# Characterizing infinite graphs allowing flexible frameworks

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A planar framework, which is a graph together with a map of its vertices to the plane, is flexible if it allows a continuous deformation preserving the distances between adjacent vertices. It has been shown recently that a finite graph admits a flexible framework if and only if it has a so called NAC-coloring, which is an edge coloring with a condition on cycles. We extend this result to the case of frameworks with countable vertex sets.

A realizations of a graph in the plane is a map associating to each vertex a point in the plane such that adjacent vertices are mapped to distinct points. A graph together with one of its realizations is called a framework. If a framework can be continuously deformed in the plane keeping the distances between adjacent vertices, then we say that such framework is flexible, see Figure 1; otherwise, the framework is called rigid.

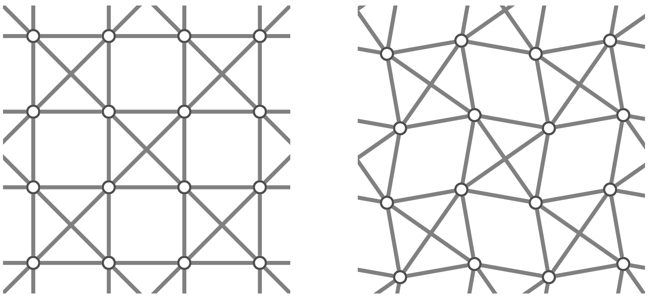

Figure 1: A flexible infinite periodic framework.

Rigidity and flexibility of finite frameworks are well-studied topics and plenty of results have been obtained so far. For example, finite rigid frameworks for which the realization

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is general (e.g. in which the coordinates of the points are algebraically independent) are combinatorially characterized by the so-called *Laman theorem* [9, 12], or via an iterative construction involving the so-called *Henneberg steps* [6].

To our knowledge, flexibility and rigidity of infinite graphs started to be investigated only recently [11], with particular interest regarding periodic frameworks [1, 10]. A Laman-type theorem for the rigidity of general infinite framework is proven in [7].

In the case of finite graphs, the existence of a flexible framework can be inferred from the existence of a particular edge 2-coloring of the graphs, called *NAC-coloring* [3]. Notice that such framework may well be not general. This technique was adapted to infinite periodic frameworks in [2].

In this paper, we show that the natural analogue of NAC-colorings for infinite graphs provides a combinatorial characterization of the existence of a flexible framework. More precisely, we show (Theorem 2.5):

**Theorem.** *A countably infinite connected graph admits a flexible framework if and only if it has a NAC-coloring.*

The proof of the main theorem relies on two different techniques. The "easy" implication is the construction of a flexible framework from a NAC-coloring: this is a straightforward extension of the analogue statement for finite fields. For the other implication, we consider an ascending tower of finite subgraphs that covers the whole graph and we recursively apply the known result about NAC-colorings for finite graphs. In particular, once a NAC-coloring is found for one subgraph of the tower, it can be extended to the next subgraph using properties of NAC-colorings described in [5].

Section 1 provides the needed background notions and results and Section 2 contains the proof of the main statement.

## 1 Preliminaries

We recall the notions of realizations, frameworks, and flexibility, which are usually given for finite graphs, but extend naturally to graphs with countably many vertices and edges.

**Definition 1.1.** Let $G$ be a connected graph, where $V_G$ is finite or countable. A map $\rho\colon V_G \to \mathbb{R}^2$ such that $\rho(u) \neq \rho(v)$ for all edges $uv \in E_G$ is a *realization*. The pair $(G, \rho)$ is called a *framework*. We define *induced edge lengths* $(\lambda_e)_{e\in E_G}$ by $\lambda_{uv} = ||\rho(u) - \rho(v)||$.

**Definition 1.2.** A *flex* of the framework $(G, \rho)$ is a continuous path $t \mapsto \rho_t$, $t \in [0, 1)$, in the space of realizations of $G$ such that $\rho_0 = \rho$ and each $(G, \rho_t)$ induces the same edge lengths as $(G, \rho)$. The flex is called *trivial* if for all $t \in [0, 1)$ there exists a Euclidean isometry $M_t$ of $\mathbb{R}^2$ such that $M_t\rho_t(v) = \rho(v)$ for all $v \in V_G$.

We define a framework to be *flexible* if there is a non-trivial flex in $\mathbb{R}^2$. Otherwise, it is called *rigid*.

We use algebraic tools to investigate necessary conditions for flexibility. Hence, we want to transform the information about the existence of a flex into existence of a positive-dimensional algebraic set consisting of realizations. Notice that this can be done easily, i.e., using just standard notions in algebraic geometry, only for finite graphs.

**Definition 1.3.** Let $H$ be a finite graph and let $(H, \rho)$ be a flexible framework with induced edge lengths $(\lambda_e)_{e \in E_H}$. Let $\bar{u}\bar{v}$ be an edge of $H$. We consider the following polynomial system in $\mathbb{C}^{2|V_H|}$ for unknown coordinates $(x_u, y_u)$ where $u \in V_H$:

$$\begin{aligned} x_{\bar{u}} &= 0\,, \quad y_{\bar{u}} = 0\,, \\ x_{\bar{v}} &= \lambda_{\bar{u}\bar{v}}\,, \quad y_{\bar{v}} = 0\,, \\ (x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda_{uv}^2 \quad \text{for all } uv \in E_H\,. \end{aligned} \tag{1}$$

Notice that the first four equations impose that for all realizations that satisfy (1), the vertices of the edge $\bar{u}\bar{v}$ have a fixed position. An irreducible complex algebraic curve $\mathcal{C}$ in the zero set of (1) is called an *algebraic motion of* $(H, \rho)$ *w.r.t.* $\bar{u}\bar{v}$. Notice that by a *curve* we mean any quasi-projective variety of dimension one, i.e., we allow Zariski open subsets of zero sets of polynomial equations.

Note that if $(H, \rho)$ has a flex, then the system (1) has infinitely many solutions and so $(H, \rho)$ admits an algebraic motion.

The algebraic technique employed in [3] proves that a finite graph has a flexible framework if and only if it admits a so-called NAC-coloring. We extend the notion of NAC-colorings to countable infinite graphs in a straightforward manner.

**Definition 1.4.** Let $G$ be a graph (finite or countable). A coloring of edges $\delta \colon E_G \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle* in $G$, either all edges have the same color, or there are at least two edges in each color. A NAC-coloring $\delta$ induces two subgraphs of $G$:

$$G^\delta_{\text{red}} = (V_G, \{e \in E_G \colon \delta(e) = \text{red}\}) \quad \text{and} \quad G^\delta_{\text{blue}} = (V_G, \{e \in E_G \colon \delta(e) = \text{blue}\})\,.$$

The set of NAC-colorings of $G$ is denoted by $\mathrm{NAC}_G$. See Figure 2 for an example.

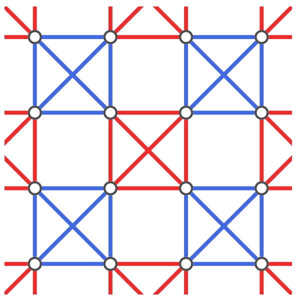

Figure 2: A NAC-coloring of the graph in Figure 1

The way how the existence of a flex determines a NAC-coloring is that NAC-colorings are constructed starting from valuations on the function field of an algebraic motion.

*Although the graph can be infinite, cycles are always finite.

**Definition 1.5.** Let $H$ be a finite graph and let $(H,\rho)$ be a flexible framework. Let $\mathcal{C}$ be an algebraic motion of $(H,\rho)$. The NAC-colorings constructed following [4, Definition 2.9] from valuations on the function field of $\mathcal{C}$ are called *active w.r.t.* $\mathcal{C}$, and we denote them by $\mathrm{NAC}_H(\mathcal{C})$.

Active NAC-colorings behave nicely under restriction to subgraphs.

**Lemma 1.6** ([5, Corollary 3.4])**.** *Let $\mathcal{C}$ be an algebraic motion of $(H,\rho)$ and $H'$ be a subgraph of $H$. If $\mathcal{C}'$ is the algebraic motion obtained by the projection of $\mathcal{C}$ to the vertices of $H'$, then*

$$\mathrm{NAC}_{H'}(\mathcal{C}') = \{\delta|_{E_{H'}} \in \mathrm{NAC}_{H'} \colon \delta \in \mathrm{NAC}_H(\mathcal{C})\}\,.$$

*In particular, a NAC-coloring of $H'$ active w.r.t. $\mathcal{C}'$ can be extended to a NAC-coloring of $H$ active w.r.t. $\mathcal{C}$.*

## 2 Main result

In this section, we prove the main result of the paper, namely Theorem 2.5. As for the finite case, the fact that the existence of a NAC-coloring implies the existence of a flexible framework is shown by a concrete construction. The idea for the opposite implication is the following: we consider an ascending tower of finite subgraphs. We construct a sequence of algebraic motions for these subgraphs so that they project to each other. Using Lemma 1.6, we get a chain of NAC-colorings, which can be used to define a NAC-coloring of the original graph.

The following statement was proved for the case of finite graphs in [3, Theorem 3.1]. The same idea of the proof works also in the infinite case.

**Proposition 2.1.** *Let $G$ be a countably infinite graph. If $G$ has a NAC-coloring, then there is a flexible framework $(G,\rho)$.*

*Proof.* Let $\delta$ be a NAC-coloring of $G$. Let $R_1, R_2, \ldots$ be the sets of vertices of connected components of the graph $G^\delta_{\text{red}}$ and $B_1, B_2, \ldots$ be the sets of vertices of connected components of the graph $G^\delta_{\text{blue}}$. Let $r_1, r_2 \ldots$ and $b_1, b_2, \ldots$ be pairwise distinct vectors in $\mathbb{R}^2$ such that $r_1 = b_1 = (0,0)$. For $\alpha \in [0,2\pi)$, we define a map $\rho_\alpha \colon V_G \to \mathbb{R}^2$ by

$$\rho_\alpha(v) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \cdot b_j + r_i\,,$$

where $i$ and $j$ are such that $v \in R_i \cap B_j$. We show that the induced edge lengths $\lambda_\alpha$ are are the same for all $\alpha$. We have

$$\lambda_\alpha(uv) = \|\rho_\alpha(u) - \rho_\alpha(v)\| = \left\| \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \cdot (b_j - b_l) + (r_i - r_k) \right\|$$

for all edges $uv \in E_G$ , where $i,j,k,l$ are such that $u \in R_i \cap B_j$ and $v \in R_k \cap B_l$. If $uv$ is red, then $i = k$ and we have

$$\lambda_\alpha(uv) = \left\| \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \cdot (b_j - b_l) \right\| = \|b_j - b_l\|\,.$$

The length $\lambda_\alpha(uv)$ is non-zero, otherwise $j = l$, namely, $u$ and $v$ are in the same blue components, which is not possible as there would be a cycle with one red edge and all others blue. If $uv$ is blue, then $j = l$ and hence

$$\lambda_\alpha(uv) = \|r_i - r_k\|\,.$$

The length $\lambda_\alpha(uv)$ is non-zero by an analogous reason as above. To conclude, the framework $(G, \rho_0)$ is flexible since $\alpha \mapsto \rho_\alpha$ is a non-trivial flex by surjectivity of $\delta$. ☐

We specify what we mean by a tower of subgraphs.

**Definition 2.2.** Let $G$ be a connected graph. A sequence $(G_i)_{i\geq 0}$ of induced finite connected subgraphs of $G$ is a *subgraph tower* in $G$ if $G_i$ is a proper subgraph of $G_{i+1}$ for all $i \geq 0$, and $\bigcup_{i\geq 0} V_{G_i} = V_G$.

Notice that for us, differently as for [7], all subgraphs in a subgraph tower are induced and the tower is vertex spanning by definition.

We will need the following algebro-geometric statement to construct a tower of subgraphs with a sequence of algebraic motions.

**Lemma 2.3.** *Let $X \subseteq \mathbb{C}^m$ and $Y \subseteq \mathbb{C}^n$, with $m, n \in \mathbb{N}$, be quasi-projective irreducible varieties and suppose that $Y$ is a curve. Let $\pi\colon X \to Y$ be a dominant regular map. Then there exists an irreducible curve $D \subseteq X$ such that $\pi|_D\colon D \to Y$ is dominant.*[†]

*Proof.* Let $k := \dim(X)$ and let $L$ be a $(m-k+1)$-dimensional affine space in $\mathbb{C}^m$. Let $G$ be the group of affinities of $\mathbb{C}^m$. Then by Kleiman's transversality theorem [8, Theorem 2], there exists an open dense subset $U \subseteq G$ such that $g \cdot L$ intersects $X$ in a curve for every $g \in U$. Let $F$ be a general fiber of $\pi$, of dimension $k-1$. Again by Kleiman's theorem, there exists an open dense subset $V \subseteq G$ such that $g \cdot L$ intersects $F$ in finitely many points for every $g \in V$. We now pick any element $g \in U \cap V$ and define $D' := (g \cdot L) \cap X$. Then $D'$ is a curve, and $\pi|_{D'}\colon D' \to C$ is dominant because by construction it is a map between curves and one of its fibers is finite. If $D'$ is irreducible, we take $D := D'$; otherwise, there is a component of $D'$ such that the restriction of $\pi$ is still dominant, and we take that component as $D$. ☐

**Lemma 2.4.** *Let $(G, \rho)$ be a countably infinite flexible framework. There is a subgraph tower $(G_i)_{i\geq 0}$ and a sequence $(\mathcal{C}_i)_{i\geq 0}$ such that for all $i \geq 0$, the curve $\mathcal{C}_i$ is an algebraic motion of $(G_i, \rho|_{V_{G_i}})$ and the projection $\pi_i\colon \mathcal{C}_i \to \mathcal{C}_{i-1}$ to the coordinates of vertices of $G_i$ is a dominant map.*

*Proof.* Let $G = (V_G, E_G)$ be a graph admitting a flexible framework and $\bar{u}\bar{v}$ be an edge of $G$. Let $f\colon [0,1) \to (\mathbb{R}^2)^{V_G}$ be the corresponding non-trivial flex. Without loss of generality, we can assume that the flex $f$ is constant on $\bar{u}\bar{v}$. Since the flex is non-trivial, there exists a vertex $w$ of $G$ for which the flex is nonconstant. Let $G_0$ be an induced connected subgraph of $G$ containing all $\bar{u}$, $\bar{v}$, and $w$. Fix any subgraph tower $(G_i)_{i\geq 0}$

[†]We thank Valentina Beorchia for useful discussions about the proof of this lemma.

starting from $G_0$. For any $i \geq 0$, let $f_i$ be the composition of the flex $f$ with the projection $(\mathbb{R}^2)^{V_G} \to (\mathbb{R}^2)^{V_{G_i}}$, which is hence a flex of $G_i$. Let

$$\pi_i \colon \mathbb{C}^{2|V_{G_i}|} \to \mathbb{C}^{2|V_{G_{i-1}}|}$$

be the natural projection that forgets the coordinates corresponding to the vertices of $V_{G_i} \setminus V_{G_{i-1}}$. By construction, we have $f_{i-1} = \pi_i \circ f_i$. Let $\rho_i$ be the realization $f_i(0)$. For any $i \geq 0$, let $F_i$ be the Zariski closure of $f_i([0,1))$ in the variety defined by (1) for $G_i$. By construction, all the $F_i$ are positive-dimensional, and the restrictions $\pi_i|_{F_i} \colon F_i \to F_{i-1}$ are dominant. In fact, since in general for a continuous function $g \colon A \to B$ and $Z \subseteq A$ we have $f(\overline{Z}) \subseteq \overline{f(Z)}$, we have

$$\pi_i(F_i) = \pi_i(\overline{f_i([0,1))}) \subseteq \overline{\pi_i(f_i([0,1)))} = F_{i-1}$$

and

$$\pi_i(F_i) \supseteq f_{i-1}([0,1)) \quad \text{hence} \quad \overline{\pi_i(F_i)} \supset \overline{f_{i-1}([0,1))} = F_{i-1}\,.$$

We refine our sequence of varieties $(F_i)_{i \geq 0}$ in such a way that the restrictions of the projections $\pi_i$ become surjective. To do so, for every $i > j$ define $\eta_{i \to j}$ to be the composition $\pi_j \circ \cdots \circ \pi_i$. Let $T_{i \to j}$ to be the image of $F_i$ under $\eta_{i \to j}$. Define

$$U_j := \bigcap_{i > j} T_{i \to j}\,.$$

By construction, each $U_j$ contains $f_j([0,1))$ because every $T_{i \to j}$ does so for $i > j$. Therefore, all the varieties in the sequence $(U_j)_{j \geq 0}$ are positive dimensional and by construction the restrictions $\pi_j|_{U_j} \colon U_j \to U_{j-1}$ are surjective.

Hence, there exists an algebraic motion $\mathcal{C}_0$ w.r.t. the edge $\bar{u}\bar{v}$ of the framework $(G_0, \rho_0)$ contained in $U_0$. Now, let $\mathcal{D}_1$ be an irreducible component of $\left(\pi_1|_{U_1}\right)^{-1}(\mathcal{C}_0)$ such that $\pi_1|_{\mathcal{D}_1} \colon \mathcal{D}_1 \to \mathcal{C}_0$ is dominant. We apply Lemma 2.3 to this situation and we obtain an algebraic motion $\mathcal{C}_1$ of the framework $(G_1, \rho_1)$ contained in $\mathcal{D}_1 \subseteq U_1$ that projects dominantly on $\mathcal{C}_0$. We now proceed inductively and we construct a sequence of algebraic motions with the desired properties. $\square$

Finally, we can prove the main result.

**Theorem 2.5.** *A countably infinite connected graph admits a flexible framework if and only if it has a NAC-coloring.*

*Proof.* The "if" part of the statement is Proposition 2.1. Hence, we are left to prove the "only if" part of the statement. Consider then a countably infinite connected graph $G = (V_G, E_G)$ admitting a flexible framework $(G, \rho)$. Using Lemma 2.4, let $(G_i)_{i \geq 0}$ be a subgraph tower of the graph $G$ with a sequence of algebraic motions $(\mathcal{C}_i)_{i \geq 0}$. We construct a sequence $(\delta_i)_{i \in \mathbb{N}}$ of NAC-colorings such that $\delta_i$ is a NAC-coloring of $G_i$ and the restriction of $\delta_{i+1}$ to $E_{G_i}$ is $\delta_i$. We proceed inductively. Let $\delta_0$ be an active NAC-coloring of $G_0$ w.r.t. $\mathcal{C}_0$. Such NAC-coloring exists by [3, Theorem 3.1]; see also [4, Theorem 2.8]. By Lemma 1.6, if $\delta_i$ is an active NAC-coloring of $G_i$ w.r.t. $\mathcal{C}_i$, then there exists an active NAC-coloring $\delta_{i+1}$ of $\mathcal{C}_{i+1}$ such that the restriction of $\delta_{i+1}$ to the edges of $G_i$ is $\delta_i$.

This sequence of NAC-colorings gives a well-defined coloring for $G$, since each edge in $G$ is eventually an edge of a subgraph in the tower. Such a coloring is a NAC-coloring since every cycle of $G$ is contained in some $G_j$ in the subgraph tower. This completes the proof. □

## References


[1] Ciprian S. Borcea and Ileana Streinu. Periodic frameworks and flexibility. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2121):2633–2649, 2010. doi:10.1098/rspa.2009.0676.

[2] Sean Dewar. Flexible placements of periodic graphs in the plane, 2019. arXiv:1911.05634.

[3] Georg Grasegger, Jan Legerský, and Josef Schicho. Graphs with Flexible Labelings. *Discrete & Computational Geometry*, 62(2):461–480, 2019. doi:10.1007/s00454-018-0026-9.

[4] Georg Grasegger, Jan Legerský, and Josef Schicho. Graphs with Flexible Labelings allowing Injective Realizations. *Discrete Mathematics*, 343(6):Art. 111713, 2020. doi:10.1016/j.disc.2019.111713.

[5] Georg Grasegger, Jan Legerský, and Josef Schicho. On the Classification of Motions of Paradoxically Movable Graphs. *Journal of Computational Geometry*, 11(1):548–575, 2020. arXiv:2003.11416, doi:10.20382/jocg.v11i1a22.

[6] L. Henneberg. Die graphische Statik der starren Körper. In *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, volume IV, pages 345–434. 1903.

[7] Derek Kitson and Stephen C. Power. The Rigidity of Infinite Graphs. *Discrete & Computational Geometry*, 60(3):531–557, 2018. doi:10.1007/s00454-018-9993-0.

[8] Steven L. Kleiman. The transversality of a general translate. *Compositio Mathematica*, 28(3):287–297, 1974. URL: http://eudml.org/doc/89215.

[9] G. Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, 4:331–340, 1970. doi:10.1007/BF01534980.

[10] Justin Malestein and Louis Theran. Generic combinatorial rigidity of periodic frameworks. *Advances in Mathematics*, 233(1):291 – 331, 2013. doi:10.1016/j.aim.2012.10.007.

[11] J. C. Owen and Stephen Power. Infinite bar-joint frameworks, crystals and operator theory. *New York Journal of Mathematics*, 17:445–490, 2011.

[12] H. Pollaczek-Geiringer. Über die Gliederung ebener Fachwerke. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, 7:58–72, 1927. doi:10.1002/zamm.19270070107.

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