Exact Moment Representation in Polynomial Optimization*†

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Abstract

We investigate the problem of representation of moment sequences by measures in Polynomial Optimization Problems, consisting in finding the infimum f^* of a real polynomial f on a real semialgebraic set S. We analyse the Moment Matrix (MoM) relaxations, dual to the Sum of Squares (SoS) relaxations, which are hierarchies of convex cones introduced by Lasserre to approximate measures and positive polynomials. We investigate the property of MoM exactness: this means that the MoM relaxation converges in finitely many steps, and the minimizing linear functionals are coming from evaluations at the minimizers of f.

We show that the MoM relaxation coincides with the dual of the SoS relaxation extended with the real radical of the support of the associated quadratic module Q. We prove that the vanishing ideal of the semialgebraic set S is generated by the kernel of the Hankel operator associated to a generic element of the truncated moment cone for a sufficiently high order of the MoM relaxation. We prove the exactness of MoM relaxation when S is finite and when regularity conditions, known as Boundary Hessian Conditions, hold on the minimizers. This implies that MoM exactness holds generically.

1 Introduction

Let $f, g_1, \ldots, g_s \in \mathbb{R}[X_1, \ldots, X_n]$ be polynomials in the indeterminates $X_1, \ldots, X_n$ with real coefficients. The goal of Polynomial Optimization is to find:

$$f^* := \inf \left\{ f(x) \in \mathbb{R} \mid x \in \mathbb{R}^n,\ g_i(x) \geq 0 \text{ for } i = 1, \ldots, s \right\}. \tag{1}$$

that is the infimum f^* of the *objective function* f on the *basic semialgebraic set* $S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } i = 1, \ldots, s\}$.

It is a general problem, which appears in many contexts (e.g. real solution of polynomial equations, ...) and with many applications. To cite a few of them: in combinatorics, network optimization design, control, ... See for instance [Las10].

To solve this NP hard problem, Lasserre [Las01] proposed to use two hierarchies of finite dimensional convex cones depending on an order $d \in \mathbb{N}$. The first hierarchy replaces non-negative polynomials by Sums of Squares (SoS) and non-negative polynomials on S by polynomials of degree $\leq d$ in the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$ generated by $\mathbf{g} = \{g_1, \ldots, g_s\}$. The second and dual hierarchy replaces positive measures by linear functionals $\in \mathcal{L}_d(\mathbf{g})$ which are non-negative on the polynomials of the truncated quadratic module $\mathcal{Q}_d(\mathbf{g})$. This condition is checked by testing positive semidefiniteness of (localized) Moment Matrices (MoM).

Hereafter $(\mathbb{R}[\mathbf{X}])^* = \hom_{\mathbb{R}}(\mathbb{R}[\mathbf{X}], \mathbb{R})$ denotes real valued linear funcionals on $\mathbb{R}[\mathbf{X}]$. We denote $\mathbb{R}[\mathbf{X}]_t$ the polynomials of degree $\leq t$, and $\sigma^{[t]}$ the restriction of the linear functional σ to $\mathbb{R}[\mathbf{X}]_t$.

*This extended abstract is based on the preprint [BM20], submitted for publication: we refer to it for the proofs and more details.

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Let $\Sigma^2 = \Sigma^2[\mathbf{X}] := \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N},\ g_i \in \mathbb{R}[\mathbf{X}] \colon f = g_1^2 + \cdots + g_r^2 \right\}$ be the convex cone of *Sum of Squares polynomials* (SoS). $Q \subset \mathbb{R}[\mathbf{X}]$ is called *quadratic module* if $1 \in Q$, $\Sigma^2 \cdot Q \subset Q$ and $Q + Q \subset Q$. A quadratic module Q is *Archimedean* if $\exists\ 0 \le r \in \mathbb{R} \colon r - \|\mathbf{X}\|^2 \in Q$. For a quadratic module Q, we define the *support* of Q as the ideal $\operatorname{supp} Q := Q \cap -Q$. For an ideal $I \subset \mathbb{R}[\mathbf{X}]$ we denote $\sqrt[\mathbb{R}]{I}$ its *real radical*: $\sqrt[\mathbb{R}]{I} := \{ f \in \mathbb{R}[\mathbf{X}] \mid \exists h \in \mathbb{N},\ s \in \Sigma^2 \quad f^{2h} + s \in I \}$.

We describe now the Lasserre SoS and MoM relaxations and we define the *exactness* property. These relaxations converge to f^* under the Archimedean hypothesis [Las01]. Hereafter we assume that the minimum f^* of the objective function f is always attained on S, that is: $S^{\min} := \{ x \in S \mid f(x) = f^* \} \neq \emptyset$.

We define the *SoS relaxation of order d* of problem (1) as $\mathcal{Q}_{2d}(\mathbf{g})$ and:

$$f^*_{\mathrm{SoS},d} := \sup \left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\}, \tag{2}$$

where $\mathcal{Q}_t(\mathbf{g}) := \left\{ s_0 + \sum_{j=1}^r s_j g_j \in \mathbb{R}[\mathbf{X}]_t \mid r \in \mathbb{N},\ g_j \in \mathbf{g},\ s_0 \in \Sigma^2_t,\ s_j \in \Sigma^2_{t - \deg g_j} \right\}$ is the truncated quadratic module generated by $\mathbf{g}$.

We define the *MoM relaxation of order d* of problem (1) as $\mathcal{L}_{2d}(\mathbf{g})$ and:

$$f^*_{\mathrm{MoM},d} := \inf \left\{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}_{2d}(\mathbf{g}),\ \langle \sigma | 1 \rangle = 1 \right\}, \tag{3}$$

where $\mathcal{L}_t(\mathbf{g}) = \{ \sigma \in (\mathbb{R}[\mathbf{X}]_t)^* \mid \forall q \in \mathcal{Q}_t(\mathbf{g}),\ \langle \sigma | q \rangle \ge 0 \}$ is the dual convex cone of $\mathcal{Q}_t(\mathbf{g})$.

Two questions arise naturally: if we can reach the minimum f^* for some order d of the relaxations, and in this case if sup = max in Equation (2). Notice that inf = min for all d in Equation (3), since $\mathcal{L}^{(1)}_d(\mathbf{g})$ is closed and $S^{\min} \neq \emptyset$. A remarkable type of minimizing linear functional are the evaluations at the minimizers: for $x^* \in S^{\min}$, the evaluation $\mathbf{e}_{x^*} \colon f \in \mathbb{R}[\mathbf{X}] \mapsto \langle \mathbf{e}_{x^*} | f \rangle = f(x^*)$ is such that $\mathbf{e}^{[2d]}_{x^*} \in \mathcal{L}_{2d}(\mathbf{g})$ and $\langle \mathbf{e}_{x^*} | f \rangle = f^*$.

Definition 1.1 (Finite Convergence)**.** We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ (resp. the MoM relaxation $\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ has the *Finite Convergence* property for f if $\exists k \in \mathbb{N}$ such that for every $d \ge k$, $f^*_{\mathrm{SoS},d} = f^*$ (resp. $f^*_{\mathrm{MoM},d} = f^*$).

Definition 1.2 (SoS Exactness)**.** We say that the SoS relaxation $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is *exact* for f if it has the finite convergence property and for all d big enough, we have $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ (in other words sup = max in the definition of $f^*_{\mathrm{SoS},d}$).

We can ask a stronger property for the MoM relaxation. We are interested, in particular, in the linear functionals that realize the minimum: $\mathcal{L}^{\min}_{2d}(\mathbf{g}) := \left\{ \sigma \in \mathcal{L}_{2d}(\mathbf{g}) \mid \langle \sigma | 1 \rangle = 1,\ \langle \sigma | f \rangle = f^* \right\}$.

Definition 1.3 (MoM Exactness)**.** We say that the MoM relaxation $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ is *exact* for f on the basic closed semialgebraic set S if:

- it has the finite convergence property;
- for every $k \in \mathbb{N}$ big enough, for $d = d(k) \in \mathbb{N}$ big enough, every truncated functional minimizer is coming from a probability measure supported on S, i.e. $\mathcal{L}^{\min}_{2d}(\mathbf{g})^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$.

MoM exactness may be considered as a particular instance of the so called *Moment Problem* (i.e. asking if $\sigma \in \mathbb{R}[\mathbf{X}]^*$ is coming from a measure) or more precisely of the *Truncated Strong Moment Problem* (i.e. considering truncated linear functionals and asking that the measure has a specified support).

Several works have been developed over the last decades to tackle these problems. Though many works focussed on the SoS relaxation and on the representation of positive polynomials with sums of squares, the MoM relaxation has been much less studied. It has interesting features, that deserve a deeper exploration: the convex cones $\mathcal{L}_d(\mathbf{g})$ of truncated non-negative linear functionals are closed; finite convergence can be decided by flat extension tests on moment matrices [CF98], [LM09]; finite minimizers can be extracted from moment matrices [HL05], [Mou18].

Exactness and finite convergence for the SoS and MoM are related, but many different situations may occur (we refer to [BM20; Nie13b; NDS06; DNP07; Mar06; Mar09; Sch00] for examples and details). There exist examples where nor the SoS nor the MoM have finite convergence (and thus they are not exact). Since $f^*_{\mathrm{SoS},d} \le f^*_{\mathrm{MoM},d}$, finite convergence for the SoS implies finite convergence for the MoM. The SoS and the MoM relaxations may have finite convergence but be not exact. There are examples where: the SoS and the MoM are exact; the SoS is exact and the MoM is not exact; the SoS is not exact and the MoM is exact.

Hereafter we investigate the truncated moment relaxation from a new perspective, developing a theoretical and computational study of truncated positive linear functionals. In Section 2 we analyse in details the properties of moment relaxations. We describe the kernel of truncated Hankel operators associated to generic positive linear functionals, and we present new results on the representation of moments of positive linear functionals as moments of measures. In Section 3 we apply the previous results to prove exactness: if the semialgebraic set is finite (section 3.1), and for generic objective function and constraints (section 3.2). We propose additional constraints to achieve exactness in a problem with singular minimizers (Section 3.3). Effective numerical computations illustrate this MoM exactness property (Section 4)

2 Geometry of Moments

We study the convex cones $\mathcal{L}_d(\mathbf{g})$ and, when the semialgebraic set is finite, we describe them in terms of evaluations. These results will be applied in Section 3 to prove exactness of the MoM relaxations.

By conic duality the (euclidean) closure $\overline{\mathcal{Q}_d(\mathbf{g})}$ of $\mathcal{Q}_d(\mathbf{g}) \subset \mathbb{R}[\mathbf{X}]_d \cong \mathbb{R}^{\binom{n+d}{d}}$ is equal to the dual convex cone of $\mathcal{L}_d(\mathbf{g})$. For the study of the MoM relaxation it is thus natural to study $\overline{\mathcal{Q}_d(\mathbf{g})}$ instead of $\mathcal{Q}_d(\mathbf{g})$.

Definition 2.1. Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. We define $\widetilde{Q} = \bigcup_d \overline{\mathcal{Q}_d(\mathbf{g})}$.

The description of $\widetilde{Q}$ given in the following theorem (and its truncated part) is important for the study of $\mathcal{L}_d(\mathbf{g})$.

Theorem 2.2. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module and let $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then $\widetilde{Q} = Q + J$ and $\operatorname{supp} \widetilde{Q} = J$. In particular, $\widetilde{Q}$ is a finitely generated quadratic module and does not depend on the particular choice of generators of Q.*

A first consequence is the description of the kernel of Moment Matrices (or Hankel operators) associated to *generic* linear functionals, i.e. linear functionals lying in the interior of $\mathcal{L}_d(\mathbf{g})$.

Definition 2.3. Let $\sigma \in \mathbb{R}[\mathbf{X}]^*_{2t}$. We define the *Hankel operator* $H^t_\sigma \colon \mathbb{R}[\mathbf{X}]_t \to \mathbb{R}[\mathbf{X}]^*_t$, $g \mapsto (g \star \sigma)^{[t]}$, where $\langle g \star \sigma | f \rangle := \langle \sigma | f g \rangle$.

Definition 2.4. We say that $\sigma^* \in \mathcal{L}_t(\mathbf{g})$ is *generic* if $\operatorname{rank} H^t_{\sigma^*} = \max\{\operatorname{rank} H^t_\eta \mid \eta \in \mathcal{L}_t(\mathbf{g})\}$.

Theorem 2.5. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then there exists $d, t \in \mathbb{N}$ such that for $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ generic, we have $J = (\ker H^t_{\sigma^*})$.*

The particular case of zero dimensional ideals was investigated in [Lau07], [LLR08], [Las+13]. In general, $\operatorname{supp} \widetilde{Q} = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ defines a variety strictly bigger than the Zariski closure of S, except when $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} \le 1$ (see [Mar08, cor. 7.4.2 (3)]) or when Q is a preordering (i.e. $Q \cdot Q \subset Q$, see [Mar08, p. 26]: it is the Real Nullstellensatz). One can work with preorderings substituting $\mathbf{g}$ with $\Pi\mathbf{g} := \prod_{j \in J} g_j \colon J \subset \{1, \dots, t\}$: we can then apply Theorem 2.5 to compute equations for the Zariski closure of S.

We focus on the case of a finite semialgebraic set S.

Theorem 2.6. *Suppose that* $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$*. Then,* $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\}$ *is non-empty and finite and there exists* $d \in \mathbb{N}$ *such that* $\forall k \in \mathbb{N}$*:*

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[2(\rho-1)+k]}.$$

where $\rho = \rho(\xi_1, \dots, \xi_r)$ *is the regularity of* S*.*

Theorem 2.6 says that all the truncated positive linear functionals are coming from evalations at points of S.

3 Applications to Polynomial Optimization

We apply Theorem 2.5 and Theorem 2.6 to prove exactess of MoM relaxations, when the semialgebraic set is finite and when a generic regularity condition at the minimizers is satisfied.

3.1 Finite semialgebraic set

We prove MoM exactness when $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} Q} = 0$.

Theorem 3.1. *Let* f^* *denote the infimum of* f *on* $S = \mathcal{S}(\mathbf{g})$ *and let* $Q = \mathcal{Q}(\mathbf{g})$*. Suppose that* $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} Q} = 0$*. Then the moment relaxation* $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ *is exact. For* $t \in \mathbb{N}$ *and* $d \geq \frac{t}{2}$ *big enough,*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_l})^{[t]},$$

where $\{\xi_1, \dots, \xi_l\} \subset \mathbb{R}^n$ *is the finite set of minimizers of* f *on* S*. Moreover, if* $d \geq t \geq \rho = \rho(\xi_1, \dots, \xi_l)$ *and* $\sigma \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ *is generic, then* $(\ker H_\sigma^t) = \mathcal{I}(\xi_1, \dots, \xi_l)$ *is the vanishing ideal of the minimizers* $\{\xi_1, \dots, \xi_l\}$ *of* f *on* S*.*

3.2 Boundary Hessian Conditions and Generic Exactness

The Boundary Hessian Conditions (introduced by Marshall in [Mar06] and [Mar09], see also [Sch09]) are regularity conditions of f and $\mathbf{g}$ at the minimizers. We show that if they hold we have exactness for the MoM relaxations. As a corollary we prove that MoM exactness is a generic property.

Definition 3.2 (Boundary Hessian Conditions)**.** Let $V \subset \mathbb{R}^n$ be a variety, and let Q be a (finitely generated) quadratic module such that $Q + \mathcal{I}(V)$ is Archimedean. Let $S = \mathcal{S}(Q) \cap V$ and $f \in \operatorname{Pos}(S)$. We say that the *Boundary Hessian Conditions* holds at $x \in \mathcal{V}(f) \cap S$ if there exists $t_1, \dots, t_m \in Q$ such that:

- $t_1, \dots, t_m$ are part of a regular system of parameters for V at x;
- $\nabla f(x) = a_1 \nabla t_1(x) + \cdots + a_m \nabla f(x)$, where a_i are strictly positive real numbers;
- the Hessian of f restricted to $\mathcal{V}(t_1, \dots, t_m) \cap V$ is positive definite at x.

Theorem 3.3. *Let* $f \in \mathbb{R}[\mathbf{X}]$*,* $Q = \mathcal{Q}(\mathbf{g})$ *be an Archimedean finitely generated quadratic module and assume that the BHC hold at every minimizer of* f *on* $S = \mathcal{S}(\mathbf{g})$*. Then the moment relaxation* $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ *is exact. For* $t \in \mathbb{N}$ *and* $d, e \geq \frac{t}{2}$ *big enough:*

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]} = \mathcal{L}_{2e}(\mathbf{g}, \pm(f - f^*))^{[t]} = \operatorname{conv}(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})^{[t]}.$$

where $\{\xi_1, \dots, \xi_r\}$ *is the finite set of minimizers of* f *on* S*. Moreover, if* $d \geq t \geq \rho(\xi_1, \dots, \xi_r)$ *and* $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ *is generic, then* $(\ker H_{\sigma^*}^t) = \mathcal{I}(\xi_1, \dots, \xi_r)$ *is the vanishing ideal of the minimizers of* f *on* S*.*

For generic $f \in \mathbb{R}[\mathbf{X}]_d$ and $g_1 \in \mathbb{R}[\mathbf{X}]_{d_1}, \dots, g_s \in \mathbb{R}[\mathbf{X}]_{d_s}$ the BHC hold, as Nie shows in [Nie14, th. 1.2]. Then exactness of the MoM relaxation is a generic condition.

In Example 4.1 BHC conditions hold and we effectively apply Theorem 3.3.

3.3 Gradient, KKT and Polar ideals

To achieve finite convergence and exactness we can add constraits to the initial problem. For global optimization we can consider the gradient equations (see [NDS06]). For constrained optimization we can consider Karush–Kuhn–Tucker (KKT) constraints, adding new variables (see [DNP07]) or projecting them to the variables $\mathbf{X}$ (Jacobian equations, see [Nie13a]). We shortly describe them.

Let $g_1,\dots,g_r,h_1,\dots,h_s \in \mathbb{R}[\mathbf{X}]$ defining $S = \mathcal{S}(\mathbf{g},\pm\mathbf{h})$, and let $f \in \mathbb{R}[\mathbf{X}]$ be the objective function. Let $\mathbf{\Lambda} = (\Lambda_1,\dots,\Lambda_r)$ and $\mathbf{\Gamma} = (\Gamma_1,\dots,\Gamma_s)$ be variables representing the *Lagrange multipliers* associated with $\mathbf{g}$ and $\mathbf{h}$. The *KKT constraints* associated to the optimization problem $\min f(x)\colon x \in \mathcal{S}(\mathbf{g},\pm\mathbf{h})$ are:

$$\begin{cases} \dfrac{\partial f}{\partial X_i} - \sum_{k=1}^{r} \Lambda_k^2 \dfrac{\partial g_k}{\partial X_i} - \sum_{j=1}^{s} \Gamma_j \dfrac{\partial h_j}{\partial X_i} = 0 & \forall i \\ \Lambda_k g_k = 0, \quad h_j = 0, \quad g_k \geq 0 & \forall j,k, \end{cases} \tag{4}$$

where the polynomials belong to $\mathbb{R}[\mathbf{X},\mathbf{\Gamma},\mathbf{\Lambda}]$.

A problem of this approach is that these are sufficient but not necessary for $x^* \in S$ being a minimizer (they are also necessary if Linear Independence Constraint Qualification holds at the minimizers, see [NW06, th. 12.1]). To solve this problem we define the *polar ideal*.

Definition 3.4. For $f,g_1,\dots,g_r,h_1,\dots,h_s \in \mathbb{R}[\mathbf{X}]$ as before, the *polar ideal* is defined as follows:

$$J := (\mathbf{h}) + \prod_{\{a_1,\dots,a_k\}\subset\{1,\dots r\}} \Big((g_{a_1},\dots,g_{a_k}) + \Big(\operatorname{rank}\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})\Big) < s+k+1\Big).$$

where $\Big(\Big(\operatorname{rank}\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})\Big) < l\Big)$ is the ideal generated by the $l\times l$ minors of the Jacobian matrix $\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})$. The generators of J besides $\mathbf{h}$ are the product of active constraints and the generators of rank ideals.

In this definition, we could replace the product of ideals by their intersection and the $l \times l$ minors of the Jacobian matrices by polynomials defining the same varieties.

We prove that every minimizer belongs to $\mathcal{V}_{\mathbb{R}}(J)$, and as an application of Theorem 3.1 we obtain that, if the polar variety is finite, then the MoM relaxation extended with the polar ideal is exact.

Lemma 3.5. *Let x^* be a minimizer of f on $S = \mathcal{S}(\mathbf{g},\pm\mathbf{h})$. Then $x^* \in \mathcal{V}_{\mathbb{R}}(J)$.*

Theorem 3.6. *Let $Q = \mathcal{Q}(\mathbf{g},\pm\mathbf{h})$ and $J = (\mathbf{h}')$ be the polar ideal, where $\mathbf{h}' \subset \mathbb{R}[\mathbf{X}]$ is a finite set of generators. If $\mathcal{V}_{\mathbb{R}}(J)$ is finite then the moment relaxation $(\mathcal{L}_{2d}(\mathbf{g},\pm\mathbf{h}'))_{d\in\mathbb{N}}$ is exact.*

The assumption in [NDS06], [DNP07] and [Nie13a] for finite convergence and SoS exactness are smoothness conditions or radicality assumptions on the associated complex variety. Our condition for MoM exactness is of a different nature, since it is on the finiteness of the real polar variety (see Example 4.2).

4 Examples

We give examples where we compute the minimum and the minimizers from an exact MoM relaxation, when BHC are satisfied (Example 4.1) and for the polar ideal (Example 4.2). Computations were performed with the Julia package `MomentTools.jl`[1] using the SDP solver `Mosek`, based on an interior point method.

Example 4.1 (Robinson form)**.** We find the minimizers of the Robinson form $f = x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4(y^2+z^2) - y^4(x^2+z^2) - z^4(x^2+y^2)$ on the unit sphere $h = x^2+y^2+z^2-1$. The Robinson

[1] https://gitlab.inria.fr/AlgebraicGeometricModeling/MomentTools.jl

polynomial has minimum $f^* = 0$ (globally and on the unit sphere), and the minimizers on $\mathcal{V}_{\mathbb{R}}(h)$ are:

$$\frac{\sqrt{3}}{3}(\pm 1, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(0, \pm 1, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, 0, \pm 1), \frac{\sqrt{2}}{2}(\pm 1, \pm 1, 0).$$

BHC are satisfied at every minimizer (see [Nie14, ex. 3.2]) and we can recover the minimizers by Theorem 3.3.

```
v, M = minimize(f, [h], [], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M)
```

Here $f^*_{\mathrm{MoM},5} \approx v = -1.27211 \cdot 10^{-7}$ and the minimizers with positive coordinates are (all the twenty minimizers are found):

$\xi_{\mathbf{x}}$:	0.577351068999	$8.812477930640\,10^{-12}$	0.707107158043	0.707107157553
$\xi_{\mathbf{y}}$:	0.577351069076	0.707107158048	$1.271729446125\,10^{-13}$	0.707107157555
$\xi_{\mathbf{z}}$:	0.577351066102	0.707107158048	0.707107158042	$2.478771201340\,10^{-9}$

Example 4.2 (Singular minimizer)**.** We minimize $f = x$ on the compact semialgebraic set $S = \mathcal{S}(x^3 - y^2, 1 - x^2 - y^2)$. The only minimizer is the origin, which is a singular point of the boundary of S. Thus BHC do not hold. The regularity conditions for the Jacobian and KKT constraints are not satisfied, but the real polar variety is finite. Adding the polar constraints, we have an exact MoM relaxation. We can recover an approximation of the minimizer from the MoM relaxation of order 5:

```
v, M = polar_minimize(f, [], [x^3-y^2,1-x^2-y^2], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M, 2.e-3)
```

The approximation of the minimum $f^* = 0$ is $v = -0.0045$, and the decomposition of the moment sequence with a threshold of $2 \cdot 10^{-3}$ gives the following approximation of the minimizer (the origin):

$$\xi = (-0.004514367348787526, 2.1341684460860045\,10^{-21}).$$

The error of approximation on the minimizer is of the same order than the error on the minimum f^*.

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