# Identifiability of rank-3 tensors

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Core problem: Understand if a given tensor $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$ admit a unique decomposition as a sum of pure tensors

$$T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i} \qquad \left(v_{j,i} \in \mathbb{C}^{n_j+1}, j = 1, \ldots, k\right).$$

Applications

| | |
|---|---|
| Phylogenetic | (Allman, Rhodes, Sullivant...) |
| S. Processing | (Jiang, Sidiropoulos...) |

Pure mathematics

| | |
|---|---|
| Generic | (Ciliberto, Chiantini, Galuppi, Hauenstein, Mella, Oeding, Ottaviani, Sommese...) |
| Specific | [Kruskal'70], [De Lathauwer-Domanov'13], [Chiantini-Ottaviani-Vannieuwenhoven'17], [Lovitz-Petrov'21]. |

# Overview

# Tensors in the projective space

We work over $\mathbb{C}$.

- Let $V_1, \ldots, V_k$ be vectors spaces, $\dim(V_i) = n_i + 1$.
  The Segre variety is the image of

$$\nu : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k) \to \mathbb{P}(V_1 \otimes \cdots \otimes V_k) := \mathbb{P}^N$$

$$([v_1], \ldots, [v_k]) \mapsto [v_1 \otimes \cdots \otimes v_k].$$

- $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $X := \nu(Y) \subset \mathbb{P}^N$.
- The rank of a tensor $q \in \mathbb{P}^N$ is

$$r(q) := \min\{r \in \mathbb{N} \mid q \in \langle p_1, \ldots, p_r \rangle,\ p_i \in X\}.$$

## Secant varieties

Fix $r > 0$. The $r$-th secant variety $\sigma_r(X)$ of $X \subset \mathbb{P}^N$ is

$$\sigma_r(X) := \overline{\bigcup_{p_1,\ldots,p_r \in X} \langle p_1, \ldots, p_r \rangle}.$$

# Identifiability of Tensors

- A tensor $q \in \mathbb{P}^N$ of rank $r > 0$ is identifiable if there exists a unique $r$-uple of points $p_1, \ldots, p_r \in X$ such that $q \in \langle p_1, \ldots, p_r \rangle$.
- For any $q \in \mathbb{P}^N$, we define the space of solutions of $q$ as

$$\mathcal{S}(Y, q) := \{A \subset Y \mid \#(A) = r(q) \text{ and } q \in \langle \nu(A) \rangle\}.$$

  - If q is identifiable then $\#\mathcal{S}(Y, q) = 1$.
  - If $A \in \mathcal{S}(Y, q)$, then $A$ evinces the rank of $q$.

### Lemma

[1] *For any* $q \in \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$*, there is a unique minimal multiprojective space* $Y' \simeq \mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_k} \subseteq Y \simeq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *with* $n'_i \leq n_i$, $i = 1, \ldots, k$ *such that* $\mathcal{S}(Y, q) = \mathcal{S}(Y', q)$.

### Definition (concise Segre)

Given a point $q \in \mathbb{P}^N$, we will call concise Segre the variety $X_q := \nu(Y')$ where $Y' \subseteq Y$ is the minimal multiprojective space $Y' \subseteq Y$ such that $q \in \langle \nu(Y') \rangle$ as in Concision/Autarky Lemma.

[1] e.g. J. M. Landsberg. *Tensors: Geometry and Applications.* Graduate Studies in Mathematics. Amer. Math. Soc. Providence, 128 (2012).

# Building the concise Segre...

- Let $\pi_i \colon Y \longrightarrow \mathbb{P}^{n_i}$ be the projection onto the $i$-th factor of $Y$.

The minimal $Y'$ defining the concise Segre of a point $q$ can be obtained as follows.

- Fix any $A \in \mathcal{S}(Y,q)$, set $A_i := \pi_i(A) \subset \mathbb{P}^{n_i}$, for all $i = 1, \ldots, k$.
- Each $\langle A_i \rangle \subseteq \mathbb{P}^{n_i}$ is a well-defined projective subspace of dimension at most $\min\{n_i, r(q)-1\}$.
- By Concision/Autarky we have $Y' = \prod_{i=1}^{k} \langle A_i \rangle$.
- If for one $A \in \mathcal{S}(Y,q)$ the set $\pi_i(A)$ is a single point then the $i$-th factor won't appear in the concise Segre.

# Rank-2 tensors

With rank-2 tensors we reduce to work with $Y = (\mathbb{P}^1)^k$ thanks to Concision/Autarky. For the general case everything was already known.

- Matrix case and 3-factor case are classical.
- $k \geq 4$. [2]

## Proposition

*Let $q \in \sigma_2^0(X)$. Then $|\mathcal{S}(Y, q)| > 1$ if and only if the concise Segre $X_q$ of $q$ is $X_q = \nu(\mathbb{P}^1 \times \mathbb{P}^1)$.*

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[2] C. Bocci, L. Chiantini and G. Ottaviani. *Refined methods for the identifiability of tensors.* Annali di Matematica Pura ed Applicata. 193 (2013).

# Rank-3 tensors

Let $q \in \mathbb{P}^N$ be a rank-3 tensor.

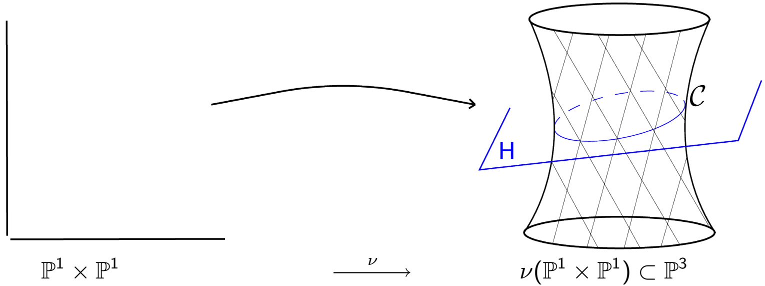
- Fix $A \in \mathcal{S}(Y, q)$, where $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.
- Note that for all $i = 1, \ldots, k$

$$\dim\langle \pi_i(A)\rangle = \begin{cases} 0 & \Longrightarrow \text{ we get rid of the } i\text{-th factor} \\ 1 & \Longrightarrow \text{ the } i\text{-th factor becomes } \mathbb{P}^1 \\ 2 & \Longrightarrow \text{ the } i\text{-th factor becomes } \mathbb{P}^2 \end{cases}$$

Therefore we reduce to work with multiprojective spaces given by products of projective lines and planes.

# Examples 1 and 2

Let $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Consider the Segre embedding on the last two factors and take a hyperplane section which intersects $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ in a conic $\mathcal{C}$.




Take a point $q \in \langle \nu(\mathbb{P}^2 \times \mathcal{C}) \rangle$. We distinguish two different cases depending on whether $\mathcal{C}$ is irreducible or not.

The previous construction is equivalent to consider an irreducible divisor $G \in |\mathcal{O}_Y(0,1,1)|$.

- $G \cong \mathbb{P}^2 \times \mathbb{P}^1$ embedded via $\mathcal{O}(1,2)$.
- Therefore $\dim \sigma_2(\nu(G)) = 7$ and thus $\sigma_2(\nu(G)) \subsetneq \langle \nu(G) \rangle \simeq \mathbb{P}^8$.

As a direct consequence we get that a general point $q \in \langle \nu(G) \rangle$ has $\nu(G)$-rank 3 and it is not-identifiable because of the non-identifiability of the points on $\langle \mathcal{C} \rangle$.
Moreover we proved that $\mathcal{S}(Y, q) = \mathcal{S}(G, q)$.

# Example 2

## Reducible $\mathcal{C}$

Let $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Take $G = G_1 \cup G_2$ a reducible element of $|\mathcal{O}_Y(0,1,1)|$, where

- $G_1 \in |\mathcal{O}_Y(0,0,1)|$, i.e. $G_1 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \{\mathrm{pt}\}$,
- $G_2 \in |\mathcal{O}_Y(0,1,0)|$, i.e. $G_2 \cong \mathbb{P}^2 \times \{\mathrm{pt}\} \times \mathbb{P}^1$.

We proved that

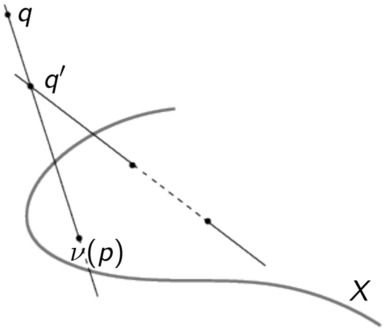
$$\langle \nu(G) \rangle = \mathrm{Join}(\sigma_2(\nu(G_1)), \nu(G_2)) = \mathrm{Join}(\sigma_2(\nu(G_2)), \nu(G_1)).$$

A general $q \in \langle \nu(G) \rangle = \mathrm{Join}(\sigma_2(\nu(G_1)), \nu(G_2))$ has rank 3 and for the subsets evincing its rank we have a 4-dimensional family of sets $A$ such that $\sharp(A) = 3$, $\# A \cap G_1 = 2$, $\# A \cap G_2 = 1$
$A \cap G_1 \cap G_2 = \emptyset$ (analogously, by looking at q as an element of $\mathrm{Join}(\sigma_2(\nu(G_2)), \nu(G_1))$ ).
Also in this case $\mathcal{S}(Y, q) = \mathcal{S}(G, q)$.

## Example 3

Take $Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\} \subseteq Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \geq 2$, $n_1, n_2 \leq 2$, $n3 = \cdots = n_k = 1$.
Take $q' \in \langle \nu(Y') \rangle \setminus \nu(Y')$, $A \in \mathcal{S}(Y', q')$ and $p \in Y \setminus Y'$.
Assume that $Y$ is the minimal multiprojective space containing $A \cup \{p\}$ and take $q \in \langle \{q', \nu(p)\} \rangle \setminus \{q', \nu(p)\}$.

- If $k \geq 3$ and $\sum_{i=1}^{k} n_i \geq 4$ then $r_{\nu(Y)}(q) = 3$ and $\mathcal{S}(Y, q) = \{\{p\} \cup A\}_{A \in \mathcal{S}(Y', q')}$.



Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the multiprojective space of the concise Segre of a rank-3 tensor $q$. The rank-3 tensor $q$ is identifiable except in the following cases:

1. $q$ is a rank-3 matrix, in this case $\dim(\mathcal{S}(Y, q)) = 6$;
2. $q$ belongs to a tangent space of the Segre embedding of $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(\mathcal{S}(Y, q)) \geq 2$;
3. $q$ is an order-4 tensor of $\sigma_3^0(Y)$ with $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(\mathcal{S}(Y, q)) \geq 1$. [3]

[3] M.V. Catalisano, A.V. Geramita, A. Gimigliano. Ranks of tensors, secant varieties of Segre varieties and fat points. Linear Algebra Appl. 355 (2002).

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3. $q$ is an order-4 tensor of $\sigma_3^0(Y)$ with $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(\mathcal{S}(Y, q)) \geq 1$.
4. $q$ is as in Example 1 where $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(\mathcal{S}(Y, q)) = 3$;
5. $q$ is as in Example 2 where $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\mathcal{S}(Y, q)$ contains two different 4-dimensional families;
6. $q$ is as in Example 3. In this case $\dim(\mathcal{S}(Y, q)) \geq 2$ and if $n_1 + n_2 + k \geq 6$ then $\dim(\mathcal{S}(Y, q)) = 2$.

## Outline of the proof

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where all $n_i \in \{1, 2\}$. Let $q \in \mathbb{P}^N$, assume $A, B \in \mathcal{S}(Y, q)$ and call $S := A \cup B$.

We proved that $\# A \cap B \leq 1$.

### Main Tool

$\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $\hat{\varepsilon}_i = (1, \ldots, 1, 0, 1, \ldots, 1)$.

**Lemma (BBCG[a])**

*Let $k \geq 2$ and consider $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where all $n_i \geq 1$. Let $q \in \mathbb{P}^N$, $A, B \in \mathcal{S}(Y, q)$ be two different subsets evincing the rank of $q$ and write $S = A \cup B$. Let $D \in |\mathcal{O}_Y(\varepsilon)|$ be an effective Cartier divisor such that $A \cap B \subset D$, where $\varepsilon = \sum_{i \in I} \varepsilon_i$ for some $I \subset \{1, \ldots, k\}$. If $h^1(\mathcal{I}_{S \setminus S \cap D}(\hat{\varepsilon})) = 0$ then $S \subset D$.*

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[a] E. Ballico, A. Bernardi, M. Christandl and F. Gesmundo. *On the partially symmetric rank of tensor product of W-states and other symmetric tensors.* Rend. Lincei Math. Appl. 30, 93-124 (2019).

Thank you!

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