Components of symmetric wide-matrix varieties

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Motivation

- **FI**-ideals arise naturally in many areas of mathematics: commutative algebra, algebraic statistics, representation theory, etc.
- Important stabilization results have been established for such ideals.
- It is natural to study their asymptotic behaviors.

Notation

Denote by **FI** the category

- whose objects are finite sets, and
- whose morphisms are injections.

Let $[n] := \{1, ..., n\}$. Observe that **FI** is equivalent to the category with objects $[n]$ for $n \geq 0$ and morphisms being injective maps $\pi : [n] \to [m]$.

$\mathbf{FI}^{\mathbf{op}}$ denotes the category opposite to **FI**.

Notation II

- Let K be a Noetherian ring.
- For a fixed positive integer c, let $\mathbf{I} = (I_n)_{n \geq 0}$ be a sequence, where each I_n is an ideal in the polynomial ring

$$K[X_{c \times n}] := K[x_{i,j} \mid 1 \leq i \leq c, 1 \leq j \leq n]$$

  such that:

  1. I_n is preserved by the natural action of the symmetric group $\mathrm{Sym}([n])$ on $K[X_{c \times n}]$ by permuting the columns $\pi x_{i,j} = x_{i,\pi(j)}$; and
  2. $I_n \subseteq I_{n+1}$.

- Such a sequence $\mathbf{I} = (I_n)_{n \geq 1}$ of ideals is called an **FI**-ideal.

Asymptotic behavior

Let $\mathbf{I} = (I_n)_{n\geq 1}$ be an **FI**-ideal.

Theorem (Nagel-Römer - ...)

- $\mathbf{I} = (I_n)_{n\geq 1}$ *stablizes: there exist* $n_0 \geq 1$ *such that*

$$I_n = \langle Sym([n])(I_m)\rangle \quad \text{for all } \; n \geq m \geq n_0.$$

- *There exist integer constants a and b such that*

$$codim(I_n) = an + b \;\; \text{for all } \; n \gg 0.$$

Conjecture

Projective dimension pd(I_n) and Castelnuovo-Mumford Regularity reg(I_n) are eventually linear functions.

A motivating question

How does the number of irreducible components of the variety $V(I_n)$ eventually grow (up to symmetry)?

Example

Let, for each $n \in \mathbb{Z}_{\geq 0}$,

$$I_n := \langle x_i^2 - 1 \mid i \in [n] \rangle \subset \mathbb{C}[x_1, ..., x_n].$$

- $(I_n)_{n \geq 1}$ satisfy the above two conditions.
- $V(I_n) = \{(p_1, ..., p_n) \mid p_i \in \{1, -1\} \text{ for all } i\}$
- The number of irreducible components of $V(I_n)$ is 2^n.
- Two points P and Q of $V(I_n)$ are in the same Sym($[n]$)-orbit if and opnly if P and Q have same number of $-1's$.
- Hence the number, up to symmetry, of irreducible components of $V(I_n)$ is $n + 1$.

$\mathbf{FI}^{\mathbf{op}}$-schemes

Dual to an **FI**-ideal $\mathbf{I} = (I_n)_{n \geq 1}$, we have a sequence $\mathbf{X} = (X_n)_{n \geq 1}$ of schemes, where

$$X_n = Spec(K[X_{c \times n}]/I_n),$$

a closed subscheme of scheme of matrices $A_n = \mathrm{Spec}(K[X_{c \times n}])$.
Then the two conditions above express that

1. X_n is preserved by the induced action of $\mathrm{Sym}([n])$ on A_n by permuting the columns; and
2. forgetting the last column maps X_{n+1} into X_n.

Such a sequence $\mathbf{X} = (X_n)_n$ of symmetric wide matrix schemes is called an $\mathbf{FI}^{\mathbf{op}}$-scheme over K.

Main Theorem

- Let $\mathbf{X} = (X_n)_{n\geq 1}$ be an $\mathbf{FI}^{\mathbf{op}}$-scheme over a noetherian ring K.
- Then the action of $\mathrm{Sym}([n])$ on X_n induces an action of $\mathrm{Sym}([n])$ on the set $\mathcal{C}(X_n)$ of irreducible components of X_n.

Theorem (Draisma-Eggermont-A.)

The number $|\mathcal{C}(X_n)/\,\mathrm{Sym}([n])|$ of $\mathrm{Sym}([n])$-orbits on $\mathcal{C}(X_n)$ is a quasipolynomial in n when n is sufficiently large.

- A *quasipolynomial* is a function $f : \mathbb{Z} \to \mathbb{R}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$$

where each $c_i : \mathbb{Z} \to \mathbb{R}$ is periodic with integral period.

Example

Set $K := \mathbb{C}(t)$, where t is a variable. Fix c=1, and let

$$I_n := \langle x_i^2 - t \mid i \in [n] \rangle \subset K[X_n] = K[x_1, ..., x_n].$$

- The set $\mathcal{C}(X_n)$ of irreducible components of $X_n = Spec(K[X_n]/I_n)$ is in bijection with the set of minimal prime ideals containing I_n.
- I_1 is a prime ideal, so is minimal containing itself. Thus $\mathcal{C}(X_1) = \{I_1\}$.
- For $n \geq 2$ and two distinct $i, j \in [n]$ any minimal prime ideal P containing I_n also contains $(x_i^2 - t) - (x_j^2 - t) = (x_i - x_j)(x_i + x_j)$ and hence

  $$\text{either } x_i - x_j \in P \text{ or } x_i + x_j \in P,$$

  but not both as P is minimal.

continued...

- Define a bijective map from $\mathcal{C}(X_n)$ to the set of unordered partitions $\{A, B\}$ of $[n]$ into two parts that sends the prime ideal P to

$$\{A = \{j \mid x_1 - x_j \in P\}, B = \{j \mid x_1 + x_j \in P\}\}$$

  where A, B are disjoint subsets of $[n]$ whose union is $[n]$.
- Thus $|\mathcal{C}(X_n)| = 2^{n-1}$ as the number of unordered partitions of $[n]$ is 2^{n-1}.
- Two unordered partitions $\{A, B\}$, and $\{\tilde{A}, \tilde{B}\}$ are in the same $\mathrm{Sym}([n])$-orbit if and only if

$$\min\{|A|, |B|\} = \min\{|\tilde{A}|, |\tilde{B}|\},$$

  and this number takes any of the values in $\{0, \ldots, \lfloor n/2 \rfloor\}$.
- Hence $|\mathcal{C}(X_n)/\,\mathrm{Sym}([n])| = \lfloor n/2 \rfloor + 1$ for all $n \geq 1$.

An application

Let K be a field, let $S \subseteq K$ be a finite subset, and let k be a natural number. For every $n \in \mathbb{Z}_{\geq 0}$, define

$$M_n := \{A \in S^{n \times n} \subseteq K^{n \times n} \mid rk(A) \leq k\},$$

the set of all rank-$\leq k$ matrices all of whose entries are in S. Let $\mathrm{Sym}([n])$ act by simultaneous row and column permutations on M_n. Then $|M_n / \mathrm{Sym}([n])|$ is a quasipolynomial in n for $n \gg 0$.

Proof

- Consider the morphism $\varphi : \mathbb{A}_K^{k \times n} \times \mathbb{A}_K^{k \times n} \to \mathbb{A}_K^{n \times n}$ given by $(A, B) \mapsto A^T \cdot B$ with image Y_n, the subvariety of rank-$\leq k$ matrices.
- By classical invariant theory, φ is the quotient map for the action of the reductive K-group GL_k acting via $(g, (A, B)) \mapsto (g^{-T} A, gB)$.

continued...

- By properties of the quotient map, φ yields a bijection between closed subsets of Y_n and closed GL_k-stable subsets of $\mathbb{A}_K^{k\times n} \times \mathbb{A}_K^{k\times n} \cong \mathbb{A}_K^{2k\times n}$.
- Let $X_n = \varphi^{-1}(M_n) \subseteq \mathbb{A}_K^{2k\times n}$ correspond to M_n under this bijection. Then $(X_n)_{n\geq 1}$ is a wide-matrix scheme, and the irreducible components of X_n are in bijection with the points of M_n.
- Hence $|M_n/\operatorname{Sym}([n])|$ is a quasipolynomial in n for $n \gg 0$.

Idea of the proof of the Main Theorem

Let $\mathbf{X} = (X_n)_{n\geq 1}$ be an $\mathbf{FI}^{\mathbf{op}}$-scheme.

- Define $X_n' := X_{n_0+n}$ the "shift" over n_0.
- For a suitable n_0 and h in the base ring K of $\mathbf{X}'$, $\mathbf{X}'[1/h]$ is of the product form, that is $X_n'[1/h] = Z^n$ with the obvious $\mathrm{Sym}([n])$ action.
- Prove the main theorem for wide matrix varieties of the product form by relating their components to integral points in rational polyhedra and using a result by Stanley.
- Do the general case by
  - counting the components of $\mathbf{X}$ where $h = 0$ (Noetherian induction)
  - counting integral points in rational polyhedra acted upon by certain groupoids and using Stanley's result and the orbit count formula for groupoids.

Thank You