots,W_d)$ called *gauge subgroup* on the edge e. The following is immediate from the definitions:

Lemma 3.5. *The gauge subgroup $PGL_{m_e} \subseteq G(W_1,\dots,W_d)$ stabilizes $T(\Gamma,\mathbf{m})$.*

Proof. Let $e = \{i_1, i_2\}$, so that PGL_{m_e} only acts on the copy of $U_e \otimes U_e^* \subseteq W_{i_1} \otimes W_{i_2}$. In fact, because of the structure of $T(\Gamma,\mathbf{m})$, PGL_{m_e} only acts on the Kronecker factor $\mathbf{u}_e = \mathrm{Id}^{(i_1,i_2)}_{m_e} \otimes \left(\bigotimes_{j\neq i_1,i_2} v_0^{(j)}\right) \in U_e \otimes U_e^* \otimes \bigotimes_{j\neq i_1,i_2}\mathbb{C}^1$.

For $A \in PGL_{m_e}$, we have $A\cdot \mathbf{u}_e = (A^{-1}\mathrm{Id}^{(i_1,i_2)}_{m_e} A) \otimes \left(\bigotimes_{j\neq i_1,i_2} v_0^{(j)}\right) = \mathbf{u}_e$. Therefore PGL_{m_e} stabilizes $\mathbf{u}_e$. □

The image of the homomorphism Ψ restricted to $\bigtimes_e GL^{\Delta}_{m_e}$ is a subgroup

$$\mathcal{G}_{\Gamma,\mathbf{m}} \simeq \bigtimes_{e\in\mathbf{e}(\Gamma)} PGL_{m_e} \subseteq G(W_k : k \in \mathbf{v}(\Gamma))$$

called *gauge subgroup* of Γ with bond dimensions $\mathbf{m}$. A consequence of Lemma 3.5 is that the graph tensor $T(\Gamma,\mathbf{m})$ is stabilized by $\mathcal{G}_{\Gamma,\mathbf{m}}$.

Denote by $\mathbf{g}_{\Gamma,\mathbf{m}}$ the Lie algebra of the gauge subgroup $\mathcal{G}_{\Gamma,\mathbf{m}}$ of Γ.

3.3. **Isotropy group of graph tensors.** The main result of this section is that the identity component of the isotropy group $G_{T(\Gamma,\mathbf{m})}$ of a graph tensor coincides with the gauge subgroup. This generalizes the known results for the iterated matrix multiplication tensor [dG78, Ges16], that is the graph tensor associated to the cycle graph. We prove a more general form of this fact in Theorem 3.6; the result on graph tensors will follow via an inductive argument in Corollary 3.7.

Theorem 3.6. *Let $T' \in \mathbb{C}^1 \otimes \bigotimes_{j=1}^d W_j'$ be a concise tensor of order $d+1$. Let $\Sigma = (\mathbf{v}(\Sigma), \mathbf{e}(\Sigma))$ be a the graph on $d+1$ vertices $\mathbf{v}(\Sigma) = \{0,\dots,d\}$ with edge set $\mathbf{e}(\Sigma) = \{e_1,\dots,e_k\}$, where $e_j = \{0, j\}$. Let $\mathbf{m} = (m_j : j = 1,\dots,k)$ be a set of bond dimensions on Σ. Let*

$$S := T(\Sigma,\mathbf{m}) \in \mathbb{C}^{m_1\cdots m_k} \otimes \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k} \otimes \mathbb{C}^1 \otimes \cdots \otimes \mathbb{C}^1$$

be the associated graph tensor. Let $T = S \boxtimes T' \in V_0 \otimes \cdots \otimes V_d$. Then

$$\mathfrak{g}_T = \mathfrak{h}_{T'} + \mathbf{g}_{\Sigma,\mathbf{m}} \subseteq \mathfrak{g}(V_0,\dots,V_d)$$

where $\mathbf{g}_{\Sigma,\mathbf{m}}$ is the Lie algebra of the gauge subgroup $\mathcal{G}_{\Sigma,\mathbf{m}}$ of Σ and $\mathfrak{h}_{T'}$ is the isotropy Lie algebra of T' in $\mathfrak{g}(\mathbb{C}^1, W_1',\dots,W_d')$.

Proof. The inclusion $\mathfrak{h}_{T'} + \mathbf{g}_{\Sigma,\mathbf{m}} \subseteq \mathfrak{g}_T$ is immediate.

For $j = 1,\dots,k$, write $V_j = U_j \otimes W_j'$ where $U_j = \mathbb{C}^{m_j}$. Write $V_0 = \mathbb{C}^1 \otimes U_1^* \otimes \cdots \otimes U_k^*$. For $j = 1,\dots,k$, let $\{u^j_{i_j} : i_j = 1,\dots,m_j\}$ be a basis of the U_j; let $\{u^0_{i_1,\dots,i_k} : i_j = 1,\dots,m_j\}$ be

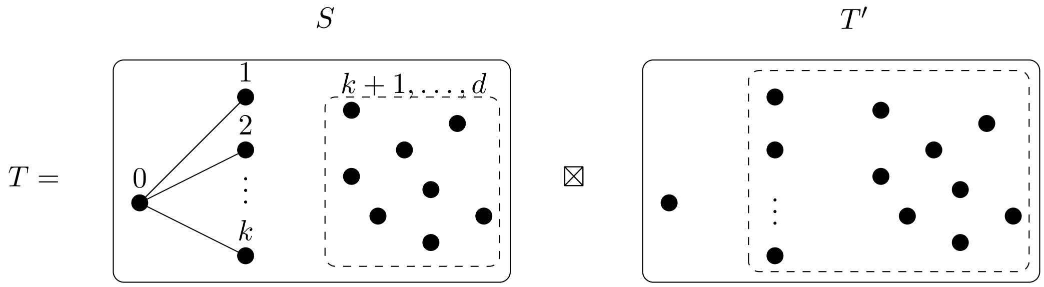


Figure 3. The tensor T in Theorem 3.6: the Kronecker product of a star tensor S centered at vertex 0 and a tensor T' whose 0-th factor is 1-dimensional

the basis of $V_0 \simeq U_1^* \otimes \cdots \otimes U_k^*$ dual to the induced basis $\{u^1_{i_1} \otimes \cdots \otimes u^k_{i_k} : i_j = 1, \dots, m_j\}$ of $U_1 \otimes \cdots \otimes U_k$. Therefore

$$S = \sum_{i_1,\dots,i_k} u^{(0)}_{i_1,\dots,i_k} \otimes u^{(1)}_{i_1} \otimes \cdots \otimes u^{(k)}_{i_k} \ \otimes u_0^{k+1} \otimes \cdots \otimes u_0^d$$

where for $j = k+1, \dots, d$, u_0^j is a generator of the corresponding $\mathbb{C}^1$ factor.

Let $\mathbf{X} = (X_0, \dots, X_d) \in \mathfrak{g}(V_0, \dots, V_d)$. Suppose $\mathbf{X} \in \mathfrak{g}_T$, that is $\mathbf{X}.T = 0$. By Leibniz's rule $\mathbf{X}.T = \sum_{j=0}^d X_j.T = 0$.

Write $X_0 = ((x^0)^{i'_1,\dots,i'_k}_{i_1,\dots,i_k})$ in the chosen basis: we have

$$X_0.T = (X_0.S) \boxtimes T' = \left[\sum_{\substack{i_1,\dots,i_k \\ i'_1,\dots,i'_k}} (x^0)^{i'_1,\dots,i'_k}_{i_1,\dots,i_k} u^0_{i'_1,\dots,i'_k} \otimes u^1_{i_1} \otimes \cdots \otimes u^k_{i_k} \right] \boxtimes T'.$$

For $j = 1, \dots, k$, write $X_j \in \mathfrak{gl}(V_j)$ as $X_j = \sum \Delta_j^{(\rho)} \boxtimes \Theta_j^{(\rho)}$ where $\Delta_j^{(\rho)} = ((\delta^{\rho,j})^{i_j}_{i'_j}) \in \mathfrak{gl}(U_j)$ and $\Theta_j^{(\rho)} \in \mathfrak{gl}(W'_j)$; then

$$X_j.T = \sum_\rho \left[\Delta_j^{(\rho)}.S\right] \boxtimes \left[\Theta_j^{(\rho)}.T'\right] =$$

$$= \sum_\rho \left[\sum_{i_1,\dots,i_k,i'_j} u^0_{i_1,\dots,i_k} \otimes u^1_{i_1} \otimes \cdots \otimes (\delta^{\rho,j})^{i_j}_{i'_j} u^j_{i_j} \otimes \cdots \otimes u^k_{i_k} \right] \boxtimes \left[\Theta_j^{(\rho)}.T'\right].$$

If $j > k$, then $V_j = \mathbb{C}^1 \otimes W'_j$ and we have $X_j.T = S \otimes X_j.T'$.

For indices $i_1^*, \dots, i_k^*, \widetilde{\imath}_1, \dots, \widetilde{\imath}_k$, write $\mathbf{X}.T = u^0_{i_1^*,\dots,i_k^*} \otimes u^1_{\widetilde{\imath}_1} \otimes \cdots \otimes u^k_{\widetilde{\imath}_k} \boxtimes T^{i_1^*,\dots,i_k^*}_{\widetilde{\imath}_1,\dots,\widetilde{\imath}_k}$ for tensors $T^{i_1^*,\dots,i_k^*}_{\widetilde{\imath}_1,\dots,\widetilde{\imath}_k} \in W'_1 \otimes \cdots \otimes W'_d$. Since $u^0_{i_1^*,\dots,i_k^*} \otimes u^1_{\widetilde{\imath}_1} \otimes \cdots \otimes u^k_{\widetilde{\imath}_k}$ are linearly independent, the condition $\mathbf{X}.T = 0$ is equivalent to $T^{i_1^*,\dots,i_k^*}_{\widetilde{\imath}_1,\dots,\widetilde{\imath}_k} = 0$ for every $i_1^*, \dots, i_k^*, \widetilde{\imath}_1, \dots, \widetilde{\imath}_k$.

Note that if $(i_1^*, \dots, i_k^*)$ and $(\widetilde{\imath}_1, \dots, \widetilde{\imath}_k)$ differ in at least two entries, then $T^{i_1^*,\dots,i_k^*}_{\widetilde{\imath}_1,\dots,\widetilde{\imath}_k}$ only depends on X_0: indeed, the summands $X_j.T$ for $j \neq 0$ only give rise to terms where $(i_1^*, \dots, i_k^*)$ and $(\widetilde{\imath}_1, \dots, \widetilde{\imath}_k)$ differ in at most one entry. Write $X_0 = X'_0 + X''_0$ where X'_0 is the component where $(i_1^*, \dots, i_k^*)$ and $(\widetilde{\imath}_1, \dots, \widetilde{\imath}_k)$ differ in at least two entries and X''_0 is the complementary component. In particular, X_0' is the only component of $\mathbf{X}$ which contributes to $T^{i_1^*,\dots,i_k^*}_{\widetilde{i}_1,\dots,\widetilde{i}_k}$ when the two sets of indices differ in at least two entries. By linearity, the discussion above shows $X_0'.T = 0$. Since T is concise, Lemma 3.4 implies that $X_0' = 0$. This shows

$$X_0 = Y_1 \otimes \mathrm{id}_{U_2^*\otimes\cdots\otimes U_k^*} + \cdots + \mathrm{id}_{U_1^*\otimes\cdots\otimes U_{k-1}^*} \otimes Y_k$$

with $Y_j \in \mathfrak{gl}(U_j^*)$. Hence, we may renormalize $\mathbf{X}$ using $\mathbf{g}_{\Gamma,\mathbf{m}}$ and obtain $X_0 = 0$. In particular, we reduced the analysis to $\mathbf{X} \in \mathfrak{g}(V_j : j \neq 0)$.

Consider $\mathbf{X} \in \mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0)$. By Lemma 3.4, we have

$$\mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0) = \bigcap_{R \in \mathrm{Im}\ (\mathrm{Flat}(T))} \mathfrak{g}_R, \tag{5}$$

where $\mathrm{Flat}(T) : V_0^* \to V_1 \otimes \cdots \otimes V_d$ is the 0-th flattening map. For indices $(i_1,\dots,i_k)$, write $T'(i_1,\dots,i_k) = \mathrm{Flat}(T)(u^{(0)}_{i_1,\dots,i_k}) = u^{(1)}_{i_1} \otimes\cdots\otimes u^{(k)}_{i_k} \boxtimes T'$. The intersection in (5) can be reduced to a set of generators of $\mathrm{Im}\ (\mathrm{Flat}(T))$; therefore we obtain

$$\mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0) = \bigcap_{i_1,\dots,i_k} \mathfrak{g}_{T'(i_1,\dots,i_k)}.$$

Since $T'(i_1,\dots,i_k)$ is not concise in $V_1 \otimes \cdots \otimes V_d$, we have $\mathfrak{g}_{T'(i_1,\dots,i_k)} = \mathfrak{h}_{T'(i_1,\dots,i_k)} \oplus \mathfrak{p}_{i_1,\dots,i_k}$, where $\mathfrak{h}_{T'(i_1,\dots,i_k)}$ is the annihilator of $T'(i_1,\dots,i_k)$ in $\mathfrak{gl}(\langle u^1_{i_1}\rangle \otimes W_1') \oplus \cdots \oplus \mathfrak{gl}(\langle u^k_{i_k}\rangle \otimes W_k') \oplus \mathfrak{gl}(V_{k+1}) \oplus \cdots \oplus \mathfrak{gl}(V_d)$ and $\mathfrak{p}_{i_1,\dots,i_k}$ is the parabolic subspace which annihilates $(u^1_{i_1} \otimes W_1') \otimes \cdots \otimes (u^k_{i_k} \otimes W_k') \otimes V_{k+1} \otimes \cdots \otimes V_d$, that is

$$\mathfrak{p}_{i_1,\dots,i_k} = \left[(\langle u^1_{i_1}\rangle^\perp \otimes W_1'^*) \otimes (U_1 \otimes W_1')\right] \oplus \cdots \oplus \left[(\langle u^k_{i_k}\rangle^\perp \otimes W_k'^*) \otimes (U_k \otimes W_k')\right].$$

Since $T'(i_1,\dots,i_k) = u^1_{i_1} \otimes \cdots \otimes u^k_{i_k} \otimes T'$, we have

$$\mathfrak{h}_{T'(i_1,\dots,i_k)} = \mathrm{Id}_{\langle u^1_{i_1}\rangle \otimes\cdots\otimes \langle u^k_{i_k}\rangle} \otimes \mathfrak{g}_{T'},$$

regarded as a subalgebra acting on the subspace $u^1_{i_1} \otimes\cdots\otimes u^k_{i_k} \otimes W_1' \otimes\cdots\otimes W_k' \otimes V_{k+1} \otimes\cdots\otimes V_d$.

Observe that, as a subspace of $\mathrm{End}(V_1 \otimes \cdots \otimes V_d)$, we have

$$\mathfrak{g}_{T'(i_1,\dots,i_k)} = \left[\mathrm{Id}_{\langle u^1_{i_1}\rangle \otimes\cdots\otimes \langle u^k_{i_k}\rangle} \otimes \mathfrak{g}_{T'}\right] \oplus \mathfrak{p}_{i_1,\dots,i_k} = \left[\mathrm{Id}_{U_1 \otimes\cdots\otimes U_k} \otimes \mathfrak{g}_{T'}\right] \oplus \mathfrak{p}_{i_1,\dots,i_k}.$$

This follows directly from Leibniz rule and the fact that, for every $i_1,\dots,i_k$, $\mathrm{Id}_{U_1\otimes\cdots\otimes U_k} = \mathrm{Id}_{\langle u^1_{i_1} \otimes\cdots\otimes u^k_{i_k}\rangle} + P_{i_1,\dots,i_k}$ where $P_{i_1,\dots,i_k} \in \mathfrak{p}_{i_1,\dots,i_k}$. We deduce

$$\mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0) = \bigcap_{i_1,\dots,i_k} \left[(\mathrm{Id}_{U_1\otimes\cdots\otimes U_k} \otimes \mathfrak{g}_{T'}) \oplus \mathfrak{p}_{i_1,\dots,i_k}\right]$$

and by Lemma 3.2, we have $\mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0) = (\mathrm{Id}_{U_1\otimes\cdots\otimes U_k} \otimes \mathfrak{g}_{T'}) \oplus \bigcap_{i_1,\dots,i_k} \mathfrak{p}_{i_1,\dots,i_k} = \mathrm{Id}_{U_1\otimes\cdots\otimes U_k} \otimes \mathfrak{g}_{T'}$ because $\bigcap_{i_1,\dots,i_k} \mathfrak{p}_{i_1,\dots,i_k} = 0$.

This concludes the proof, as we showed

$$\mathfrak{g}_T = \mathfrak{g}_T + \mathbf{g}_{\Sigma,\mathbf{m}} = \mathfrak{g}_T \cap \mathfrak{g}(V_j : j \neq 0) + \mathbf{g}_{\Sigma,\mathbf{m}} = \mathfrak{g}_{T'} + \mathbf{g}_{\Sigma,\mathbf{m}}. \qquad \square$$

Applying Theorem 3.6 to graph tensors, we deduce the following result:

Corollary 3.7. *Let $\Gamma = (\mathbf{v}(\Gamma), \mathbf{e}(\Gamma))$ be a graph with d vertices and let $\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$ be a set of bond dimensions on Γ. Let $T := T(\Gamma, \mathbf{m}) \in \bigotimes_{j=1}^d V_j$ be the associated graph tensor. Then the isotropy Lie algebra of T coincides with Lie algebra of the gauge subgroup of Γ; in symbols*

$$\mathfrak{g}_T = \mathbf{g}_{\Gamma,\mathbf{m}}.$$

Proof. We proceed by induction on the number of vertices d. If $d = 1$, the statement is clear as T is a single vector, with trivial isotropy Lie algebra.

Suppose Γ is a graph with $d+1$ vertices and write $\mathbf{v}(\Gamma) = \{0, \ldots, d\}$. Let Σ be the subgraph of Γ given by the edges incident to the vertex 0. In other words $\mathbf{v}(\Sigma) = \{0, \ldots, d\}$, $\mathbf{e}(\Sigma) = \{e \in \mathbf{e}(\Gamma) : 0 \in e\}$. Let Γ' be the graph with $\mathbf{v}(\Gamma) = \{0, \ldots, d\}$ and $\mathbf{e}(\Gamma') = \mathbf{e}(\Gamma) \setminus \mathbf{e}(\Sigma)$ and let $\mathbf{m}', \mathbf{m}''$ be the corresponding subsets of the collection of bond dimensions $\mathbf{m}$. Write $S = T(\Sigma, \mathbf{m}'')$ and $T' = T(\Gamma', \mathbf{m}')$; then

$$T = S \boxtimes T'.$$

By the induction hypothesis, $\mathfrak{g}_{T'} = \mathbf{g}_{\Gamma',\mathbf{m}'}$ and $\mathfrak{g}_S = \mathbf{g}_{\Sigma,\mathbf{m}''}$. By Theorem 3.6

$$\mathfrak{g}_T = \mathfrak{g}_{T'} + \mathbf{g}_{\Sigma,\mathbf{m}''} = \mathbf{g}_{\Gamma',\mathbf{m}'} + \mathbf{g}_{\Sigma,\mathbf{m}''} = \mathbf{g}_{\Gamma,\mathbf{m}},$$

and this concludes the proof. □

3.4. **Additional results on isotropy groups.** In this section, we prove a generalization of [CGL+20, Thm. 4.1(iii)]; it does not have a direct application in this work but it is of interest on its own right.

Given two spaces V, W, there is a natural embedding $GL(V) \to GL(V \otimes W)$ defined by $g \mapsto g \otimes \mathrm{Id}_W$; correspondingly the Lie algebra $\mathfrak{gl}(V)$ can be regarded as a subalgebra of $\mathfrak{gl}(V \otimes W)$. In particular, if $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a subalgebra, then $\mathfrak{g}$ is naturally identified with a subalgebra of $\mathfrak{gl}(V \otimes W)$.

Proposition 3.8. *Let $T \in V_1 \otimes \cdots \otimes V_d$ and $S \in W_1 \otimes \cdots \otimes W_d$ be concise tensors. Assume $\mathfrak{g}_T = \{0\} \subseteq \mathfrak{g}(V_1, \ldots, V_d)$. Then*

$$\mathfrak{g}_{T \boxtimes S} = \mathfrak{g}_S$$

regarded as a subalgebra of $\mathfrak{g}(V_1 \otimes W_1, \ldots, V_d \otimes W_d)$.

Proof. The inclusion

$$\mathfrak{g}_S \subseteq \mathfrak{g}_{T \boxtimes S}$$

is immediate from the definition of Kronecker product.

Let $\mathbf{X} \in \mathfrak{g}_{T \boxtimes S}$. Write $\mathbf{X} = (X_1, \ldots, X_d)$ with $X_k \in \mathfrak{gl}(V_k \otimes W_k)$. Our goal is to show that $X_k = \mathrm{Id}_{V_k} \otimes Z_k$ for some $Z_k \in \mathfrak{gl}(W_k)$ with $\mathbf{Z} := (Z_1, \ldots, Z_d) \in \mathfrak{g}_S$.

For every $p = 1, \ldots, d$, fix bases $\{v_j^p : j = 1, \ldots, \dim V_p\}$ of V_p and similarly for W_p. Write

$$T = \sum T^{i_1,\ldots,i_d} v_{i_1}^1 \otimes \cdots \otimes v_{i_d}^d,$$
$$S = \sum S^{j_1,\ldots,j_d} w_{j_1}^1 \otimes \cdots \otimes w_{j_d}^d.$$

For $k = 1, \dots, d$, write $(x_k)^{ij}_{i'j'}$ for the entries of X_k with respect to the basis $v^k_i \otimes w^k_j$. By Leibniz's rule, the condition $\mathbf{X}.(T \boxtimes S) = 0$ is equivalent to

$$\sum_{k=1}^{d} (x_k)^{i_k j_k}_{i'_k j'_k} T^{i_1,\dots,i'_k,\dots,i_d} S^{j_1,\dots,j'_k,\dots,j_d} = 0 \quad \text{for every } i_1,\dots,i_d,j_1,\dots,j_d, \tag{6}$$

where we use the summation convention that repeated upper and lower indices are to be summed over their range.

For every $j_1,\dots,j_d$, and for every $k = 1,\dots,d$, define $Y_k(j_1,\dots,j_d) \in \mathfrak{gl}(V_k)$ by

$$(y_k(j_1,\dots,j_d))^{i_k}_{i'_k} = (x_k)^{i_k j_k}_{i'_k j'_k} S^{j_1,\dots,j'_k,\dots,j_d}.$$

Regard $\mathbf{Y}(j_1,\dots,j_d) = (Y_1(j_1,\dots,j_d),\dots,Y_d(j_1,\dots,j_d))$ as an element of $\mathfrak{g}(V_1,\dots,V_d)$. From (6), we deduce that $\mathbf{Y}(j_1,\dots,j_d)$ satisfies $\mathbf{Y}.T = 0$ and therefore $\mathbf{Y} \in \mathfrak{g}_T$. From the hypothesis $\mathfrak{g}_T = \{0\}$ and therefore, for every k, there exists $\lambda_k(j_1,\dots,j_d)$ such that $Y_k(j_1,\dots,j_d) = \lambda_k(j_1,\dots,j_d)\mathrm{Id}_{V_k}$ and $\sum_k \lambda_k(j_1,\dots,j_d) = 0$.

Since $Y_k(j_1,\dots,j_d)$ is a multiple of the identity, we have

$$0 = (y_k(j_1,\dots,j_d))^{i_k}_{i'_k} = (x_k)^{i_k j_k}_{i'_k j'_k} S^{j_1,\dots,j'_k,\dots,j_d} \quad \text{for } i_k \neq i'_k,$$
$$0 = (y_k(j_1,\dots,j_d))^{i_k}_{i_k} - (y_k(j_1,\dots,j_d))^{1}_{1} = [(x_k)^{i_k j_k}_{i_k j'_k} - (x_k)^{1 j_k}_{1 j'_k}] S^{j_1,\dots,j'_k,\dots,j_d}.$$

In other words, if $i_k \neq i'_k$, setting $Z_k(i_k,i'_k) \in \mathfrak{gl}(W_k)$ to be defined by $(z_k(i_k,i'_k))^{j_k}_{j'_k} = (x_k)^{i_k j_k}_{i'_k j'_k}$, we have $Z_k(i_k,i'_k).S = 0$. This means that $Z_k(i_k,i'_k) \in \mathfrak{g}_S \cap \mathfrak{gl}(W_k)$: since S is concise, Lemma 3.4 implies $Z_k(i_k,i'_k) = 0$. This shows that $(x_k)^{i_k j_k}_{i'_k j'_k} = 0$ whenever $i_k \neq i'_k$. Similarly, if $i_k \geq 2$, setting $(z_k(i_k))^{j_k}_{j'_k} = (x_k)^{i_k j_k}_{i_k j'_k} - (x_k)^{1 j_k}_{1 j'_k}$, we have $Z_k(i_k).S = 0$, hence $Z_k(i_k) = 0$ and therefore $(x_k)^{i_k j_k}_{i_k j'_k} = (x_k)^{1 j_k}_{1 j'_k}$ for every i_k.

We deduce that $X_k = \mathrm{Id}_{V_k} \otimes Z_k$ for some $Z_k \in \mathfrak{gl}(W_k)$. Now, let $\mathbf{Z} = (Z_1,\dots,Z_k)$. We conclude

$$0 = \mathbf{X}.(T \boxtimes S) = \mathbf{Z}.(T \boxtimes S) = T \boxtimes \mathbf{Z}.S$$

and therefore $\mathbf{Z} \in \mathfrak{g}_S$. This concludes the proof. □

4. Dimension of Tensor Network varieties

We provide an upper bound on $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ for every $\mathbf{m}$ and $\mathbf{n}$.

First, we give the following definitions following [LQY12].

Definition 4.1. Let $(\Gamma,\mathbf{m},\mathbf{n})$ be a tensor network. A vertex $v \in \mathbf{v}$ is called

- *subcritical* if $\prod_{e \ni v} m_e \geq n_v$; *strictly subcritical* if the inequality is strict;
- *supercritical* if $\prod_{e \ni v} m_e \leq n_v$; *strictly supercritical* if the inequality is strict;
- *critical* if v is both subcritical and supercritical.

The tensor network $(\Gamma,\mathbf{m},\mathbf{n})$ is called [strictly] subcritical (resp. supercritical) if all its vertices are [strictly] subcritical (resp. supercritical).

First, we determine a reduction which allows us to assume that the bond dimensions associated to the edges incident a fixed vertex are *balanced*, in a way made precise in Lemma 4.2.

Then, we provide a second reduction, proving that the tensor network variety of a tensor network having strictly supercritical vertices can be realized via a vector bundle construction as a natural extension of the tensor network variety where the strictly supercritical vertices are reduced to be critical.

Finally, we prove an upper bound for $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ in the subcritical range. This upper bound reduces to an equality in the critical case.

Recall that from (3), we have

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \dim \mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d) - \dim \Phi^{-1}(T).$$

4.1. **Reduction of bond dimension.** We already observed in Remark 2.1 that we may always assume bond dimensions are at least 2. Here, we show that if they are "too unbalanced", then they can be reduced without affecting the dimension of the tensor network variety.

We say that a tensor network $(\Gamma, \mathbf{m}, \mathbf{n})$ has *overabundant bond dimension* if there exist a vertex $v \in \mathbf{v}(\Gamma)$ and an edge $e \in \mathbf{e}(\Gamma)$ incident to v such that

$$m_e > n_v \prod_{e' \ni v, e' \neq e} m_{e'}. \tag{7}$$

The following result shows that overabundant bond dimensions do not contribute to the dimension of the tensor network variety.

Lemma 4.2. *Let $(\Gamma, \mathbf{m}, \mathbf{n})$ be a tensor network. Fix $v \in \mathbf{v}(\Gamma)$, let k be the degree of the vertex v and $\{e_1, \dots, e_k\}$ be the edges incident to v; assume $m_{e_1} \leq \dots \leq m_{e_k}$. If (7) holds for v, that is*

$$m_{e_k} > n_v \cdot m_{e_1} \cdots m_{e_{k-1}},$$

then

$$\mathcal{TNS}_{\mathbf{m},\mathbf{n}} = \mathcal{TNS}_{\overline{\mathbf{m}},\mathbf{n}}$$

where $\overline{\mathbf{m}}$ is defined by $\overline{m}_e = m_e$ if $e \neq e_k$ and $\overline{m}_{e_k} = n_v \cdot m_1 \cdots m_{e_{k-1}}$.

Proof. Let $T \in \mathcal{TNS}_{\mathbf{m},\mathbf{n}} \subseteq V_1 \otimes \dots \otimes V_d$ be a generic element and let $(X_1, \dots, X_d) \in \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)$ be an element such that $(X_1, \dots, X_d) \cdot T(\Gamma, \mathbf{m}) = T$.

Suppose $v = d$ and $e_j = \{d, j\}$ for $j = 1, \dots, k$. Write $U_j = \mathbb{C}^{m_j}$; let $W_d = U_1^* \otimes \dots \otimes U_k^*$, so that, for $j = 1, \dots, k$, we have $W_j = U_j \otimes W'_j$ where W'_j depends on the other edges incident to the vertex j.

Regard X_d as a tensor in $W_d^* \otimes V_d = U_1 \otimes \dots \otimes U_k \otimes V_d$. Since $m_{e_k} > m_{e_1} \cdots m_{e_{k-1}} \cdot n_v$, X_d is not concise on the factor U_k: let $\overline{U}_k \subseteq U_k$ with $\dim \overline{U}_k = m_{e_1} \cdots m_{e_{k-1}} \cdot n_v$ be a subspace such that $X_d \in U_1 \otimes \dots \otimes U_{k-1} \otimes \overline{U}_k \otimes V_d$. Correspondingly, let $\overline{U}_k^* = U_k^* / \overline{U}_k^{\perp}$. Note that $T(\Gamma, \overline{\mathbf{m}})$ coincides with the image of $T(\Gamma, \mathbf{m})$ via the projection $U_k^* \to \overline{U}_k^*$ on the d-th factor.

Now, define $\overline{W}_d = U_1^* \otimes \dots \otimes U_{k-1}^* \otimes \overline{U}_k^*$ and $\overline{W}_k = W'_k \otimes \overline{U}_k$. Let $\overline{X}_d = X_d$ be the linear map regarded as an element of $\mathrm{Hom}(\overline{W}_d, V_d)$. Moreover, the space $\mathrm{Hom}(W_k, V_k) = (W'_k \otimes U_k)^* \otimes V_k = W_k'^* \otimes U_k^* \otimes V_k$ naturally projects onto $W_k'^* \otimes \overline{U}_k^* \otimes V_k = \mathrm{Hom}(\overline{W}_k, V_k)$: let $\overline{X}_k$ be the image of X_k under this projection.

Now, one can verify that

$$T = (X_1, \dots, X_d) \cdot T(\Gamma, \mathbf{m}) = (\overline{X}_1, \dots, \overline{X}_d) \cdot T(\Gamma, \overline{\mathbf{m}})$$

where $\overline{X}_v = X_v$ if $v \neq k, d$. $\square$

4.2. **Reduction for supercritical vertices.** The reduction of this section appeared already in [LQY12]. We include it here for completeness.

For a vector space V with $\dim V = n$ and and an integer $k \leq n$, let $\mathbf{G}(k, V)$ be the Grassmannian of k-dimensional linear subspaces of V. Recall that $\dim \mathbf{G}(k, V) = k(n-k)$. The variety $\mathbf{G}(k, V)$ has a *tautological bundle*

$$\sigma : \mathcal{S} \to \mathbf{G}(k, V);$$

the fiber of $\mathcal{S}$ over a point $[E] \in \mathbf{G}(k, V)$ is the plane E itself: $\mathcal{S}_{[E]} = E$.

Proposition 4.3. *Let $(\Gamma, \mathbf{m}, \mathbf{n})$ be a tensor network. Suppose that the vertex $d \in \mathbf{v}(\Gamma)$ is supercritical and write $N = \dim W_d = \prod_{e \ni d} m_e$. Let $\mathbf{n}' = (n'_v : v \in \mathbf{v}(\Gamma))$ be the d-tuple of local dimensions defined by $n'_v = n_v$ if $v \neq d$ and $n'_d = N$.*

Then

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = N(n_d - N) + \dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}'}.$$

Proof. Let $\mathcal{S}_d^{V_1 \otimes \cdots \otimes V_{d-1}}$ be the vector bundle over the Grassmannian $\mathbf{G}(N, V_d)$ whose fiber over a plane $[E]$ is $V_1 \otimes \cdots \otimes V_{d-1} \otimes E$; this is the tautological bundle augmented by the trivial bundle with constant fiber $V_1 \otimes \cdots \otimes V_{d-1}$. Consider the diagram

$$\begin{tikzcd}
& \mathcal{S}_d^{V_1 \otimes \cdots \otimes V_{d-1}} \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
\mathbf{G}(N, V_d) & & V_1 \otimes \cdots \otimes V_d
\end{tikzcd}$$

where the second projection π maps an element of the bundle to its fiber component: $([E], T) \mapsto T$. By conciseness, this projection is generically one-to-one.

Consider the subbundle of $\mathcal{S}_d^{V_1 \otimes \cdots \otimes V_{d-1}}$ whose fiber at $[E]$ is $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}'}$ where the d-th factor is identified with E. Let $\underline{\mathcal{TNS}}^{\Gamma}_{\mathbf{m},\mathcal{S}}$ be the total space of this subbundle. We have

$$\dim \underline{\mathcal{TNS}}^{\Gamma}_{\mathbf{m},\mathcal{S}} = \dim \mathbf{G}(N, V_d) + \dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}'} = N(n_d - N) + \dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}'}.$$

The projection π is generically one-to-one and maps $\underline{\mathcal{TNS}}^{\Gamma}_{\mathbf{m},\mathcal{S}}$ surjectively onto $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$. Therefore $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \dim \underline{\mathcal{TNS}}^{\Gamma}_{\mathbf{m},\mathcal{S}}$ and this concludes the proof. $\square$

Iteratively applying Proposition 4.3, one can reduce all strictly supercritical vertices to critical vertices.

Theorem 4.4. *Let $(\Gamma, \mathbf{m}, \mathbf{n})$ be a tensor network. For every $v \in \mathbf{v}(\Gamma)$ let $N_v = \prod_{e \ni v} m_e$. Let $\mathbf{n}'$ be the set of local dimensions defined by $n'_v = \min\{N_v, n_v\}$. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \sum_{v \in \mathbf{v}(\Gamma)} n'_v(n_v - n'_v) + \dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}'}.$$

Note that the tensor network $(\Gamma, \mathbf{m}, \mathbf{n}')$ appearing in Theorem 4.4 is, by definition, subcritical. It remains to understand $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ in the subcritical range.

4.3. **Subcritical range.** We will provide an upper bound for $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ when the tensor network $(\Gamma, \mathbf{m}, \mathbf{n})$ is subcritical. The upper bound is obtained, following (3), by determining a lower bound on the dimension of the generic fiber of the parametrization of $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$.

Theorem 4.5. *Let* $(\Gamma, \mathbf{m}, \mathbf{n})$ *be a subcritical tensor network. Then the dimension of the generic fiber of the map* Φ *is bounded from below by the dimension of the* $\mathcal{G}_{\Gamma,\mathbf{m}}$*-orbit of a generic element of* $\mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)$.

Proof. Let $T = (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathbf{m})$, with $X_1 \otimes \cdots \otimes X_d \in \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)$ a generic element. The fiber of $\Phi : \mathrm{Hom}(W_1, \dots, W_d; V_1 \otimes \cdots \otimes V_d) \to \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$ over T is

$$\Phi^{-1}(T) = \{Y_1 \otimes \cdots \otimes Y_d \in \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d) : (Y_1 \otimes \cdots \otimes Y_d) \cdot T(\Gamma, \mathbf{m}) = T\}$$

Since every vertex is subcritical, for every j, a generic element of $\mathrm{Hom}(W_j, V_j)$ is surjective. Let $Y_1 \otimes \cdots \otimes Y_d \in \Phi^{-1}(T)$. By conciseness, Y_j has the same image as X_j, therefore Y_j is surjective as well, and there exists $g \in GL(W_j)$ such that $Y_j = X_j g_j$.

For $X = X_1 \otimes \cdots \otimes X_d$, and $g = g_1 \otimes \cdots \otimes g_d \in G(W_1, \dots, W_d)$, write $g.X = X_1 g_1 \otimes \cdots \otimes X_d g_d$. In particular, if $g \in \mathcal{G}_{\Gamma,\mathbf{m}}$ then

$$Y \cdot T(\Gamma, \mathbf{m}) = (g.X) \cdot T(\Gamma, \mathbf{m}) = (X_1 \otimes \cdots \otimes X_d)(g_1 \otimes \cdots \otimes g_d) \cdot T(\Gamma, \mathbf{m}) = X \cdot T(\Gamma, \mathbf{m}) = T,$$

and the dimension of the fiber is bounded by

$$\begin{aligned} \dim \Phi^{-1}(T) &= \dim\{Y : Y \cdot T(\Gamma, \mathbf{m}) = T\} \\ &= \dim\{g.X : g \in G(W_1, \dots, W_d), (g.X) \cdot T(\Gamma, \mathbf{m}) = T\} \\ &\geq \dim\{g.X : g \in \mathcal{G}_{\Gamma,\mathbf{m}}\} \\ &= \dim(\mathcal{G}_{\Gamma,\mathbf{m}} \cdot X). \end{aligned} \tag{8}$$

Therefore the dimension of the generic fiber is bounded from below by the dimension of the $\mathcal{G}_{\Gamma,\mathbf{m}}$-orbit of a generic element of $\mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)$. □

Applying the Theorem of the Dimension of the Fibers [Sha94, Thm. 1.25] to the $\mathcal{G}_{\Gamma,\mathbf{m}}$-orbit of a generic element $X \in \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)$, we deduce the following corollary, which completes the proof of Theorem 1.1.

Corollary 4.6. *Let* $(\Gamma, \mathbf{m}, \mathbf{n})$ *be a subcritical tensor network with no overabundant bond dimension. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} \leq \left[\textstyle\sum_{v \in \mathbf{v}(\Gamma)} N_v n_v - d + 1\right] - \textstyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$$

where $N_v = \prod_{e \ni v} m_e$ *and* $X = X_1 \otimes \cdots \otimes X_d$ *with* $X_v \in \mathrm{Hom}(W_v, V_v)$ *generic.*

Proof. From (3) $\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \dim \mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d) - \dim \Phi^{-1}(T)$ where T is a generic element of $\mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}}$.

Now $\dim \operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) = \sum_{v \in \mathbf{v}(\Gamma)} N_v n_v - d + 1$. By Theorem 4.5,

$$\begin{aligned}\dim \Phi^{-1}(T) \geq \dim \mathcal{G}_{\Gamma,\mathbf{m}} \cdot X = \\ \dim \mathcal{G}_{\Gamma,\mathbf{m}} - \dim \operatorname{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X) = \\ \textstyle\sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) - \dim \operatorname{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X),\end{aligned}$$

where $X \in \operatorname{Hom}(W_1, \ldots, W_d, V_1, \ldots, V_d)$ is generic. □

4.4. **Sharpening the upper bound.** In this section, we study the upper bound obtained in Corollary 4.6 and we provide sufficient conditions to have $\dim \operatorname{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X) = 0$. These results will lead us to a complete proof of Corollaries 1.2 and 1.3.

Definition 4.7. Let G be an algebraic group acting on an algebraic variety V. We say that the action is *generically stable* if there exists an element $v \in V$ such that the stabilizer $\operatorname{Stab}_G(v)$ is a finite group.

In particular, the condition that the action of $\mathcal{G}_{\Gamma,\mathbf{m}}$ on $\operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$ is generically stable is equivalent to the fact that the term $\dim \operatorname{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X)$ in Corollary 4.6 is 0.

A rich theory has been developed in the study of stable group actions (and more generally semistable actions, which are beyond the scope of this paper) starting from [KN79] and related works. We refer to [MFK94] for the theory.

First, we provide a result on the cycle graph C_d, which yields the result on matrix product states in Corollary 1.2.

Proposition 4.8. *Let $(C_d, \mathbf{m}, \mathbf{n})$ be the tensor network on the cycle graph with constant bond dimension $\mathbf{m} = (m, \ldots, m)$. Assume $n_j \geq 2$ for at least one index. Then the action of $\mathcal{G}_{C_d,\mathbf{m}}$ on $\operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$ is generically stable.*

Proof. Let $X_1 \otimes \cdots \otimes X_d \in \operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$ be a generic element. Write $W_j = U_j \otimes U_{j+1}^*$ with $U_j = U_{j+1} = \mathbb{C}^m$. Then X_j is a generic element of $U_j^* \otimes U_{j+1} \otimes V_j$, with $\dim V_j = n_j \geq 1$. For every j, write $X_j = \sum_{p=1}^{n_j} X_j^{(p)} \otimes v_p$ where $v_1, \ldots, v_{n_j}$ is a basis of V_j and $X_j^{(p)} \in U_j^* \otimes U_{j+1}$.

By genericity $X_j^{(1)}$ is a fixed isomorphism $X_j^{(1)} : U_j \to U_{j+1}$; after choosing bases in U_j, we write $X_j^{(1)} = \mathrm{Id}_{\mathbb{C}^m}$ in coordinates for $j = 1, \ldots, d-1$ and $X_d^{(1)} : U_d \to U_1$ is a generic diagonal matrix.

The stabilizer $\operatorname{Stab}_{\mathcal{G}_{C_d,\mathbf{m}}}(X)$ is contained in the stabilizer of $X_1^{(1)} \otimes \cdots \otimes X_d^{(1)}$: this is the centralizer of $X_d^{(1)}$; in coordinates this is the maximal torus $\Theta_m \subseteq PGL_m$ of diagonal matrices in PGL_m^{Δ}, where $PGL_m^{\Delta} \subseteq \mathcal{G}_{C_d,\mathbf{m}} = \times_1^d PGL(U_j)$ lies on the diagonal of the direct factors. Therefore $\operatorname{Stab}_{\mathcal{G}_{C_d,\mathbf{m}}}(X) \subseteq \Theta_m$.

Now, there exists at least one index j such that $n_j \geq 2$. Correspondingly, there is a map $X_j^{(2)} : U_j \to U_{j+1}$. Therefore $\operatorname{Stab}_{\mathcal{G}_{C_d,\mathbf{m}}}(X) \subseteq \operatorname{Stab}_{\Theta_m}(X_j^{(2)})$. By genericity, $X_j^{(2)}$ is not diagonal in the fixed basis, hence $\operatorname{Stab}_{\Theta_m}(X_j^{(2)})$ is trivial.

This shows that a generic $X \in \operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$ satisfies $\dim \operatorname{Stab}_{\mathcal{G}_{C_d,\mathbf{m}}}(X) = 0$, hence the action of $\mathcal{G}_{C_d,\mathbf{m}}$ on $\operatorname{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$ is generically stable. □

We generally believe that $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X) = 0$ in "most cases" at least when the bond dimensions are balanced. However, we cannot extend the argument of Proposition 4.8 to the general case. Instead, we further localize the action, reembedding the gauge subgroup $\mathcal{G}_{\Gamma,\mathbf{m}}$ in the product group H describe in Section 3.2. This will allow us to use results on the stability of the action on tensor spaces which in turn guarantee the stability of the action of $\mathcal{G}_{\Gamma,\mathbf{m}}$.

Recall that the subgroup $H \subseteq G(W_1, \dots, W_d)$ from Section 3.2 is the (image in $G(W_1, \dots, W_d)$ of the) group $\bigtimes_{e\in\mathbf{e}(\Gamma)} GL(U_e)\times GL(U_e^*) \simeq \bigtimes_{v\in\mathbf{v}(\Gamma)} H_v$, where $H_v = \bigtimes_{e\ni v} GL_{m_e}$. Since $\mathcal{G}_{\Gamma,\mathbf{m}} \subseteq H$, clearly $\mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X) \subseteq \mathrm{Stab}_H(X)$. Therefore, if the action of H on $\mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d)$ is generically stable, so is the action of $\mathcal{G}_{\Gamma,\mathbf{m}}$ on $\mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d)$.

We establish the following result, whose proof is immediate from the product structure of $\mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d)$ and of X.

Lemma 4.9. *If $X = X_1 \otimes \cdots \otimes X_d$ then*

$$\mathrm{Stab}_H(X) = \bigtimes_{v\in\mathbf{v}(\Gamma)} \mathrm{Stab}_{H_v}(X_v).$$

In particular, the action of H on $\mathrm{Hom}(W_1, \dots, W_d, V_1, \dots, V_d)$ is generically stable if and only if for every v the action of H_v on $\mathrm{Hom}(W_v, V_v)$ is generically stable.

Now, regard $X_v \in \mathrm{Hom}(W_v, V_v)$ as a tensor in $V_v \otimes W_v^* = V_v \otimes \left(\bigotimes_{e\ni v} U'_e\right)$ where $U'_e = U_e$ or $U'_e = U_e^*$ depending on whether U_e of U_e^* appears in W_v. The group H_v acts trivially on V_v; by Lemma 3.2, we deduce that $\mathrm{Stab}_{H_v}(X_v)$ is the simultaneous stabilizer of $\mathrm{Im}\, X_v(V_v^*) \subseteq \bigotimes_{e\ni v} U'_e$. In particular, if X_v is generic, $\mathrm{Stab}_{H_v}(X_v)$ is the simultaneous stabilizer of n_v elements of W_v^*.

Therefore, we are reduced to study the stability of the action of a product of special linear groups $SL(U_1) \times \cdots \times SL(U_k)$ on the space $U_1 \otimes \cdots \otimes U_k \otimes V$. The study of the stability of this action is characterized in the recent [DM20, DMW20] and in the special case where $\dim V = 1$ it is characterized in [BRR18]. In particular, the following result leads to the proof of Corollary 1.3, and to a wide range of generalizations.

Proposition 4.10. *Let $k \geq 3$ and consider vector spaces $U_1, \dots, U_k, V$ with $\dim U_\alpha = m$, $\dim V = n$. The action of $SL(U_1) \times \cdots \times SL(U_k)$ on $U_1 \otimes \cdots \otimes U_k \otimes V$ is generically stable unless $(k, m, n) = (3, 2, 1)$.*

Proof. The case $(k, m, n) = (3, 2, 1)$ corresponds to the action of $SL_2 \times SL_2 \times SL_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$; this is not stable since

$$9 = \dim(SL_2 \times SL_2 \times SL_2) > \dim \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = 8.$$

Except for this case, the result follows from [DMW20, Theorem 1.5 (Case 4)], since the inequality $m \leq \frac{1}{2} m^{k-1} n$ is always verified. □

4.5. **Critical case.** We conclude this section showing that the dimension of the tensor network variety in the critical case equals the upper bound of Corollary 4.6 and $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathbf{m}}}(X) = 0$. As a consequence, via Theorem 4.4, we obtain the equality in the supercritical range, which completes the proof of Corollary 1.4.

Proposition 4.11. *Let* $(\Gamma, \mathbf{m}, \mathbf{n})$ *be a supercritical tensor network. Write* $N_v = \prod_{e \ni v} m_e$. *Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} = \sum_{v \in \mathbf{v}(\Gamma)} n_v N_v - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1).$$

Proof. First consider the critical case, that is $N_v = n_v$. In this case, a generic $X_v \in \mathrm{Hom}(W_v, V_v)$ is invertible. Therefore

$$\begin{aligned}\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} &= \dim G(W_1, \dots, W_d) \cdot T(\Gamma, \mathbf{m}) = \\ &\dim G(W_1, \dots, W_d) - d + 1 - \dim \mathcal{G}_{\Gamma,\mathbf{m}} = \sum_{v \in \mathbf{v}(\Gamma)} N_v^2 - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1).\end{aligned}$$

In the supercritical case, we apply Theorem 4.4. Write $\mathbf{N} = (N_v : v \in \mathbf{v}(\Gamma))$, so that the tensor network $(\Gamma, \mathbf{m}, \mathbf{N})$ is critical. Then

$$\begin{aligned}\dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{n}} &= \dim \mathcal{TNS}^{\Gamma}_{\mathbf{m},\mathbf{N}} + \sum_{v \in \mathbf{v}(\Gamma)} N_v(n_v - N_v) = \\ &\sum_{v \in \mathbf{v}(\Gamma)} N_v^2 - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1) + \sum_{v \in \mathbf{v}(\Gamma)} N_v(n_v - N_v) = \\ &\sum_{v \in \mathbf{v}(\Gamma)} n_v N_v - d + 1 - \sum_{e \in \mathbf{e}(\Gamma)} (m_e^2 - 1).\end{aligned}$$

□

5. Analysis of small cases

In this section, we analyze few cases of tensor network varieties for small graphs and small bond dimension.

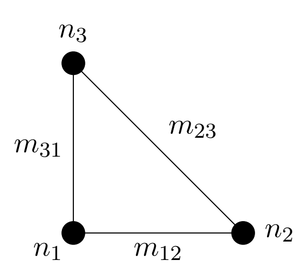
If Γ only contains two vertices, then the tensor network variety is easily described as a variety of matrices whose rank is bounded from above by the bond dimension of the unique edge.

We start our analysis with the case of three vertices.

5.1. Triangular graph.

The graph tensor associated to the triangular graph is the matrix multiplication tensor. This is the object of a rich literature, devoted to determining the value of the exponent of matrix multiplication. We refer to [Blä13, Lan17] for an overview on the subject.

Let C_3 be the triangular graph. Write $\{1, 2, 3\}$ for the three vertices and m_{12}, m_{23}, m_{31} for the three bond dimensions and (n_1, n_2, n_3) for the three local dimensions, ordered as follows:



	$\mathbf{n}$	lower bound	upper bound
	$(2,2,2)$	8	8
	$(2,2,3)$	12	12
	$(2,2,4)$	16	16
	$(2,3,3)$	18	18
*	$(2,3,4)$	22	24
*	$(2,4,4)$	26	29
	$(3,3,3)$	25	25
	$(3,3,4)$	29	29
	$(3,4,4)$	31	31
	$(4,4,4)$	37	37

Table 1. Upper and lower bound for $\dim \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}}$. The lower bound is obtained via a direct calculation. The upper bound is the value obtained in Corollary 1.2. In the cases marked with * the two bounds do not coincide.

If $\mathbf{m} = (m_{12}, m_{23}, m_{31}) = (a, b, 1)$ (in other words, the edge $\{3, 1\}$ is erased) then every tensor in $W_1 \otimes W_2 \otimes W_3$ is a restriction of the graph tensor. In particular, if $\mathbf{n} = (n_1, n_2, n_3)$ with $n_1 \leq a$, $n_2 \leq ab$, $n_3 \leq b$, then

$$\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = V_1 \otimes V_2 \otimes V_3.$$

Therefore, the first interesting case is the one with bond dimensions $\mathbf{m} = (2, 2, 2)$. We record the cases in the subcritical range in Table 1. For each of these cases, the lower bound for the dimension is obtained computing explicitly the rank of the differential of the parametrization map Φ of Section 2 at a random point. We perform this calculation in Macaulay2 [GS20]. The scripts performing the calculation are available at
`https://fulges.github.io/code/BDG-DimensionTNS.html`.

Since the point to compute the differential is chosen at random, we are confident that the number recorded as a lower bound is equal to the actual dimension of the tensor network variety $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}}$. However, from a formal point of view, the sole calculation of the rank of the differential at a random point does not provide a complete proof.

The only cases where the lower bound does not match the upper bound given in Corollary 1.2 are the ones with $\mathbf{n} = (2, 3, 4)$ and $\mathbf{n} = (2, 4, 4)$. In these cases, we prove that the dimension of the tensor network variety equals the lower bound of Table 1. We provide the following result, that we prove in general and will be used in Theorem 5.2 in the cases $(a, b, r) = (3, 4, 2)$ and $(a, b, r) = (4, 4, 2)$.

Lemma 5.1. *Let V_1, V_2, V_3 be vector spaces with $\dim V_1 = 2$, $\dim V_2 = a$, $\dim V_3 = b$. Let $\sigma_r \subseteq \mathbb{P}(V_2 \otimes V_3)$ be the variety of elements of rank at most r. Define*

$$\mathcal{Z}_{a,b,r} = \overline{\left\{ T \in V_1 \otimes V_2 \otimes V_3 : T(V_1^*) \cap \sigma_r \text{ contains at least two points} \right\}} \subseteq \mathbb{P}(V_1 \otimes V_2 \otimes V_3)$$

Then $\mathcal{Z}_{a,b,r}$ is an irreducible variety and

$$\dim \mathcal{Z}_{a,b,r} = 2r(a + b - r) + 1$$

Proof. Define the variety of secant lines

$$\mathcal{S}_{a,b,r} = \overline{\left\{ L \in \mathbf{G}(2, V_2 \otimes V_3) : \mathbb{P}L \cap \sigma_r \text{ contains at least two points} \right\}} \subseteq \mathbf{G}(2, V_2 \otimes V_3),$$

where $\mathbf{G}(2, V_2 \otimes V_3)$ denotes the Grassmannian of 2-planes in $V_2 \otimes V_3$.

Then $\mathcal{S}_{a,b,r}$ is an irreducible variety of dimension $2 \dim \sigma_r = 2[r(a+b-r)-1]$ [EH16, Section 10.3].

The variety $\mathcal{Z}_{a,b,r}$ is an $SL(V_1)$-bundle on $\mathcal{S}_{a,b,r}$. This guarantees that $\mathcal{Z}_{a,b,r}$ is irreducible and provides $\dim \mathcal{Z}_{a,b,r} = \dim \mathcal{S}_{a,b,r} + 3 = 2[r(a+b-r)-1]+3 = 2r(a+b-r)+1$ as desired. □

Theorem 5.2. *Let* $\mathbf{m} = (2,2,2)$.

- *if* $\mathbf{n} = (2,3,4)$ *then* $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = \mathcal{Z}_{3,4,2}$*; in particular* $\dim \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = 22$*;*
- *if* $\mathbf{n} = (2,4,4)$ *then* $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = \mathcal{Z}_{4,4,2}$*; in particular* $\dim \mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} = 26$*.*

Proof. The lower bound on the dimension follows from Table 1.

By Lemma 5.1, we have $\mathcal{Z}_{3,4,2} = 4 \cdot (3+4-2)+1 = 21$ and $\mathcal{Z}_{4,4,2} = 4 \cdot (4+4-2)+1 = 25$.

In the rest of the proof, we show that $\mathcal{TNS}^{C_3}_{\mathbf{m},(2,3,4)} \subseteq \widehat{\mathcal{Z}}_{3,4,2}$ and $\mathcal{TNS}^{C_3}_{\mathbf{m},(2,4,4)} \subseteq \widehat{\mathcal{Z}}_{4,4,2}$; here, if $Y \subseteq \mathbb{P}W$ is a projective variety, $\widehat{Y}$ denotes its affine cone in the space W.

Fix generic X_1, X_2, X_3 with $X_j \in \mathrm{Hom}(W_j, V_j)$ and let $T = X_1 \otimes X_2 \otimes X_3(T(C_3, \mathbf{m}))$. Let $L = T(V_1^*) \subseteq V_2 \otimes V_3$. It suffices to show that $\mathbb{P}L \cap \sigma_2$ contains at least two points in the two cases of interest.

We are free to normalize the linear maps X_1, X_2, X_3 using the gauge subgroup in $GL(W_1, W_2, W_3)$ and the action of $GL(V_1) \times GL(V_2) \times GL(V_3)$ on $V_1 \otimes V_2 \otimes V_3$.

Identify X_1 with a 2×2 matrix $B_1(v_1^{(1)}, v_2^{(1)})$ whose entries are linear combinations of the elements of a basis $\{v_1^{(1)}, v_2^{(2)}\}$ of V_1 and similarly for X_2 and X_3. In this way

$$X_1 \otimes X_2 \otimes X_3(T(C_3, \mathbf{m})) = \mathrm{trace}\Big(B_1(v_1^{(1)}, v_2^{(1)}) \cdot B_2(v_1^{(2)}, \dots, v_3^{(2)}), B_3(v_1^{(3)}, \dots, v_4^{(3)}) \Big).$$

Write $B_1(v_1^{(1)}, v_2^{(1)}) = B_1^1 v_1^{(1)} + B_1^2 v_2^{(1)}$ and similarly for the other matrices.

By genericity, the map X_3 is invertible: using the action of $GL(V_3)$, we may assume

$$B_3^1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \quad B_3^2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \quad B_3^3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \quad B_3^4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Moreover, the linear space $\langle B_1^1, B_1^2 \rangle$ contains at least one matrix of rank 1; using the action of $GL(V_1)$ and of the gauge group, we may assume $B_1^1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

With these normalizations, it is possible to verify that the line $\mathbb{P}(T(V_1^*))$ contains two rank two matrices. We provide a Macaulay2 script determining the intersection $\mathbb{P}(T(V_1^*)) \cap \sigma_2$ at `https://fulges.github.io/code/BDG-DimensionTNS.html`.

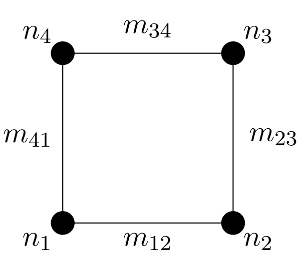
If $\mathbf{n} = (2,3,4)$, this shows $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} \subseteq \mathcal{Z}_{3,4,2}$; if $\mathbf{n} = (2,4,4)$, this shows $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} \subseteq \mathcal{Z}_{4,4,2}$.

Finally, since $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} \subseteq \mathcal{Z}_{3,4,2}$ and they are both irreducible varieties of dimension 22, equality holds. Similarly, equality holds in the inclusion $\mathcal{TNS}^{C_3}_{\mathbf{m},\mathbf{n}} \subseteq \mathcal{Z}_{4,4,2}$. □

	$\mathbf{n}$	lower bound	upper bound
*	$(2,2,2,2)$	15	16
*	$(2,2,2,3)$	20	21
*	$(2,2,2,4)$	24	25
	$(2,2,3,3)$	25	25
	$(2,2,3,4)$	29	29
	$(2,2,4,4)$	33	33
*	$(2,3,2,3)$	24	25
*	$(2,3,2,4)$	28	29
	$(2,3,3,3)$	29	29
	$(2,3,3,4)$	33	33
	$(2,3,4,3)$	33	33
	$(2,3,4,4)$	37	37
*	$(2,4,2,4)$	32	33
	$(2,4,3,4)$	37	37
	$(2,4,4,4)$	41	41
	$(3,3,3,3)$	33	33
	$(3,3,3,4)$	37	37
	$(3,3,4,4)$	41	41
	$(3,4,3,4)$	41	41
	$(3,4,4,4)$	45	45
	$(4,4,4,4)$	49	49

Table 2. Upper and lower bound for $\dim \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$. The lower bound is obtained via a direct calculation. The upper bound is the value obtained in Corollary 1.2. In the cases marked with * the two bounds do not coincide.

5.2. **Square graph.** Consider the square graph C_4 with bond dimensions $\mathbf{m} = (m_{12}, m_{23}, m_{34}, m_{41})$ and local dimensions $\mathbf{n} = (n_1, \dots, n_4)$.



We focus on the case where all bond dimensions are equal to 2. As in the previous section, we record in Table 2 the lower bound obtained computing the differential of the parametrization at a random point and the upper bound obtained via Corollary 1.2. As before, because of the random choice of point, we are confident that the value recorded as lower bound coincides with the value of $\dim \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$. We provide a formal proof for the case $\mathbf{n} = (2,2,2,2)$ in Theorem 5.3.

Theorem 5.3. *Let* $\mathbf{m} = (2,2,2,2)$ *and* $\mathbf{n} = (2,2,2,2)$*. Then*

$$\dim \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}} = 15;$$

more precisely $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ *is a hypersurface of degree* 6.

Proof. The lower bound $\dim \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}} \geq 15$ is obtained in Table 2.

Since $\dim V_1 \otimes V_2 \otimes V_3 \otimes V_4 = 16$, we obtain that either $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ is the entire space or it is a hypersurface.

We determine an irreducible equation of degree 6 vanishing on $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$.

This equation is a degree 6 invariant for the action of $GL(V_1) \times \cdots \times GL(V_4)$ on $V_1 \otimes \cdots \otimes V_4$. Its construction is described explicitly in [LT03, HLT12]. The evaluation of the invariant is performed by a Macaulay2 script [GS20] available at
`https://fulges.github.io/code/BDG-DimensionTNS.html`.

We illustrate here how to construct it and how to exploit the action of $GL(V_1) \times \cdots \times GL(V_4)$ and of the gauge group to normalize the linear maps and reduce the degrees of freedom in order to allow the script evaluate the invariant.

Given a tensor $T \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$, consider the bilinear map $T^{1,3} : V_1^* \times V_3^* \to V_2 \otimes V_4$. This makes $V_2 \otimes V_4$ into a space of 2×2 matrices depending bilinearly on $V_1 \times V_3$. Let $F(T) = \det(T^{1,3})$ be the determinant (of the 2×2 matrix $V_2 \otimes V_4$) evaluated on the image of $T^{1,3}$. So $F(T)$ is a polynomial of bidegree $(2, 2)$ in $V_1 \times V_3$, therefore it can be regarded as a bilinear form on $S^2 V_1 \times S^2 V_2$, where $S^2 W$ denotes the second symmetric power of a vector space W. Since $\dim S^2 \mathbb{C}^2 = 3$, this bilinear form has an associated 3×3 matrix. The invariant I_6 that we are interested in is the determinant of such matrix, which is a polynomial of degree 6 in the coefficients of the original tensor T.

In order to prove that I_6 vanishes identically on $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$, we apply a normalization which reduces the total degrees of freedom, then we perform the calculation symbolically in Macaulay2.

Write $T \in \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ as

$$T = \operatorname{trace}\Big(B_1(v_1^{(1)}, v_2^{(1)}) \cdots B_4(v_1^{(4)}, v_2^{(4)}) \Big)$$

where $B_p(v_1^{(p)}, v_2^{(p)}) = B_p^1 v_1^{(p)} + B_p^2 v_2^{(p)}$ are 2×2 matrices depending linearly on a fixed basis of V_p.

Since $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ is invariant under the action of $GL(V_1) \times \cdots \times GL(V_4)$ and the graph tensor is invariant under the action of the gauge subgroup, we may use these groups to normalize the matrices B_p^j. In particular, by the action of $GL(V_1)$ and $GL(V_3)$, we may assume B_1^1 and B_3^1 are rank one matrices; further, using the action of the gauge subgroup, we may assume $B_1^1 = B_3^1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

With this normalization, the evaluation of the invariant is performed and we can verify that $I_6(T) = 0$ whenever $T \in \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$.

Since I_6 is irreducible, we conclude $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ is a hypersurface of degree 6. □

If $d = 5, 6, 7$, the calculation of the differential at a random point shows that in the case of constant bond dimension 2 the dimension of tensor network varieties coincides with the upper bound of Corollary 1.2. Therefore, we propose the following conjecture:

Conjecture 5.4. *Let $d \geq 3$, $\mathbf{m} = (2, \dots, 2)$ and $\mathbf{n} = (n_1, \dots, n_d)$ with $n_j \geq 2$. Then*
$$\dim \mathcal{TNS}^{C_d}_{\mathbf{m},\mathbf{n}} = \min \left\{ 4 \left(\textstyle\sum_1^d n_j - d \right) + 1, \textstyle\prod_1^d n_j \right\}$$
except in the following cases:

- *if $d = 3$: $\mathbf{n} = (2, n_2, n_3)$, with $n_2 \geq 3$, $n_3 \geq 4$ and their cyclic permutations;*
- *if $d = 4$: $\mathbf{n} = (2, n_2, 2, n_4)$ with $n_2, n_4 \geq 2$ and their cyclic permutations.*

The results of this section, together with Theorem 4.4, confirm Conjecture 5.4 for $d = 3$. As mentioned above, a direct calculation confirms the conjecture for $d = 5, 6, 7$. In the case $d = 4$, the conjecture is confirmed in the case $\mathbf{n} = (2, 2, 2, 2)$, in all cases where the upper and lower bounds coincide in Table 2 and in the supercritical cases constructed from those.

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(A. Bernardi, C. De Lazzari) Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38123 Povo (TN), Italy

(F. Gesmundo) Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103, Leipzig, Germany

Email address, Bernardi: `alessandra.bernardi@unitn.it`

Email address, De Lazzari: `claudia.delazzari@unitn.it`

Email address, Gesmundo: `fulvio.gesmundo@mis.mpg.de`