Local effectivity in projective spaces

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Abstract

In this note we introduce a Waldschmidt decomposition of divisors which might be viewed as a generalization of Zariski decomposition based on the effectivity rather than the nefness of divisors. As an immediate application we prove a recursive formula providing new effective lower bounds on Waldschmidt constants of very general points in projective spaces. We use these bounds in order to verify Demailly's conjecture in a number of new cases.

1 Introduction

Let X be a smooth projective variety and let L be an ample line bundle on X. The concept of the *local positivity* of L has been coined by Demailly, who introduced in [6] the following invariants measuring in effect the local positivity.

Definition 1.1 (Seshadri constant)**.** Let X be a smooth projective variety and let L be an ample line bundle on X. Let $P \in X$ be a fixed point and let $f : \mathrm{Bl}_P X \to X$ be the blow up of X at P with the exceptional divisor E. The real number

$$\varepsilon(X; L, P) = \sup \{t \in \mathbb{R} : \ f^*L - tE \ \text{ is nef}\}$$

is the *Seshadri constant* of L at P.

Thus $\varepsilon(X; L, P)$ is the value of t for which the ray $f^*L - tE$ hits the boundary of the nef cone on $\mathrm{Bl}_P X$. It is natural to introduce a similar invariant, which gives the value of t, where the ray $f^*L - tE$ hits the boundary of the pseudo-effective cone on $\mathrm{Bl}_P X$. We consider this invariant (more precisely its reciprocal introduced in Definition 1.3) as a way to measure the *local effectivity* of L.

Definition 1.2 (The μ-invariant)**.** Let X be a smooth projective variety and let L be an ample line bundle on X. Let $P \in X$ be a fixed point and let $f : \mathrm{Bl}_P X \to X$ be the blow up of X at P with the exceptional divisor E. The real number

$$\mu(X; L, P) = \sup \{t \in \mathbb{R} : \ f^*L - tE \ \text{ is effective}\}$$

is the *μ-invariant* of L at P.

Both notions can be easily generalized replacing the point P by an arbitrary subscheme $Z \subset X$ and taking $f : \mathrm{Bl}_Z X \to X$ to be the blow up of X along the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$. We denote the exceptional divisor of f again by E.

Whereas the μ-invariant $\mu(X; L, Z)$ is not much present in the literature, its reciprocal is the well-known Waldschmidt constant of Z. We define first the *initial degree of Z with respect to L* as

$$\alpha(X; L, Z) = \min \{d : \ df^*L - E \ \text{ is effective}\}.$$

For an integer $m \geq 1$, let mZ denote the subscheme defined by the symbolic power $\mathcal{I}_Z^{(m)}$ of $\mathcal{I}_Z$, see [15, Definition 9.3.4]. Then the asymptotic version of the initial degree is the following.

Definition 1.3 (Waldschmidt constant)**.** Let X be a smooth projective variety and let L be an ample line bundle on X. Let $Z \subset X$ be a subscheme. The real number

$$\widehat{\alpha}(X; L, Z) = \inf_{m \geq 1} \frac{\alpha(X; L, mZ)}{m}$$

is the *Waldschmidt constant* of Z with respect to L.

Remark 1.4. Since the numbers $\alpha(X; L, mZ)$ for $m \geq 1$ form a subadditive sequence, i.e. there is

$$\alpha(X; L, (k + \ell)Z) \leq \alpha(X; L, kZ) + \alpha(X; L, \ell Z)$$

for all k and ℓ, the infimum in Definition1.3 exists and moreover we have

$$\widehat{\alpha}(X; L, Z) = \lim_{m \to \infty} \frac{\alpha(X; L, mZ)}{m}.$$

Waldschmidt constants appear in different guises in various branches of mathematics. Apparently, they were first considered in complex analysis in connection with estimates on the growth order of holomorphic functions, see [18]. In this setup X is simply $\mathbb{C}^n$ or $\mathbb{P}^n$. We prefer the homogeneous approach here. Then the polarization L is just the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. Let I be a non-zero, proper homogeneous ideal in the polynomial ring $\mathbb{C}[x_0, \ldots, x_N]$. The *initial degree* of I is

$$\alpha(\mathbb{P}^N; I) = \min \{d : (I)_d \neq 0\},$$

where $(I)_d$ denotes the degree d part of I. The Waldschmidt constant of $I \subset \mathbb{C}[x_0, \ldots, x_N]$ is then

$$\widehat{\alpha}(\mathbb{P}^N; I) = \inf_{m \geq 1} \frac{\alpha(\mathbb{P}^N; I^{(m)})}{m},$$

which of course agrees with Definition 1.3. In recent years there has been considerable interest in Waldschmidt constants in general, see e.g. [7], [2], [17], [11]. Special attention has been given to the following Conjecture stated originally by Demailly in [5, p. 101]. It has been formulated recently by Harbourne and Huneke in [13, Question 4.2.1]. Apparently the authors were not aware of Demailly's work. We use again the projective version.

Conjecture 1.5 (Demailly)**.** Let $Z \subset \mathbb{P}^N$ be a finite set of points and let I be the homogeneous saturated ideal defining Z. Then for all $m \geq 1$

$$\widehat{\alpha}(\mathbb{P}^N; I) \geq \frac{\alpha(\mathbb{P}^N; I^{(m)}) + N - 1}{m + N - 1}. \tag{1}$$

For $m = 1$ the Conjecture of Demailly reduces to the statement which is best known as the Conjecture of Chudnovsky, see [3, Problem 1], to the effect that the inequality

$$\widehat{\alpha}(\mathbb{P}^N; I) \geq \frac{\alpha(\mathbb{P}^N; I) + N - 1}{N}. \tag{2}$$

holds for all ideals defining finite sets of points in $\mathbb{P}^N$. Demailly's Conjecture for $\mathbb{P}^2$ has been proved by Esnault and Viehweg using methods of complex projective geometry, see [10, Inégalité A].

In the present note, we provide lower bounds on Waldschmidt constants of sets of general points in projective spaces. The core result of the paper is Theorem 3.2. It gives an iterative way to control Waldschmidt constants of very general points. The most important new contribution of our work is the concept of Waldschmidt decomposition introduced in Section 2. We hope it is a general tool which could become new path of approaching numerous central problems in computational algebraic geometry. As immediate application of our machinery we derive the following two results, see Theorem 4.8 and Proposition 4.4.

Theorem A. *Demailly's Conjecture 1.5 holds for $r \geq m^N$ very general points in $\mathbb{P}^N$.*

Theorem B. *Let k be a positive integer and let s be an integer in the range $1 \leq s \leq k$. Let*

$$r \geq s(k+1)^{N-1} + (k+1-s)k^{N-1}.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq k + \frac{s}{k+1}.$$

Related Singular script [1] renders in an effective way our results.

Convention and notation. We work throughout over the field $\mathbb{C}$ of complex numbers.

2 Waldschmidt decomposition

The numerical meaning of the Waldschmidt constant $\widehat{\alpha}(X; L, Z)$ is that if $D \in |kL|$ is an effective divisor vanishing along Z with multiplicity m, then

$$\frac{k}{m} \geq \widehat{\alpha}(X; L, Z).$$

This condition extends easily to effective $\mathbb{R}$-divisors. Indeed, let $D = \sum \delta_i D_i$ be an effective $\mathbb{R}$-divisor with $D \equiv \delta L$ for some $\delta > 0$. Then $\operatorname{mult}_Z D = \sum \delta_i \operatorname{mult}_Z D_i$ and

$$\frac{\delta}{\operatorname{mult}_Z D} \geq \widehat{\alpha}(X; L, Z).$$

In this section we introduce certain decomposition of a divisor, depending on its numerical properties. We call it the Waldschmidt decomposition as it is governed by Waldschmidt constants. This decomposition can be viewed as a higher dimensional version of the Bezout decomposition defined in [8, Section 2.1]. Whereas it is possible to define it on arbitrary varieties, we restrict our approach here to $\mathbb{P}^N$ and its linear subspaces. In this setting the definition is most transparent.

Definition 2.1 (Waldschmidt decomposition in $\mathbb{P}^N$)**.** Let $H \cong \mathbb{P}^{N-1}$ be a hyperplane in $\mathbb{P}^N$ and let Z be a subscheme in H. Let D be a divisor of degree d in $\mathbb{P}^N$. The *Waldschmidt decomposition of D with respect to H and Z* is the sum of $\mathbb{R}$-divisors

$$D = D' + \lambda \cdot H$$

such that $\deg(D') = d - \lambda$,

$$\frac{d-\lambda}{\operatorname{mult}_Z D'} \geq \widehat{\alpha}(H; \mathcal{O}_H(1), Z) \tag{3}$$

and λ is the least non-negative real number such that (3) is satisfied.

Of course, it may happen that $\lambda = 0$ in Definition 2.1. This number is positive, if the restriction of D to H would produce a divisor in $|\mathcal{O}_H(1)|$ violating the inequality (3). Thus λ is the least multiplicity such that H is numerically forced to be contained in D with this multiplicity. It may well happen that the divisor D' still contains H as a component.

Remark 2.2. The definition of the Waldschmidt decomposition with respect to H can be extended to a finite number of hyperplanes $H_1, \ldots, H_s$.

3 The main result

In this section we state our main result. The statement is motivated by the proof of the following lower bound on Waldschmidt constants presented in [9, Theorem 3].

Theorem 3.1 (Lower bound on Waldschmidt constants)**.** *Let I be the saturated ideal of a set of r very general points in $\mathbb{P}^N$. Then*

$$\widehat{\alpha}(\mathbb{P}^N; I) \geq \lfloor \sqrt[N]{r} \rfloor.$$

It is expected that for r sufficiently big, there is actually the equality $\widehat{\alpha}(\mathbb{P}^N; I) = \sqrt[N]{r}$ but this statement seems out of reach with present methods.

Theorem 3.2. *Let $H_1, \ldots, H_s$ be $s \geq 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1, \ldots, a_s \geq 1$ be real numbers such that*

$$1 - \sum_{j=1}^{s-1} \frac{1}{a_j} > 0 \tag{4}$$

and

$$1 - \sum_{j=1}^{s} \frac{1}{a_j} \leq 0. \tag{5}$$

Let

$$Z_i = \{P_{i,1}, \ldots, P_{i,r_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$$

be the set of r_i points such that

$$\widehat{\alpha}(H_i; Z_i) \geq a_i \tag{6}$$

and let $Z = \bigcup_{i=1}^{s} Z_i$. Finally, let

$$q := \left(1 - \sum_{j=1}^{s-1} \frac{1}{a_j}\right) \cdot a_s + s - 1. \tag{7}$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; Z) \geq q.$$

Proof. First observe that, for any $t = 1, \ldots, s-1$, by (4) we have

$$1 - \sum_{j=1}^{t} \frac{1}{a_j} > 0.$$

Multiplying by a_t, moving $a_t/a_t = 1$ to the right hand side and making some preparation we get

$$a_t - \sum_{j=1}^{t-1} \frac{a_t}{a_j} > \left(1 - \sum_{j=1}^{t-1} \frac{1}{a_j}\right) + \sum_{j=1}^{t-1} \frac{1}{a_j}.$$

Dividing by $1-\sum\limits_{j=1}^{t-1}\frac{1}{a_j}$ we get

$$a_t > 1 + \frac{\sum\limits_{j=1}^{t-1}\frac{1}{a_j}}{1-\sum\limits_{j=1}^{t-1}\frac{1}{a_j}} \tag{8}$$

for $t \le s-1$. Similarly, starting with (4), we get

$$a_s \le 1 + \frac{\sum\limits_{j=1}^{s-1}\frac{1}{a_j}}{1-\sum\limits_{j=1}^{s-1}\frac{1}{a_j}}. \tag{9}$$

We assume to the contrary that there is a divisor D of degree d in $\mathbb{P}^N$ vanishing to order at least m at all points of Z such that

$$p := \frac{d}{m} < q. \tag{10}$$

It is convenient to work with the $\mathbb{Q}$-divisor $\Gamma = \frac{1}{m}D$, which is of degree p and has multiplicities at least 1 at every point of Z.

Step 0.
Let $\Gamma = \Gamma' + \sum_{i=1}^{s}\lambda_i H_i$ be the Waldschmidt decomposition of Γ with respect to $H_1,\ldots,H_s$ and $Z_1,\ldots,Z_s$ respectively. The conditions (3) and (6) imply then that

$$\left\{\begin{array}{lrcl} (11.1) & p-\sum\limits_{i=1}^{s}\lambda_i & \ge & a_1(1-\lambda_1) \\ (11.2) & p-\sum\limits_{i=1}^{s}\lambda_i & \ge & a_2(1-\lambda_2) \\ \vdots & \vdots & & \\ (11.s) & p-\sum\limits_{i=1}^{s}\lambda_i & \ge & a_s(1-\lambda_s) \end{array}\right. \tag{11}$$

We will show that the conditions in (4), (5), (10) and (11) cannot hold simultaneously. This will provide the desired contradiction to the existence of D. The idea is first to achieve equalities in (11).

Step 1.
Our first claim is that there exists $\lambda_1' \le \lambda_1$ such that

$$\left\{\begin{array}{lrcl} (12.1) & p-\lambda_1'-\sum\limits_{i=2}^{s}\lambda_i & = & a_1(1-\lambda_1') \\ (12.2) & p-\lambda_1'-\sum\limits_{i=2}^{s}\lambda_i & \ge & a_2(1-\lambda_2) \\ \vdots & & & \\ (12.s) & p-\lambda_1'-\sum\limits_{i=2}^{s}\lambda_i & \ge & a_s(1-\lambda_s) \end{array}\right. \tag{12}$$

Indeed, we have

$$p-\lambda_1-\sum_{i=2}^{s}\lambda_i \ge a_1(1-\lambda_1)$$

from (11.1). Decreasing λ_1 by ε, the left hand side increases by ε as well, whereas the right hand side increases by $a_1 \cdot \varepsilon$. Since $a_1 > 1$ by (4.1), there must exist $\varepsilon \geq 0$ such that

$$p - (\lambda_1 - \varepsilon) - \sum_{i=2}^{s} \lambda_i \;=\; a_1(1 - (\lambda_1 - \varepsilon)).$$

We put $\lambda_1' = \lambda_1 - \varepsilon$. Note also that decreasing λ_1 preserves the inequalities with indices $j = 2, \ldots, s$ in (11) because the left hand sides of all these inequalities increase, while the right hand sides remain unaltered.

In order to alleviate the notation, we drop the prime index by the new λ_1.

Step t (the induction step). In the second step we assume that we found new $\lambda_1, \ldots, \lambda_{t-1}$ such that the following holds:

$$\begin{cases} p - \sum\limits_{i=1}^{t-1} \lambda_i - \lambda_t - \sum\limits_{i=t+1}^{s} \lambda_i & = & a_1(1 - \lambda_1) \\ & \vdots & \\ p - \sum\limits_{i=1}^{t-1} \lambda_i - \lambda_t - \sum\limits_{i=t+1}^{s} \lambda_i & = & a_{t-1}(1 - \lambda_{t-1}) \\ p - \sum\limits_{i=1}^{t-1} \lambda_i - \lambda_t - \sum\limits_{i=t+1}^{s} \lambda_i & \geq & a_t(1 - \lambda_t) \\ & \vdots & \\ p - \sum\limits_{i=1}^{t-1} \lambda_i - \lambda_t - \sum\limits_{i=t+1}^{s} \lambda_i & \geq & a_s(1 - \lambda_s) \end{cases} \tag{13}$$

Our aim is to push this one step further, to the situation, where (for new $\lambda_1, \ldots, \lambda_t$) we will have at least t equalities.

Let

$$C := p - \lambda_t - \sum_{i=t+1}^{s} \lambda_i.$$

Solve the following system of equalities with respect to $\lambda_1, \ldots, \lambda_{t-1}$ and a parameter λ_t.

$$\begin{cases} C - \sum\limits_{i=1}^{t-1} \lambda_i & = & a_1(1 - \lambda_1) \\ & \vdots & \\ C - \sum\limits_{i=1}^{t-1} \lambda_i & = & a_{t-1}(1 - \lambda_{t-1}) \end{cases} \tag{14}$$

Let $\lambda_1', \ldots, \lambda_{t-1}'$ be unique (by Lemma 5.1) solutions to that system. Again, by Lemma 5.1,

$$\sum_{i=1}^{t-1} \lambda_i' = \frac{C\left(\sum\limits_{j=1}^{t-1} \frac{1}{a_j}\right) - (t-1)}{\left(\sum\limits_{j=1}^{t-1} \frac{1}{a_j}\right) - 1}. \tag{15}$$

Since λ_t is hidden in C (as $-\lambda_t$), decreasing λ_t by ε increases $\sum\limits_{j=1}^{t-1} \lambda_i'$ by

$$\varepsilon \left(\frac{\sum\limits_{j=1}^{t-1} \frac{1}{a_j}}{\sum\limits_{j=1}^{t-1} \frac{1}{a_j} - 1} \right).$$

Thus the left hand side of the inequality (13).t increases by

$$\varepsilon\left(1+\frac{\sum\limits_{j=1}^{t-1}\frac{1}{a_j}}{1-\sum\limits_{j=1}^{t-1}\frac{1}{a_j}}\right),$$

which by (8) is strictly less than εa_t. In effect, decreasing λ_t, solving (14) for $\lambda_1,\ldots,\lambda_{t-1}$ gives a new sequence $\lambda'_1,\ldots,\lambda'_t$, with

- preserved equalities (13).1 — (13).$(t-1)$,
- left hand side of (13).t increasing faster than the right hand side,
- left hand sides of (13).$(t+1)$ — (13).s increasing, while right hand sides remain unaltered.

As in Step 1, this suffices to obtain new $\lambda_1,\ldots,\lambda_t$ with one more equality in (13).

Step s (the final step).
Assume that we have now $s-1$ equalities in (13), with the last inequality not necessaritly being an equality. We begin exactly as in the previous step. The only difference is that, by (9), decreasing λ_t forces the left hand side of the last inequality (13) to increase *faster* than the right hand side. Thus we may decrease λ_t (altering $\lambda_1,\ldots,\lambda_{t-1}$ to preserve equalities) to zero to obtain

$$\left\{\begin{array}{clcl}
(16.1) & p-\sum\limits_{i=1}^{s-1}\lambda_i & = & a_1(1-\lambda_1)\\
(16.2) & p-\sum\limits_{i=1}^{s-1}\lambda_i & = & a_2(1-\lambda_2)\\
\vdots & \vdots & & \\
(16.(s-1)) & p-\sum\limits_{i=1}^{s-1}\lambda_i & = & a_{s-1}(1-\lambda_{s-1})\\
(16.s) & p-\sum\limits_{i=1}^{s-1}\lambda_i & \geq & a_s
\end{array}\right. \tag{16}$$

It follows from Lemma 5.1 that now

$$\sum_{i=1}^{s-1}\lambda_i=\frac{pR-(s-1)}{R-1}, \tag{17}$$

where $R=\sum\limits_{j=1}^{s-1}1/a_j$. From (7) we have

$$q=(1-R)a_s+(s-1). \tag{18}$$

Taking (16.s) into account we get

$$q\leq(1-R)\left(p-\frac{pR-(s-1)}{R-1}\right)+(s-1)=p-Rp+pR-(s-1)+(s-1)=p.$$

This contradicts however clearly (10) and we are done. □

4 Applications

We will focus on Waldschmidt constants of sets of very general points in $\mathbb{P}^N$. The notation

$$\widehat{\alpha}(\mathbb{P}^N; r)$$

denotes the Waldschmidt constant $\widehat{\alpha}(\mathbb{P}^N; I)$ of a radical ideal I of r very general points in $\mathbb{P}^N$.

Theorem 4.1. *Let $N \geq 2$, let $k \geq 1$ be an integer. Assume that for some integers $r_1, \ldots, r_{k+1}$ and rational numbers $a_1, \ldots, a_{k+1}$ we have*

$$\widehat{\alpha}(\mathbb{P}^{N-1}; r_j) \geq a_j \text{ for } j = 1, \ldots, k+1,$$

$$k \leq a_j \leq k+1 \text{ for } j = 1, \ldots, k, \quad a_1 > k, \quad a_{k+1} \leq k+1.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; r_1 + \ldots + r_{k+1}) \geq \left(1 - \sum_{j=1}^{k} \frac{1}{a_j}\right) a_{k+1} + k.$$

Proof. We combine Theorem 3.2 and the specialization. We take hyperplanes $H_1, \ldots, H_{k+1}$ and specialize r_j points to a set $Z_j \subset H_j$ for $j = 1, \ldots, k+1$, so that the points in Z_j are in very general position on H_j. Hence

$$\widehat{\alpha}(H_j; Z_j) = \widehat{\alpha}(\mathbb{P}^{N-1}; r_j).$$

To check that (4) is satisfied, we compute

$$\sum_{j=1}^{k} \frac{1}{a_j} < \sum_{j=1}^{k} \frac{1}{k} = 1$$

since $a_j \geq k$ and $a_1 > k$. Similarly we check that (5) holds,

$$\sum_{j=1}^{k+1} \frac{1}{a_j} \geq \sum_{j=1}^{k+1} \frac{1}{k+1} = 1.$$

The inequalities (6) are satisfied by assumptions. Thus the Waldschmidt constant of specialized points is bounded as desired, hence for points in the very general position the bound also holds. □

Example 4.2. We bound from below $\widehat{\alpha}(\mathbb{P}^3; 20)$. Let $k = 2$ (in fact, it is very easy to find the suitable k in general; it must satisfy $k^N < r < (k+1)^N$, where r is the number of points in $\mathbb{P}^N$). Then we look for integers r_1, r_2 and r_3 and rational numbers a_1, a_2, a_3 satisfying the assumptions of Theorem 4.1. Since we want to bound $\widehat{\alpha}(\mathbb{P}^3; 20)$, it must be

$$r_1 + r_2 + r_3 \leq 20.$$

Since $\widehat{\alpha}(\mathbb{P}^2; r_1) \geq a_1 > 2$, we see that $r_1 > 4$. Similarly $r_2 \geq 4$, $r_3 \geq 1$. Moreover, from $a_j \leq 3$ we see that we may restrict ourselves to the case when $r_j \leq 9$. Since $\widehat{\alpha}(\mathbb{P}^2; r)$ is known for $r \leq 9$, it suffices to search through all the possibilities (r_1, r_2, r_3), compute (a_1, a_2, a_3) for each of them and get the bound. This can be done by hand in principle. We have used a simple computer program to do the dully calculations for us. As a result, for

$$r_1 = 8, \qquad r_2 = 8, \qquad r_3 = 4$$

we get

$$a_1 = 48/17, \qquad a_2 = 48/17, \qquad a_3 = 2.$$

Thus, from the formula, $\widehat{\alpha}(\mathbb{P}^3; 20) \geq 31/12 \simeq 2.583$. Note that the upper bound is $\sqrt[3]{20} \simeq 2.714$.

4.1 A recursive approach

Now we study a much harder example which allows us to discuss some algorithmic issues.

Example 4.3. We want to bound $\widehat{\alpha}(\mathbb{P}^4; 180)$. Since now $N = 4$, we get immediately $k = 3$, since then $k^N < 180 < (k+1)^N$. We are interested in sequences of integers

$$(r_1, r_2, r_3, r_4) \text{ with } r_1 + r_2 + r_3 + r_4 \leq 180.$$

As before, we have additional constraints. Since $\widehat{\alpha}(\mathbb{P}^{N-1}; r_j) \geq a_j \geq k$, we get (in general) that $r_j \geq k^{N-1}$. In our situation this gives $r_2, r_3 \geq 27$, $r_1 \geq 28$, $r_4 \geq 1$. It is reasonable to restrict to $r_j \leq (k+1)^{N-1}$, so in our case, $r_j \leq 64$.

The first problem we encounter here is the number of sequences (r_1, r_2, r_3, r_4) with above properties. But this can be (in the case studied here, $N = 4$, $r = 180$) easily managed by a suitable computer program. What requires much more attention is coming up with good bounds a_j for $\widehat{\alpha}(\mathbb{P}^3; r_j)$. These constants are not known, except for several cases: 2 for $\widehat{\alpha}(\mathbb{P}^3; 8)$, 3 for $\widehat{\alpha}(\mathbb{P}^3; 27)$ and 4 for $\widehat{\alpha}(\mathbb{P}^3; 64)$. So the first approach is to use only numbers r_j of the form ℓ^{N-1}, which is weak, but manageable (we will address this later, in Proposition 4.4). Taking

$$r_1 = 64, \qquad r_2 = 64, \qquad r_3 = 27, \qquad r_4 = 8$$

we get

$$a_1 = 4, \quad a_2 = 4, \quad a_3 = 3, \quad a_4 = 2, \quad \text{thus} \quad \widehat{\alpha}(\mathbb{P}^4; 180) \geq \frac{10}{3} \simeq 3.333.$$

Using again a computer program we can find, as in the previous example, all necessary bounds for $\widehat{\alpha}(\mathbb{P}^{N-1}; \widetilde{r})$ for $\widetilde{r} = 1, \ldots, (k+1)^{N-1}$. In our case it requires 64 computations to find a bound in $\mathbb{P}^3$. Each of them requires again looking for sequences satisfying certain properties and then going down to $\mathbb{P}^2$. In effect, the run time grows exponentially when N is increased. For $\widehat{\alpha}(\mathbb{P}^2; \widetilde{r})$, however, a much better idea is to use known best bounds, e.g., [14, Theorem 2.2 and discussion thereafter].

Coming back to our case, with the help of a computer program, which run several minutes, all possibilities were scanned and the best results were found taking

$$r_1 = 52, \qquad r_2 = 52, \qquad r_3 = 49, \qquad r_4 = 27.$$

Again with a computer we obtain

$$a_1 = a_2 = \frac{17457}{4816}, \quad a_3 = \frac{63495}{17974}, \quad a_4 = 3, \quad \text{thus} \quad \widehat{\alpha}(\mathbb{P}^4; 180) \geq 3.495.$$

In fact, the last number is exactly 430502824/123159135. Observe that the upper bound is $\sqrt[4]{180} \simeq 3.663$.

From the above considerations we conclude that checking all partitions of r into $k+1$ numbers would take too much time for bigger N. To make this faster and manageable even in the case, e.g., $N \geq 100$ we must drastically reduce the number of subcases. The radical idea is to consider only one distribution, and go down to $\mathbb{P}^{N-1}$ with only one case.

Observe that we look for the numbers $a_1, \ldots, a_{k+1}$ such that

$$\left(1 - \sum_{j=1}^{k} \frac{1}{a_j}\right) a_{k+1}$$

is as big as possible. The numbers a_j are good bounds for $\widehat{\alpha}(\mathbb{P}^{N-1}; r_j)$, so we may as well assume, that they are close to $\sqrt[N-1]{r_j}$ or even pretend they are equal.

We consider first the expression

$$\sum_{j=1}^{k} \frac{1}{\sqrt[N-1]{r_j}}. \tag{19}$$

For all partitions $r_1 + \cdots + r_k = const$, we want (19) to be as small as possible. Without going into details, this forces all numbers r_j to be nearly equal. Therefore we want to maximize

$$\left(1 - \frac{k}{\sqrt[N-1]{r_1}}\right) \sqrt[N-1]{r_{k+1}}$$

under the condition

$$kr_1 + r_{k+1} = r,$$

or, which is much nicer to compute, to maximize

$$\left(1 - \frac{k}{a_1}\right) a_{k+1}$$

under the condition

$$ka_1^{N-1} + a_{k+1}^{N-1} \leq r.$$

Since we want to go down with only one case, we force a_{k+1} to be an integer. Now the problem is to distribute points to r_1 and r_{k+1}. It is a matter of an easy calculation to check integer a_{k+1} with $r_1 = \lfloor (r - a_{k+1}^{N-1})/k \rfloor$ gives the best result.

In our case, $N = 4$ and $r = 180$, the following distribution was found:

$$r_1 = 51, \qquad r_2 = 51, \qquad r_3 = 51, \qquad r_4 = 27.$$

Thus we need a lower bound for $\widehat{\alpha}(\mathbb{P}^3; 51)$. Again, we use the above heuristic method to find the distribution

$$r'_1 = 14, \qquad r'_2 = 14, \qquad r'_3 = 14, \qquad r'_4 = 9.$$

We take the bound for $\widehat{\alpha}(\mathbb{P}^2; 14) \geq 86/23$. Thus

$$\widehat{\alpha}(\mathbb{P}^3; 51) \geq \frac{309}{86}, \qquad \widehat{\alpha}(\mathbb{P}^4; 180) \geq \frac{360}{103} \simeq 3.495.$$

Our previous best bound is better only by $\simeq 0.0003549$ but the run time of the algorithm outlined here is considerably shorter.

Less radical, but a better approach is to consider all distributions $kr_1 + r_{k+1} \leq r$ with r_{k+1} being a pure $(N-1)$th power. The implementation of these two approaches in Singular [4] can be found in the file `boundforWC`, [1]. Running `bound` works faster (for big N), but `boundmore` gives better bounds.

4.2 An easy way to distribute points on hyperplanes

We pass now to some general effective lower bounds.

Proposition 4.4. *Let k be a positive integer and let s be an integer in the range $1 \leq s \leq k$. Let*

$$r \geq s(k+1)^{N-1} + (k+1-s)k^{N-1}.$$

Then

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq k + \frac{s}{k+1}.$$

Proof. This is an easy consequence of Theorem 4.1. Namely, taking

$$r_1 = \ldots = r_s = (k+1)^{N-1}, \qquad r_{s+1} = \ldots = r_{k+1} = k^{N-1},$$

we get by Theorem 3.1

$$a_1 = \ldots = a_s = k+1, \qquad a_{s+1} = \ldots = a_{k+1} = k.$$

Consequently,

$$\widehat{\alpha}(r) \geq \left(1 - \frac{s}{k+1} - \frac{k-s}{k}\right) k + k = s - \frac{sk}{k+1} + k = k + \frac{s}{k+1}.$$

$\square$

Example 4.5. Without the above proposition, the general available lower bound for $\widehat{\alpha}(\mathbb{P}^5; 1024)$ is 4. It requires at least $r \geq 3125$ points to pass to the better bound $\widehat{\alpha}(\mathbb{P}^5; r) \geq 5$. But with Proposition 4.4 we can take $s = 1$, $k = 4$ to get

$$\widehat{\alpha}(\mathbb{P}^5; 1649) \geq 4 + \frac{1}{5}.$$

Similarly, we need only 2018 points to get $4 + \frac{2}{5}$, only 2387 to get $4 + \frac{3}{5}$ and only 2756 to get $4 + \frac{4}{5}$.

Proposition 4.6. *Let $r \leq (k+1)^N$. Then*

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq \frac{r}{(k+1)^{N-1}}.$$

Proof. We will use induction. Consider three cases.

Case $k^N \leq r \leq k(k+1)^{N-1}$.
Note that

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq k = \frac{k(k+1)^{N-1}}{(k+1)^{N-1}} \geq \frac{r}{(k+1)^{N-1}}.$$

Case $r \leq k^N$.
By induction on k we have

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq \frac{r}{k^{N-1}} \geq \frac{r}{(k+1)^{N-1}}.$$

Case $r > k(k+1)^{N-1}$.
Take

$$r_1 = \ldots = r_k = (k+1)^{N-1}, \qquad r_{k+1} = r - k(k+1)^{N-1}.$$

Observe that

$$r - k(k+1)^{N-1} \leq (k+1)^N - k(k+1)^{N-1} = (k+1)^{N-1},$$

thus by Theorem 3.1 and induction (on N) we get

$$a_1 = \ldots = a_k = k+1, \qquad a_{k+1} = \frac{r - k(k+1)^{N-1}}{(k+1)^{N-2}}.$$

By Theorem 4.1 we get

$$\widehat{\alpha}(\mathbb{P}^N; r) \geq \left(1 - \frac{k}{k+1}\right) \frac{r - k(k+1)^{N-1}}{(k+1)^{N-2}} + k = \frac{r}{(k+1)^{N-1}}.$$

$\square$

Example 4.7. We can now give very accurate bounds for $\widehat{\alpha}(\mathbb{P}^5; r)$ for r close to 3125. Since $3125 = 5^5$, we have

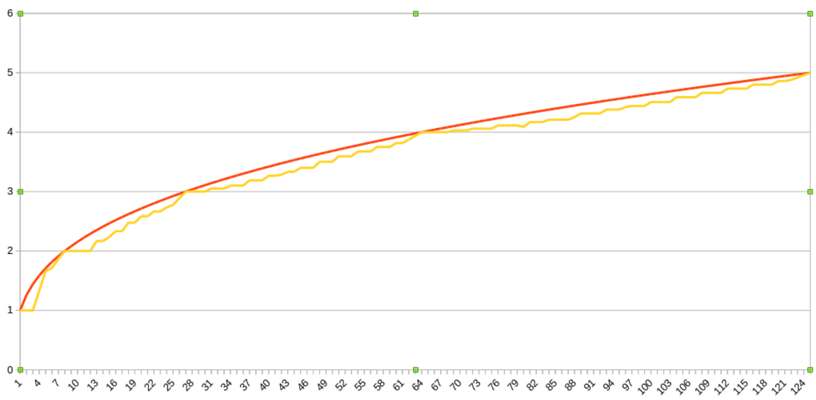
$$\widehat{\alpha}(\mathbb{P}^5; 3124) \geq 5 - \frac{1}{625}, \qquad \widehat{\alpha}(\mathbb{P}^5; 3123) \geq 5 - \frac{2}{625}$$

and so on.

4.3 Discussion on the accuracy

By Theorem 3.1 it is obvious that we can locate every $\widehat{\alpha}(\mathbb{P}^N; r)$ in an interval of length at most 1. It is interesting to know what is the difference between the upper bound (which is conjectured to be the actual bound for $r \geq 2^N$) and the lower bound obtained by our algorithm. In Figure 1 we present the upper and lower bounds for $r = 1, \ldots, 125$ points in $\mathbb{P}^3$.

Figure 1: Upper and lower bounds for $\widehat{\alpha}(\mathbb{P}^3; r)$



In Table 1 we present the maximal difference δ between the lower and upper bound.

N	3	3	4	4	5	5	6	7
r	$8-125$	$125-1000$	$16-256$	$256-1296$	$32-243$	$243-1024$	$64-729$	$128-2187$
δ	0.289	0.186	0.295	0.259	0.305	0.277	0.305	0.301

Table 1: Maximal differences for lower and upper bounds in various intervals of the number of points in projective spaces of low dimensions

4.4 Towards Demailly's Conjecture

As an important consequence of Theorem 3.2 we obtain the following result.

Theorem 4.8. *Demailly's Conjecture 1.5 holds for $r \geq m^N$ very general points in $\mathbb{P}^N$.*

Proof. The Main Theorem in [16] states that Conjecture 1.5 holds for $r \geq (m+1)^N$ very general points in $\mathbb{P}^N$. Hence it is enough to deal with sets Z containing r very general points with r in the range $m^N \leq r < (m+1)^N$. The general yoga of our proof is the following: We use lower bounds on the Waldschmidt constant of Z provided either by Theorem 3.1 or by Proposition 4.4 and check, by naive conditions count, that $\alpha(mZ)$ is small enough for the inequality (2) to be satisfied.
Case 1. For $N \geq 4$ and $m \geq 3$, it follows from Lemma 5.2 that there exists a hypersurface

in $\mathbb{P}^N$ of degree $m(m+N-1)-N+1$ vanishing to order at least m at all points of Z. Since in any case it is $\widehat{\alpha}(Z) \geq m$ by Theorem 3.1, it follows that

$$\widehat{\alpha}(Z) \geq m \geq \frac{\alpha(mZ)+N-1}{m+N-1}$$

and we are done in this case.
Case 2. Let $N=3$ and let $2 \leq m = 2n+\varepsilon$ with $\varepsilon \in \{0,1\}$. Assume that

$$m^3 \leq r \leq (n+1+\varepsilon)(m+1)^2 + (n-1)m^2.$$

It follows again from the naive conditions count that there exists a surface in $\mathbb{P}^3$ of degree m^2+2m-2 passing with multiplicity at least m through all points in Z. Hence $\alpha(mZ) \leq m(m+2)-2$ and thus

$$\frac{\alpha(mZ)+2}{m+2} \leq m \leq \widehat{\alpha}(Z),$$

which is exactly (1).
If the number of points is in the range

$$(n+1+\varepsilon)(m+1)^2 + (n-1)m^2 \leq r < (m+1)^3,$$

then Proposition 4.4 implies that

$$\widehat{\alpha}(Z) \geq m+1 - \frac{m+1-n-1-\varepsilon}{m+1} \geq m + \frac{1}{2}.$$

If $m=2n$ is even, then there exists a surface of degree $4n^2+9n+2$ vanishing at all points of Z to order at least m. Indeed, this follows from the inequality

$$\binom{4n^2+9n+5}{3} \geq (m+1)^3 \binom{m+2}{3},$$

which is equivalent to

$$8n^5 + (62/3)n^4 + (39/2)n^3 + (47/6)n^2 + n \geq 0.$$

Hence $\alpha(mZ) \leq (m+\frac{1}{2})(m+2)-2$, which gives

$$\widehat{\alpha}(Z) \geq m + \frac{1}{2} \geq \frac{\alpha(mZ)+2}{m+2},$$

hence (1) holds.
The case $m = 2n+\varepsilon$ is similar and we leave it as a simple exercise.
Case 3. Let $m=2$ and let Z be a set of r very general points in $\mathbb{P}^N$ with $2^N \leq r < 3^N$. In any case it is $\widehat{\alpha}(Z) \geq 2$ by Theorem 3.1. For $N \geq 7$ this bound suffices to conclude Conjecture 1.5. Indeed, since

$$\binom{2N+3}{N} \geq 3^N(N+1) \quad \text{holds for } N \geq 7,$$

there is a hypersurface of degree $N+3$ singular in points of Z. Hence $\alpha(2Z) \leq N+3$ and this implies

$$\widehat{\alpha}(Z) \geq 2 \geq \frac{\alpha(2Z)+N-1}{N+1}.$$

For $4 \leq m \leq 6$ we split the argument in two cases:

a) $r \leq 2 \cdot 3^{N-1} + 2^{N-1}$ and

b) $r \geq 2 \cdot 3^{N-1} + 2^{N-1}$.

In case a) the previous argument works. There is a hypersurface of degree $N+3$ in $\mathbb{P}^N$ singular in points of Z. In case b) we apply Proposition 4.4 with $s = 2$ and $k = 2$. It follows then that $\widehat{\alpha}(Z) \geq 8/3$. By elementary conditions count, there is a hypersurface of degree $2N+1$ singular at Z, so that $\alpha(2Z) \leq 2N+1$. Hence

$$\widehat{\alpha}(Z) \geq \frac{8}{3} \geq \frac{2N+1+N-1}{N+1}$$

holds as $N \leq 7$.

Case 4. Finally we are left with $m = 1$ but this has been proved for all N in [9] and independently in [12]. □

Remark 4.9. Using similar methods one can easily check if the bound for $\widehat{\alpha}(\mathbb{P}^N; r)$ is sufficient to prove the Demailly Conjecture for a given N, m and r. We wrote an appropriate procedure (`Demailly` in `boundforWC`) and check that, for example, the Conjecture holds for all $N \leq 3$, $m \leq 3$ and any number of very general points.

5 Auxiliary results

Lemma 5.1. *Assume that positive real numbers $a_1, \ldots, a_{t-1}$ are given, satisfying*

$$1 - \sum_{j=1}^{t-1} \frac{1}{a_j} \neq 0.$$

Let C be a real number. Consider the following system of linear equations:

$$\begin{cases} C - \sum\limits_{i=1}^{t-1} x_i &= a_k(1 - x_k) \text{ for } k = 1, \ldots, t-1 \\ y &= \sum\limits_{i=1}^{t-1} x_i. \end{cases}$$

Then there is the unique solution for $(x_1, \ldots, x_{t-1}, y)$ to this system. In particular

$$y = \frac{C(\sum\limits_{j=1}^{t-1} \frac{1}{a_j}) - (t-1)}{\sum\limits_{j=1}^{t-1} \frac{1}{a_j} - 1}.$$

Proof. We look for the (unique) solution for y, thus we use Cramer's rule. The matrix of this system (after some reorganisation: the variable y is placed in the first column, then $x_1, \ldots, x_{t-1}$, then non-linear part) is equal to

$$M := \begin{bmatrix} 0 & 1-a_1 & 1 & 1 & \ldots & 1 & C - a_1 \\ 0 & 1 & 1-a_2 & 1 & \ldots & 1 & C - a_2 \\ 0 & 1 & 1 & 1-a_3 & \ldots & 1 & C - a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & \ldots & 1-a_{t-1} & C - a_{t-1} \\ -1 & 1 & 1 & 1 & \ldots & 1 & 0. \end{bmatrix}.$$

We denote the columns of M by $[A_0 A_1 \dots A_{t-1} B]$.

To compute the determinant of the main matrix $[A_0 A_1 \dots A_{t-1}]$ we subtract the last row from the others, obtaining the matrix with the first column and last rows filled with 1 (except -1 in the left bottom corner), and then $-a_1, -a_2, \dots$ over the diagonal. Applying Laplace rule we compute this determinant to be equal to

$$D_1 = \left(\sum_{i=1}^{t-1} a_1 \dots \widehat{a_i} \dots a_{t-1} \right) - a_1 \dots a_{t-1} = a_1 \dots a_{t-1} \cdot \left(\sum_{j=1}^{t-1} \frac{1}{a_j} - 1 \right)$$

which is non-zero (by the assumption). Hence the solution is unique.

To compute the determinant of the matrix $[B A_1 A_2 \dots A_{t-1}]$ we "kill" all 1's using the last row, then "kill" all a_i's in the first column using other columns, obtaining the matrix with

$$\begin{bmatrix} C & -a_1 & & \\ \vdots & & \ddots & \\ C & & & -a_{t-1} \\ t & 1 & \dots & 1 \end{bmatrix}.$$

By the Laplace rule, the determinant

$$D_2 = C \left(\sum_{i=1}^{t-1} a_1 \dots \widehat{a_i} \dots a_{t-1} \right) - (t-1) a_1 \dots a_{t-1} = a_1 \dots a_{t-1} \left(C \left(\sum_{j=1}^{t-1} \frac{1}{a_j} \right) - (t-1) \right).$$

By the Cramer's rule, the claim follows. □

Lemma 5.2. *For all $N \geq 4$, $m \geq 3$ there is*

$$\binom{m(m+N-1)+1}{N} > \binom{m+N-1}{N} (m+1)^N. \tag{20}$$

Proof. With $m \geq 3$ fixed, the proof goes by induction on N. In the initial case $N = 4$ it is elementary to check that the claim is equivalent to the inequality

$$m^2(2m^5 + 11m^4 - 89m^2 - 146m - 42) > 0,$$

which is fulfilled for all $m \geq 3$.

For the induction step, we assume that (20) holds and we want to show that

$$\binom{m(m+N)+1}{N+1} > \binom{m+N}{N+1} (m+1)^{N+1} \tag{21}$$

holds as well. It is convenient to abbreviate $A = m(m+N)$. Using the induction assumption and after elementary operations we get

$$\binom{m(m+N)+1}{N+1} > \binom{m+N}{N+1} (m+1)^{N+1} \cdot$$
$$\cdot \frac{(A+1)A(A-1)\dots(A-m)}{(m+1)(m+N)(A-N)(A-N-1)\dots(A-N-m+2)},$$

so that in order to get (21), it suffices to show

$$(A+1)A(A-1)\dots(A-m) \geq (m+1)(m+N)(A-N)(A-N-1)\dots(A-N-m+2),$$

which follows by comparing both sides term by term (there are $(m+1)$ terms on both sides of the inequality). □

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