ALGEBRAIC AND PUISEUX SERIES SOLUTIONS OF SYSTEMS OF AUTONOMOUS ALGEBRAIC ODES OF DIMENSION ONE IN SEVERAL VARIABLES

JOSÉ CANO, SEBASTIAN FALKENSTEINER, AND J.RAFAEL SENDRA

Abstract. In this paper we show that all formal Puiseux series solutions of systems of autonomous algebraic ODEs of dimension one in several differential indeterminate are convergent in a certain neighborhood. Moreover, if the first component of a solution vector is an algebraic Puiseux series, we prove that the other components are algebraic as well. We present an algorithm for computing algebraic solution vectors, represented by its minimal polynomials, that uses polynomial factorization methods. As ongoing work we are dealing with the question of avoiding factorization and deciding which roots of a list of minimal polynomials indeed define a solution.

keywords Algebraic autonomous ordinary differential equation, formal Puiseux series solution, algebraic solutions, convergent solution, algebraic space curve.

1. Introduction

In [5] we have studied local solutions of first order autonomous algebraic ordinary differential equations (shortly AODEs). Therein we prove that every fractional power series solution of such equations is convergent, and an algorithm for computing these solutions is provided. In [4] we generalized these results to the case of systems of higher order autonomous AODEs in one unknown function which associated algebraic set is of dimension one, i.e. the algebraic set is a finite union of curves and, maybe, points. Here we show that every component of a fractional power series solution vector of a dimension one system of higher order autonomous AODEs in several unknown functions is convergent.

We follow the idea from [4] and triangularize the given system by using regular chains (see [3] and references therein) and derive for every solution component a corresponding equation by a simple differential elimination process.

In the case where the first solution component is an algebraic Puiseux series, the coefficients of the derived equations for the next solution components are algebraic and can be expressed in a closed form. This enables to compute for every component of a possible solution vector its minimal polynomials and check whether they indeed define a solution. These results can be also be seen as a generalization of [7] to algebraic solutions and several differential unknowns.

2. Systems of Dimension One

In this paper we consider systems of differential equations in several differential indeterminates, i.e. systems of the type

$$\widetilde{\mathcal{S}} = \{F_j(y_1, \dots, y_1^{(m_1-1)}, \dots, y_p, \dots, y_p^{(m_p-1)}) = 0\}_{1 \leq j \leq M}, \tag{2.1}$$

where $F_1, \dots, F_M \in \mathbb{C}[y_1, \dots, y_p^{(m_p-1)}]$ with $p \geq 1$ and $m_1 > 1, m_2, \dots, m_p \geq 1$. Let $m = m_1 + \cdots + m_p$ and let us denote the algebraic set implicitly defined by $\widetilde{\mathcal{S}}$ by

$$\mathbb{V}_{\mathbb{C}}(\widetilde{\mathcal{S}}) = \{(a_0, \dots, a_m) \in \mathbb{C}^{m+1} \mid F(a_0, \dots, a_m) = 0 \text{ for all } F \in \widetilde{\mathcal{S}}\}.$$

Throughout this paper we will assume that the dimension of $\mathbb{V}_{\mathbb{C}}(\widetilde{\mathcal{S}})$ is one.

Let $\mathbb{C}\langle\langle x\rangle\rangle$ be the field of formal Puiseux series expanded around any $x_0 \in \mathbb{C}_\infty$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the one-point compactification of the field of complex numbers. Since the considered equations are invariant under translation of the independent variable, we can assume without loss of generality that the formal Puiseux series are expanded around zero or at infinity such as in [4]. We are interested in non-constant formal Puiseux series solution vectors $(y_1(x), \dots, y_p(x)) \in \mathbb{C}\langle\langle x\rangle\rangle^p$ of (2.1) expanded around zero or at infinity such that $(y_1(x_0), \dots, y_p(x_0)) = (a_0, \dots, a_p) \in \mathbb{C}_\infty^p$. Additionally, we may assume that each component has non-negative order. Otherwise define $I \subseteq \{1, \dots, p\}$ as the set of indexes where the order is negative and perform the change of variable $\tilde{y}_i = 1/y_i$ in $\widetilde{\mathcal{S}}$ for every $i \in I$. It can be proven that the resulting system is again of dimension one and of the type (2.1).

For our reasonings it is essential to decompose the given system (2.1) into a (finite) union of regular chains [6]. In [8][Theorem 5.2.2] it is shown that the regular zeros of the given system of algebraic equations and that of its decomposition into regular sets are the same. By imposing that the first equation is square-free, this is also true in the case of differential systems of dimension one:

Lemma 2.1. *For every $\widetilde{\mathcal{S}}$ as in (2.1) we can compute a finite union of regular chains with the same solution vectors $(y_1(x), \dots, y_p(x)) \in (\mathbb{C}\langle\langle x\rangle\rangle \setminus \mathbb{C})^p$.*

Based on Lemma 2.1, the strategy of determining the solutions of (2.1) is to compute a regular chain decomposition with respect to the term ordering

$$y_1 < \cdots < y_1^{(m_1-1)} < \cdots < y_p < \cdots < y_p^{(m_p-1)},$$

which results in subsystems $\mathcal{S}$ of the form

$$(2.2)\qquad \begin{cases} G_1(y_1, y_1') = \sum_{j=0}^{r_1} G_{1,j}(y_1) \cdot (y_1')^j = 0 \\ G_2(y_1, y_1', y_1'') = \sum_{j=0}^{r_2} G_{2,j}(y_1, y_1') \cdot (y_1'')^j = 0 \\ \quad\vdots \\ G_{m_1}(y_1, \dots, y_1^{(m_1-1)}, y_2) = \sum_{j=0}^{r_{m_1}} G_{m_1,j}(y_1, \dots, y_1^{(m_1-1)}) \cdot y_2^j = 0 \\ \quad\vdots \\ G_{m-1}(y_1, \dots, y_p^{(m_p-1)}) = \sum_{j=0}^{r_{m-1}} G_{m-1,j}(y_1, \dots, y_p^{(m_p-2)}) \cdot (y_p^{(m_p-1)})^j = 0 \end{cases}$$

with $r_j \geq 1$ and $\mathrm{init}(G_j) = G_{j,r_j} \neq 0$ for every $1 \leq j \leq m$. Solution vectors with a constant term might not be covered in this decomposition, but they can be found by direct computations. Without loss of generality we also impose that in (2.2) the equation $G_1 = 0$ and its separant have no common (non-constant) differential solutions by considering $G_1 \in \mathbb{C}[y_1, y_1']$ to be square-free and with no factor in $\mathbb{C}[y_1]$ or $\mathbb{C}[y_1']$ (compare to [4]).

Let us set $m_0 = 1$. For a given system $\mathcal{S}$ as in (2.2) we will consider the subsystems

$$\mathcal{S}_j = \{G_{m_1+\cdots+m_{j-1}} = 0, \dots, G_{m_1+\cdots+m_j-1} = 0\}$$

involving only $y_1, \dots, y_j$ and its derivatives. Then we have the disjoint union

$$\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_p.$$

Let us set in the following $K_0 = \mathbb{C}(x)$ and let $y_1(x), y_2(x), \ldots \in \mathbb{C}\langle\langle x\rangle\rangle$. For $j \in \mathbb{N}^*$ we iteratively define K_j as the field obtained by adjoining to K_{j-1} the element $y_j(x)$, i.e.

$$K_j = K_{j-1}(y_j(x)).$$

Obviously the relation

$$\mathbb{C}(x) = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_j \subseteq \cdots \subseteq \mathbb{C}\langle\langle x\rangle\rangle$$

holds. Let us note that if $y_1(x), y_2(x), \ldots$ are algebraic over $\mathbb{C}(x)$, then $K_1, K_2, \ldots$ are algebraic extension fields of $\mathbb{C}(x)$.

Theorem 2.2. *All components of a formal Puiseux series solution vector of system (2.2), expanded around a finite point or at infinity, are convergent.*

Proof. First we consider in (2.2) the subsystem $\mathcal{S}_1$ involving only the differential indeterminate y_1. Since $\mathcal{S}_1$ is the first part of a regular chain and has dimension one, we can use Theorem 4 in [4] for $\mathcal{S}_1$ and hence, the first component $y_1(x)$ is convergent.

Let us now evaluate the equations in (2.2) at $y_1(x)$ and its derivatives and consider

$$P_2(x, y_2) = G_{m_1}(y_1(x), \ldots, y_1^{(m_1-1)}(x), y_2) = 0$$

with $P_2 \in K_1[y_2]$ and the coefficients of P_2 are convergent Puiseux series in x. Since (2.2) is a regular chain and non-constant solutions define a regular zero,

$$\operatorname{init}(G_{m_1})(y_1(x), \ldots, y_1^{(m_1-1)}(x)) = G_{m_1, r_{m_1}}(y_1(x), \ldots, y_1^{(m_1-1)}(x)) \neq 0$$

and therefore, P_2 is non-trivial in y_2. Then, by Puiseux's Theorem, all solutions $y_2(x)$ of $P_2 = 0$ are convergent.

We can continue this process up to $G_{m-m_p} = 0$ and its Puiseux expansion for $y_p(x)$ to obtain that all components of the solution vector $(y_1(x), \ldots, y_p(x))$ are convergent Puiseux series. □

In the proof of Theorem 2.2 we have constructed polynomials with coefficients in the field extensions K_j. In general these polynomials cannot be expressed in a closed form and therefore the roots can not be computed explicitly and a zero-check for testing whether the component extends to a solution vector is in general impossible. If $y_1(x)$ is algebraic over $\mathbb{C}(x)$, however, this can be done.

3. Algebraic Solutions

In this subsection we consider a subclass of formal Puiseux series, namely algebraic series. These are $y(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ such that there exists a non-zero polynomial $G \in \mathbb{C}[x, y]$ with $G(x, y(x)) = 0$. Note that since the field of formal Puiseux series is algebraically closed, all algebraic solutions can be represented as (formal) Puiseux series.

Based on the reasonings in [1], algebraic solutions of first order autonomous AODEs can be found algorithmically. Moreover, if there exists a non-constant algebraic solution then all solutions are algebraic and they have the same minimal polynomial up to a shift in the independent variable as we recall in the following theorems.

Theorem 3.1. *Let $F \in \mathbb{C}[y, y']$ be an irreducible polynomial with a non-constant solution $y(x) \in \mathbb{C}\langle\langle x \rangle\rangle$, algebraic over $\mathbb{C}(x)$. Then all non-constant formal Puiseux series solution of $F(y, y') = 0$ are algebraic over $\mathbb{C}(x)$.*

Theorem 3.2. *Let $F \in \mathbb{C}[y, y']$ be irreducible and let $y(x)$ be a non-constant solution of $F = 0$, algebraic over $\mathbb{C}(x)$ with minimal polynomial $G \in \mathbb{C}[x, y]$. Then all non-constant formal Puiseux series solutions of $F = 0$ are algebraic and given by $G(x + c, y)$, where $c \in \mathbb{C}$.*

Theorem 3.3. *Let $(y_1(x), \ldots, y_p(x)) \in \mathbb{C}\langle\langle x \rangle\rangle^p$ be a solution vector of system (2.2) such that $y_1(x)$ is algebraic over $\mathbb{C}(x)$. Then $y_2(x), \ldots, y_p(x)$ are algebraic over $\mathbb{C}(x)$.*

Proof. Let us use the notation introduced in the proof of Theorem 2.2. Let $Q_1(x, y_1) \in \mathbb{C}[x, y_1]$ be the minimal polynomial of $y_1(x) \in \mathbb{C}\langle\langle x \rangle\rangle$. Now $y_2(x)$ is a root of the non-zero polynomial

$$P_2(x, y_2) = G_{m_1}(y_1(x), \ldots, y_1^{(m_1-1)}(x), y_2) \in K_1[y_2].$$

By computing the derivative of $Q_1(x, y_1(x)) = 0$ with respect to x, we obtain

$$\frac{\partial Q_1}{\partial y_1}(x, y_1(x)) \cdot y_1'(x) + \frac{\partial Q_1}{\partial x}(x, y_1(x)) = 0. \tag{3.1}$$

Since $\frac{\partial Q_1}{\partial y_1} \in \mathbb{C}[x, y_1]$ is a non-zero polynomial of less degree than the minimal polynomial of $y_1(x)$, it follows that

$$\frac{\partial Q_1}{\partial y_1}(x, y_1(x)) \neq 0.$$

Hence, by (3.1), $y_1'(x) \in K_1$. By taking further derivatives of (3.1), it follows that also $y_1^{(2)}(x), \ldots, y_1^{(m_1-1)}(x) \in K_1$. Therefore,

$$P_2(x, y_2) = G_{m_1}(y_1(x), \ldots, y_1^{(m_1-1)}(x), y_2) \in K_1[y_2]$$

with $P_2(x, y_2(x)) = 0$. As we have seen in the proof of Theorem 2.2, P_2 effectively depends on y_2 and therefore, $y_2(x)$ is algebraic over $\mathbb{C}(x)$. By continuing this process iteratively, also the next solution components are algebraic over $\mathbb{C}(x)$ and the statement follows. □

We have assumed that $(y_1(x), \ldots, y_p(x))$ is indeed a solution vector of the original system $\mathcal{S}$, but this is unknown in advance. In the following we present an algorithmic procedure for restricting algebraic solutions to a finite number of solution candidates.

Based on Theorem 3.3, let us assume that $y_1(x), \ldots, y_{j-1}(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ are algebraic with minimal polynomials

$$Q_1(x, y_1) \in \mathbb{C}[x, y_1], \ldots, Q_{j-1}(x, y_{j-1}) \in \mathbb{C}[x, y_{j-1}]$$

and let us define

$$\tilde{P}_j(y_1, y_j) = \operatorname{Res}(G_{m_1+\cdots+m_{j-1}}, \{G_1, \ldots, G_{m_1+\cdots+m_{j-1}-1}\}),$$

where the resultant is computed consecutively with respect to the leading variable. Now set

$$P_j(x, y_j) = \operatorname{Res}_{y_1}(\tilde{P}_j, Q_1) \tag{3.2}$$

and let us denote by $\mathscr{Q}_j$ the set of irreducible factors of P_j in $\mathbb{C}[x, y_j]$. We call the elements in $\mathscr{Q}_j$ the *solution candidates* of $\mathcal{S}_j$.

Let us note that $\tilde{P}_j$, and therefore P_j, is non-zero and effectively depending on y_j since $\mathcal{S}$ is a regular chain (see [3][Theorem 21]) and Q_1 is irreducible. By a similar argumentation, the following condition for the existence of solution candidates can be derived.

Lemma 3.4. *Let $(Q_1, \ldots, Q_{j-1}) \in \mathbb{C}[x, y_1] \times \cdots \times \mathbb{C}[x, y_j]$ be the minimal polynomials of an algebraic solution vector of the subsystem $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{j-1}$ of (2.1) and let $Q_j \in \mathscr{Q}_j$ be a solution candidate of $\mathcal{S}_j$. There exists an algebraic solution vector of the subsystem $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_j$ with the minimal polynomials $(Q_1, \ldots, Q_j)$ if and only if the differential system*

$$\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_j \cup \{Q_1, \ldots, Q_j\}$$

is consistent.

Note that the consistency of the differential system in Lemma 3.4 can be decided by using the `Rosenfeld-Gröbner` algorithm (see [2]). Therefore, the above results can be used to construct an algorithm deciding whether systems of the type (2.1) have a solution formed by algebraic Puiseux series and in the affirmative case return their minimal polynomials $Q_1(x, y_1), \ldots, Q_p(x, y_p)$ (over the field of rational functions). Let us remark that in this case not all vectors $(y_1(x), \ldots, y_p(x))$, formed by roots of the minimal polynomials $Q_1, \ldots, Q_p$, are necessary solutions of the original system. In the case of rational solution vectors, however, every minimal polynomial Q_j has degree one in y_j and a unique root such that this problem does not occur.

There are several implementations for performing computations with regular chains and in particular computing regular chain decompositions of (2.1) such as in the `Maple`-package `RegularChains`. In [4] an algorithm to derive for the subsystem $\mathcal{S}_1$ a reduced differential equation $F(y, y') = 0$ is presented. Recently there has been implemented a `Maple`-package `FirstOrderSolve` (see *https://risc.jku.at/sw/firstordersolve/*) that includes an algorithm deciding whether $F = 0$ has algebraic solutions and computing them in the affirmative case. Using Theorem 3.3, where it is shown that if the subsystem $\mathcal{S}_1$ has an algebraic solution, then all other solution components are algebraic, and Lemma 3.4, where it is given a necessary and sufficient condition for solution candidates, all algebraic solutions of a given system (2.1) can be found among the solution candidates. Let us illustrate this in the following example.

Example 3.5. Let us consider the system of differential equations given by

$$\widetilde{\mathcal{S}} = \begin{cases} yy'y'' + y'^3 - yy'' - y'^2 = 0 \\ z^3 - 2y'^2 + yy' - 1 = 0 \\ z^3 + yy'' - y'^2 = 0 \\ 3z^2z' - 4y'y'' = 0 \end{cases} \tag{3.3}$$

The system $\widetilde{\mathcal{S}}$ has a regular chain decomposition

$$\mathcal{S} = \begin{cases} G_1 = yy' - 1 = 0 \\ G_2 = y'^2 + yy'' = 0 \\ G_3 = z^3 - 2y'^2 + yy' - 1 = 0 \\ G_4 = 3z^2z' - 4y'y'' = 0 \end{cases} \quad \text{and} \quad \begin{cases} y' - 1 = 0 \\ 2 - y + yy'' = 0 \\ z^3 + y - 3 = 0 \\ 3z^2z' - 4y'' = 0 \end{cases}$$

and systems where the first equation depends only on y.

The system on the right does not have any solution vector, because the reduced differential equation of $\{y' - 1 = 0, 2 - y + yy'' = 0\}$ is constantly 1.

The system $\mathcal{S}$ does not have a solution vector where the first component is constant. The subsystem $\mathcal{S}_1 = \{G_1 = 0, G_2 = 0\}$ of $\mathcal{S}$ has the reduced differential equation $G_1 = 0$ having the algebraic solutions given by the minimal polynomial

$$Q_1(x, y) = y^2 - 2(x + c),$$

where $c \in \mathbb{C}$.

Let us compute the solution candidates by considering the irreducible factors of

$$P_2(x, y_2) = \operatorname{Res}(G_3, \{Q_1, G_1, G_2\}) = (-2(x + c)z^3 + 2)^2,$$

which is

$$Q_2 = (x + c)z^3 - 1.$$

We check the necessary condition from Lemma 3.4 by computing the Rosenfeld-Gröbner decomposition of the system $\{G_1, G_2, G_3, Q_1, Q_2\}$ with the `Maple`-package `DifferentialAlgebra`:

$$[\; c\, z(x)^3 + x\, z(x)^3 - 1,\; -2\,x - 2\,c + y(x)^2 \;].$$

Since this is non-empty, the system $\mathcal{S}$ has at least one algebraic solution $(y(x), z(x))$ with minimal polynomials (Q_1, Q_2).

Acknowledgements

First author partially supported by MTM2016-77642-C2-1-P (AEI/FEDER, UE). Second and third authors partially supported by FEDER/Ministerio de Ciencia, Innovación y Universidades Agencia Estatal de Investigación/MTM2017-88796-P (Symbolic Computation: new challenges in Algebra and Geometry together with its applications). Second author also supported by the Austrian Science Fund (FWF): P 31327-N32. Third author is a member of the Research Group ASYNACS (Ref.CT-CE2019/683).

References

[1] Aroca, J., Cano, J., Feng, R., and Gao, X.-S. Algebraic General Solutions of Algebraic Ordinary Differential Equations. In *Proceedings of the 2005 international symposium on Symbolic and algebraic computation* (2005), ACM, pp. 29–36.

[2] Boulier, F., Lazard, D., Ollivier, F., and Petitot, M. Computing representations for radicals of finitely generated differential ideal. *Applicable Algebra in Engineering, Communicationand Computing 20* (2009), 73–121.

[3] Boulier, F., Lemaire, F., Poteaux, A., and Maza, M. M. An equivalence theorem for regular differential chains. *Journal of Symbolic Computation 93* (2019), 34–55.

[4] Cano, J., Falkensteiner, S., and Sendra, J. R. Algebraic, Rational and Puiseux Series Solutions of Systems of Autonomous Algebraic ODEs of Dimension One. *Mathematics in Computer Science* (2020).

[5] Cano, J., Falkensteiner, S., and Sendra, J. R. Existence and convergence of puiseux series solutions for first order autonomous differential equations. *Journal of Symbolic Computation* (2020).

[6] Chen, C., and Moreno Maza, M. Algorithms for Computing Triangular Decompositions of Polynomial Systems. In *Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation* (New York, NY, USA, 2011), ISSAC '11, Association for Computing Machinery, p. 8390.

[7] Lastra, A., Sendra, J., Ngo, L., and Winkler, F. Rational General Solutions of Systems of Autonomous Ordinary Differential Equations of Algebro-Geometric Dimension One. *Publicationes Mathematicae Debrecen 86* (2015), 49–69.

[8] Wang, D. *Elimination Methods*. Springer Science & Business Media, 2012.

Dpto. Algebra, análisis Matemático, Geometría y Topología, Universidad de Valladolid, Spain.
E-mail address: jcano@agt.uva.es

Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Austria.
E-mail address: falkensteiner@risc.jku.at

Dpto. de Física y Matemáticas, Universidad de Alcalá, Madrid, Spain.
E-mail address: rafael.sendra@uah.es