ON THE REGULARITY OF CACTUS SCHEMES

EXTENDED ABSTRACT

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This has to be considered as a work in progress. It has not been submitted to arXiv nor to any journal and we plan to continue this work during the next months.

Abstract. In this work we employ the generalized additive decomposition (GAD) of a homogeneous polynomial F of degree d, as introduced by Iarrobino-Kanev. We prove that the scheme naturally associated to any GAD of F is regular in degree d. As a consequence, we show that for any such F, there always exists at least a 0-dimensional scheme evincing the cactus rank of F that is regular in degree d. This leads to a remarkable improvement of a cactus decomposition algorithm.

1. Motivation

Establishing the regularity degree of a given algebra is a topic of weighty interest for both algebraic geometry and commutative algebra. In a nutshell, for a graded commutative algebra $\mathcal{A}$ it amounts to determining the least positive integer after which the Hilbert function of $\mathcal{A}$ agrees with its Hilbert polynomial.

In this work, we address this challenging problem for the graded algebra $\mathcal{A}$ to be the quotient ring of a zero-dimensional scheme Z whose ideal is contained in the apolar ideal of a homogeneous polynomial F, see [IK99, Ger96]. In that case, we say that Z is apolar to F. We recall that the latter algebras have been largely studied in the commutative algebra framework also due to the Macaulay's Theorem, which states that all artinian Gorenstein algebras are isomorphic to the quotient ring of an apolar ideal, see [IK99, Lemma 2.14].

In the case of zero-dimensional schemes, the Hilbert polynomial is equal to the scalar given by the degree of the scheme. The minimal degree of an apolar scheme to a given polynomial F was classically introduced by Iarrobino and Kanev as "scheme length" of F; see [IK99, Definition 5.1]. Nowadays, it is well-known as "cactus rank" of F. This notion have been popularized mainly by Buczyńska and Buczyńsky because its wide interest, e.g., see [BB14]. Hence, a natural question is the following.

Question 1. *Let Z be a cactus scheme of a homogeneous polynomial F of degree d. Is Z regular in degree d, i.e. does the Hilbert function of Z stabilize to*

$$\operatorname{codim} I(Z)_d = \dim(\Bbbk[\mathbf{x}]/I)_d = \deg(Z)?$$

We provide a partially positive answer to this question as a consequence of our main result (Theorem 3.5), which shows that when Z evinces a generalized additive decomposition (1) for a degree-d polynomial, then it is regular in degree d. Such a decomposition, which has been already explored in the literature [IK99, BBM14, BT20], is always evinced by at least one among the apolar schemes of minimal size, i.e. among the cactus schemes of F by [BBM14, Theorem 3.7].

This partial result has already consequences of practical interest for the execution of algorithms that share the task of testing different bases for a given quotient space $\Bbbk[x_1, \dots, x_n]/I$, where I is a prescribed zero-dimensional ideal. As a concrete example, for achieving a symbolic computation of a generalized additive decomposition for a given polynomial F, in [BT20] the ideal I arises from the kernel of a Hankel operator associated to F. In such cases, the known bound on the degrees of elements in a basis of $\Bbbk[x_1, \dots, x_n]/I$ was not sharp in all but the "easy" cases (i.e. those corresponding to Waring decompositions). The current work shows that the degree of these elements may always be assumed to be not greater than $\deg F$, which remarkably reduces the number of bases to be tested during the algorithm execution.

A complete solution to Question 1 is still a line of open research: both providing a positive answer or an instance of a non-regular scheme computing the cactus rank of some polynomial are considered noteworthy results.

This extended abstract is organized as follows. In Section 2 the standard notation and results employed in the present work are recalled. Afterwards, in Section 3 generalized additive decompositions are discussed and the main theorem is derived. Finally, an application of such result to the decomposition algorithm of [BT20] is presented in Section 4.

2. Regularity of apolar schemes

Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let $\mathcal{S} = \Bbbk[x_0, \dots, x_n] = \bigoplus_{d \geq 0} \mathcal{S}_d$ be the standard graded polynomial ring with coefficients in $\Bbbk$, i.e., $\mathcal{S}_d$ is the $\Bbbk$-vector space of degree-d homogeneous polynomials, or forms.

Consider the dual space $\mathcal{S}^* \cong \Bbbk[y_0, \dots, y_n]$ and the *apolar action* of $\mathcal{S}^*$ on $\mathcal{S}$ by partial derviatives, namely

$$\circ : \mathcal{S}^* \times \mathcal{S} \to \mathcal{S}, \quad (H, G) \mapsto H(\partial_0, \dots, \partial_n)G.$$

Definition 2.1. *Given $F \in \mathcal{S}$, the **apolar ideal** of F is*

$$F^\perp = \left\{ H \in \mathcal{S}^* \mid H \circ F = 0 \right\}.$$

*A 0-dimensional scheme $Z \subset \mathbb{P}\mathcal{S}_1$ is said to be **apolar to** F if $I(Z) \subset F^\perp$. The **cactus rank** of F is the minimum degree of an apolar scheme of F. We call **cactus scheme** a scheme apolar to F which computes its cactus rank.*

Given any $F \in \mathcal{S}_d$, we regard it as a functional in $\mathcal{S}_d^*$ defined by

$$F^* : \mathcal{S}_d \to \Bbbk, \quad G \mapsto F \circ G.$$

In order to study the regularity of a cactus scheme, we consider the *Veronese embedding*

$$\nu_d : \mathbb{P}\mathcal{S}_1 \to \mathbb{P}\mathcal{S}_d^*, \quad [L] \to [(L^d)^*].$$

Note that, if $L = \xi_0 x_0 + \ldots + \xi_n x_n \in \mathcal{S}_1$, then $(L^d)^* = d!\mathrm{ev}_\xi \in \mathcal{S}_d^*$ where ev_ξ is just the map of evaluation at the point $\xi = (\xi_0, \dots, \xi_n) \in \Bbbk^{n+1}$.

Lemma 2.2 (Apolarity Lemma, [IK99, Ger96])**.** *Let Z be a scheme in $\mathbb{P}\mathcal{S}_1$ defined by the ideal $I(Z) \subset \mathcal{S}^*$ and let $F \in \mathcal{S}_d$. Then, the following are equivalent:*

(1) $[F^*] \in \langle \nu_d(Z) \rangle$;

(2) $I(Z) \subset F^\perp$.

From the Apolarity Lemma, we have that codim $I(Z)_d = \dim\langle \nu_d(Z) \rangle + 1$. Hence, Question 1 can be rephrased in more geometric terms asking whether $\dim\langle \nu_d(Z) \rangle = \deg(Z) - 1$.

3. Generalized additive decompositions

By [BBM14, Theorem 3.7], we know that there exists a cactus scheme which *computes a generalized additive decomposition.*

Definition 3.1. *Let $F \in \mathcal{S}_d$ and let $L_1, \dots, L_s \in \mathcal{S}_1$. A **generalized additive decomposition** (GAD) of F **supported at** $(L_1, \dots, L_s)$ is an expression*

$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i, \quad \text{where } 0 \leq k_i \leq d, \text{ for all } i \in \{1, \dots, s\}, \tag{1}$$

where L_i does not divide G_i, for each $i \in \{1, \dots, s\}$. We also assume for every $i \neq j$ that $L_i \neq L_j$.

Following [BBM14, BJMR18], we associate a 0-dimensional scheme to any GAD as (1).

Let $L \in \mathcal{S}_1$ be a linear form. We write $\mathcal{S}^L := \mathcal{S}/(L-1)$ for the quotient ring and $\pi_L : \mathcal{S} \twoheadrightarrow \mathcal{S}^L$ for the de-homogenization map with respect to L. Given a form $F \in \mathcal{S}_d$, we consider the de-homogenization $f := \pi_L(F) \in \mathcal{S}^L$. We construct the scheme defined by the apolar ideal of f

$$Z_{F,L} := V(f^{\perp}).$$

By construction, it is a scheme in the affine chart $\mathrm{Spec}(\mathcal{S}^L)$ supported at the origin. Hence, by homogenizing again the ideal with respect to the linear form L, we obtain a 0-dimensional scheme of $\mathbb{P}\mathcal{S}_1$ supported at $[L]$. With a little abuse of notation, we still denote it by $Z_{F,L}$ and we call it the **natural scheme apolar to F at L.**

Lemma 3.2. [BJMR18, Corollary 4] *The scheme $Z_{F,L}$ is apolar to F.*

Definition 3.3. *Given a GAD as in (1) we consider the scheme*

$$Z = Z_1 \cup \ldots \cup Z_s, \quad \text{with } Z_i = Z_{L_i^{d-k_i}G_i, L_i}.$$

*We say that Z evinces the GAD. The **size** of the GAD is*

$$\sum_{i=1}^{s} \deg(Z_i).$$

*The **generalized rank** of F is the smallest size of a GAD of F. We denote it* $\mathrm{grk}(F)$.

We recall the following well-known fact, which constitutes the local formulation of our main result.

Lemma 3.4. *Let $F \in \mathcal{S}_d$ and Z is a local scheme evincing a GAD of F, then Z is regular in degree d.*

Proof. As observed in [BJMR18, Remark 3], the linear span of $\nu_d(Z_{F,L})$ in $\mathbb{P}\mathcal{S}_d^*$ can be described as follows: let $D_L := L^{\perp} \cap \mathcal{S}_1^*$ and D_L^e its e-th symmetric power, then

$$\langle \nu_d(Z_{F,L}) \rangle = \mathbb{P} \bigoplus_{e=0}^{d} \left(L^e \left(D_L^e \circ F \right) \right)^* \subset \mathbb{P}\mathcal{S}_d^*.$$

where $D_L^e \circ F$ is the $\Bbbk$-vector space of degree-$(d-e)$ partial derivatives of F. Now, if $F = L^{d-k}G$, with L that does not divide G, we look at the scheme $Z_{F,L}$. If $H \in D_L$, then $H \circ F = L^{d-k}(H \circ G)$. Therefore, we have that

$$\langle \nu_d(Z_{F,L}) \rangle = \mathbb{P} \bigoplus_{e=0}^{k} \left(L^{d-k+e} \left(D_L^e \circ G \right) \right)^* \subset \mathbb{P}\mathcal{S}_d^*; \tag{2}$$

in particular, since $\deg(G) = k \leq d$, we have that $Z_{F,L}$ is contained in the k-fat point supported at $[L]$. Since a k-fat point, and therefore all its subschemes, is regular in degree $\geq k-1$, we have that $\deg Z_{F,L} - 1 = \langle \nu_d(Z_{F,L}) \rangle$. □

By exploiting the above lemma, we derive the general result.

Theorem 3.5. *Let $F \in \mathcal{S}_d$ and Z is a scheme evincing a GAD, then Z is regular in degree d.*

Proof. Consider a GAD of $F \in \mathcal{S}_d$ as in (1) and let Z be the scheme evincing it as in Definition 3.3. For each $i \in \{1, \ldots, s\}$, we consider the vector space

$$W_i := \bigoplus_{e=0}^{k} \left(L_i^{d-k+e} (D_{L_i}^e \circ G_i) \right)^* \subset \mathcal{S}_d^*.$$

Hence, we want to prove that $W_i \cap \sum_{j \neq i} W_j = \varnothing$. Indeed, from that, we deduce

$$\dim \langle W_1, \ldots, W_r \rangle = \sum_{i=1}^{s} \dim W_i = \sum_{i=1}^{s} \deg(Z_i)$$

where the last equality follows from Lemma 3.4.

In order to prove that $W_i \cap \sum_{j \neq i} W_j = \varnothing$, we work in $\mathcal{S}^*$ and we consider, for each $i \in \{1, \dots, s\}$,

$$\bar{W}_i = \bigoplus_{e=0}^{k} \mathrm{ev}_{\xi_i} \cdot (D^e_{L_i} \circ G_i)^* \subset \mathcal{S}^*$$

where $\xi_i = (\xi_{i,0}, \dots, \xi_{i,n}) \in \Bbbk^{n+1}$ if $L_i = \xi_{i,0}x_0 + \dots + \xi_{i,n}x_n$, and, with a slight abuse of notation, we use the notation F^* to indicate the map

$$F^* : \mathcal{S} \to \mathcal{S}, \quad G \mapsto F \circ G.$$

Since $(\mathrm{ev}_{\xi_i})_{|\mathcal{S}^*_d} = d!(L_i^d)^*$, we have that $\bar{W}_i \cap \mathcal{S}_d = W_i$. Hence, it is enough to show $\bar{W}_i \cap \sum_{j \neq i} \bar{W}_j = \varnothing$. For $a \in \mathcal{S}_1$, we consider

$$M_a : \mathcal{S}^* \to \mathcal{S}^*, \quad G \mapsto M_a(G) = (H \mapsto G \circ (aH))$$

By [BT20, Theorem 6.2], the elements of $\bar{W}_i$ are generalized eigenvectors for M_{x_j} with eigenvalue $\xi_{i,j}$. Hence, we consider

$$E_i = \bigcap_{j=0}^{n} E_j[\xi_{i,j}]$$

where $E_j[\mu]$ is the generalized eigenspace of M_{x_j} with eigenvalue μ.

Finally, we observe that $E_i \cap \sum_{j \neq i} E_j \neq \varnothing$.

We proceed by induction on s. Let $s = 2$ and assume by contradiction that there exists a non-zero $v \in E_1 \cap E_2$. Thus, for each $j \in \{1, \dots, s\}$, we have that $v \in E_j[\xi_{1,j}] \cap E_j[\xi_{2,j}]$. Hence, v is an eigenvector for M_{x_j} with eigenvalue $\xi_{1,j} = \xi_{2,j}$. Therefore, $\xi_1 = \xi_2$, which contradicts Definition 1.

Assume $s > 2$. Consider $\sum_{i=1}^{s} \lambda_i v_i = 0$, with $v_i \in E_i$. Let $m_{i,s}$ be such that $[M_{x_j} - \xi_{s,j}]^{m_{j,s}}(v_s) = 0$; then,

$$0 = [M_{x_j} - \xi_{s,j}]^{m_{j,s}} \left(\sum_{i=1}^{s} \lambda_i v_i \right) = \sum_{i=1}^{s-1} \lambda_i [M_{x_j} - \xi_{s,j}]^{m_{j,s}}(v_i).$$

Each $[M_{x_j} - \xi_{s,j}]^{m_{j,s}}(v_i)$ is still an element of E_i; hence, by induction, we have that $\lambda_i = 0$ for any i such that $[M_{x_j} - \xi_{s,j}]^{m_{j,s}}(v_j) \neq 0$. However, there exists $h \in \{1, \dots, n\}$ such that $[M_{x_h} - \xi_{s,h}]^{m_{h,s}}(v_i) \neq 0$ otherwise $v_i \in E_i \cap E_s$, contradicting the case $s = 2$. ☐

Therefore, we have proved the following

Corollary 3.6. *For every $F \in \mathcal{S}_d$ there exists a cactus scheme of F which is regular in degree d.*

Proof. From [BBM14, Theorem 3.7] the set of forms of degree d with a GAD of minimal size r coincides with the set of forms with cactus rank equal to r, therefore for any GAD of minimal size the scheme evincing that GAD computes the cactus rank of F, and it is regular in degree d by Theorem 3.5. ☐

Corollary 3.7. *Let Z be a cactus scheme for $F \in \mathcal{S}_d$ such that each of its connected components is contained in a $(d+1)$-fat point. Then Z is regular in degree d.*

Proof. Let $Z = Z_1 \cup \dots \cup Z_s$ be a decomposition of Z into irreducible components where for every $i \in \{1, \dots, s\}$ the scheme Z_i is supported at $L_i \in \mathcal{S}_1$. Let $h_i + 1$ be the minimal order of a fat point containing Z_i. Since by hypothesis $h_i + 1 \leq d + 1$, there are $G_i \in S_{h_i}$ such that $F = \sum_{i=1}^{s} L_i^{d-h_i} G_i$, which is a GAD for F. Therefore by Theorem 3.5, Z is regular in degree d. ☐

The remaining question is if the latter holds for *any* cactus scheme. Indeed, a priori, there might by a cactus scheme which do not evince a GAD; in particular, there might be a cactus scheme of $F \in \mathcal{S}_d$ having a connected component which is not contained in the $(d+1)$-fat point. At the moment, we do not have an example of it nor a proof that it cannot be the case.

4. An application

For a given $F \in \mathcal{S}_d$, in [BT20] an algorithm for detecting a cactus decomposition is presented, which corresponds to a GAD of minimal size in the notation of the current work. The main subroutine of this algorithm aims at detecting the minimal positive integer r such that there exists an ideal I with $\dim \Bbbk[x_1, \ldots, x_n]/I = r$ and such that there exists a monomial basis $\{b_1, \ldots, b_r\}$ of $\Bbbk[x_1, \ldots, x_n]/I$ that makes some multiplication operators commute. These conditions are satisfied whenever there exists a GAD of F of size r, which may be reconstructed by the last part of the algorithm once such r and $\{b_1, \ldots, b_r\}$ are discovered.

As observed in [BT20, Section 7.2], a priori every monomial of degree lower than r should be considered for constructing such an admissible basis, even if it was noticed that the degree d, which is in general much lower than r, has been sufficient for every observed candidate. In fact, Corollary 3.6 shows that this is always the case, as there is at least one cactus scheme evincing a GAD for F that is regular in degree d, hence the degree-d part of $\mathcal{S}/F^{\perp}$ has dimension r. Thus, there is a basis

$$\langle b_1, \ldots, b_r \rangle = \Bbbk[x_1, \ldots, x_n]/I,$$

made of elements of degree $\deg(b_i) \leq d$.

We remark that the reduction in the number of bases to be examined grows rapidly with $r - d$, and that high-degree bases are the most expensive to be tested. As a result, this improvement is particularly significant for high rank tensors, which are often considered the most burdensome to deal with computationally.

References

[BB14] Weronika Buczyńska and Jaroslenaw Buczyński. Secant varieties to high degree veronese reembeddings, catalecticant matrices and smoothable gorenstein schemes. *Journal of Algebraic Geometry*, 23:63–90, 2014.

[BBM14] Alessandra Bernardi, Jérôme Brachat, and Bernard Mourrain. A comparison of different notions of ranks of symmetric tensors. *Linear Algebra and its Applications*, 460:205–230, 2014.

[BJMR18] Alessandra Bernardi, Joachim Jelisiejew, Pedro Macias Marques, and Kristian Ranestad. On polynomials with given hilbert function and applications. *Collectanea Mathematica*, 69(1):39–64, 2018.

[BT20] Alessandra Bernardi and Daniele Taufer. Waring, tangential and cactus decompositions. *Journal de Mathématiques Pures et Appliquées*, 143:1–30, 2020.

[Ger96] Anthony V. Geramita. Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals. In *The curves seminar at Queen's*, volume 10, pages 2–114, 1996.

[IK99] Anthony Iarrobino and Vassil Kanev. *Power sums, Gorenstein algebras, and determinantal loci.* Springer Science & Business Media, 1999.

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