Identifiability of rank-3 tensors
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Identifiability
Quick introduction

Core problem: Understand if a given tensor $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$ admit a unique decomposition as a sum of pure tensors

$$T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i} \quad (v_{j,i} \in \mathbb{C}^{n_j+1}, j = 1, \ldots, k).$$

Applications

Phylogenetic (Allman, Rhodes, Sullivant...)
S. Processing (Jiang, Sidiropoulos...)

Pure mathematics

Generic (Ciliberto, Chiantini, Galuppi, Hauenstein, Mella, Oeding, Ottaviani, Sommese...)

Specific [Kruskal’70], [De Lathauwer-Domanov’13], [Chiantini-Ottaviani-Vannieuwenhoven’17], [Lovitz-Petrov’21].
Overview

1. Notation

2. Concise Segre and identifiability of rank-2 tensors

3. Identifiability of rank-3 tensors
Tensors in the projective space

We work over \( \mathbb{C} \).

- Let \( V_1, \ldots, V_k \) be vectors spaces, \( \dim(V_i) = n_i + 1 \). The Segre variety is the image of

\[
\nu : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k) \rightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_k) := \mathbb{P}^N
\]

\[
([v_1], \ldots, [v_k]) \mapsto [v_1 \otimes \cdots \otimes v_k].
\]

- \( Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) and \( X := \nu(Y) \subset \mathbb{P}^N \).

- The rank of a tensor \( q \in \mathbb{P}^N \) is

\[
r(q) := \min\{r \in \mathbb{N} \mid q \in \langle p_1, \ldots, p_r \rangle, p_i \in X\}.
\]
Fix $r > 0$. The $r$-th secant variety $\sigma_r(X)$ of $X \subset \mathbb{P}^N$ is

$$
\sigma_r(X) := \bigcup_{p_1,\ldots,p_r \in X} \langle p_1, \ldots, p_r \rangle.
$$
A tensor $q \in \mathbb{P}^N$ of rank $r > 0$ is identifiable if there exists a unique $r$-uple of points $p_1, \ldots, p_r \in X$ such that $q \in \langle p_1, \ldots, p_r \rangle$.

- For any $q \in \mathbb{P}^N$, we define the **space of solutions** of $q$ as
  \[ S(Y, q) := \{ A \subset Y \mid \#(A) = r(q) \text{ and } q \in \langle \nu(A) \rangle \}. \]

- If $q$ is identifiable then $\#S(Y, q) = 1$.
- If $A \in S(Y, q)$, then $A$ evinces the rank of $q$. 

**Identifiability of Tensors**
Lemma

1 For any \( q \in \mathbb{P}(V_1 \otimes \cdots \otimes V_k) \), there is a unique minimal multiprojective space \( Y' \simeq \mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_k} \subseteq Y \simeq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) with \( n'_i \leq n_i \), \( i = 1, \ldots, k \) such that \( S(Y, q) = S(Y', q) \).

Definition (concise Segre)

Given a point \( q \in \mathbb{P}^N \), we will call concise Segre the variety \( X_q := \nu(Y') \) where \( Y' \subseteq Y \) is the minimal multiprojective space \( Y' \subseteq Y \) such that \( q \in \langle \nu(Y') \rangle \) as in Concision/Autarky Lemma.

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Building the concise Segre...

Let \( \pi_i : Y \rightarrow \mathbb{P}^{n_i} \) be the projection onto the \( i \)-th factor of \( Y \). The minimal \( Y' \) defining the concise Segre of a point \( q \) can be obtained as follows.

1. Fix any \( A \in S(Y, q) \), set \( A_i := \pi_i(A) \subset \mathbb{P}^{n_i} \), for all \( i = 1, \ldots, k \).
2. Each \( \langle A_i \rangle \subseteq \mathbb{P}^{n_i} \) is a well-defined projective subspace of dimension at most \( \min\{n_i, r(q) - 1\} \).
3. By Concision/Autarky we have \( Y' = \prod_{i=1}^{k} \langle A_i \rangle \).
4. If for one \( A \in S(Y, q) \) the set \( \pi_i(A) \) is a single point then the \( i \)-th factor won’t appear in the concise Segre.
Rank-2 tensors

With rank-2 tensors we reduce to work with $Y = (\mathbb{P}^1)^k$ thanks to Concision/Autarky. For the general case everything was already known.

- Matrix case and 3-factor case are classical.
- $k \geq 4$. ²

Proposition

Let $q \in \sigma^0_2(X)$. Then $|S(Y, q)| > 1$ if and only if the concise Segre $X_q$ of $q$ is $X_q = \nu(\mathbb{P}^1 \times \mathbb{P}^1)$.

Let \( q \in \mathbb{P}^N \) be a rank-3 tensor.

- Fix \( A \in \mathcal{S}(Y, q) \), where \( Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \).
- Note that for all \( i = 1, \ldots, k \)

\[
\dim \langle \pi_i(A) \rangle = \begin{cases} 
0 & \implies \text{we get rid of the } i\text{-th factor} \\
1 & \implies \text{the } i\text{-th factor becomes } \mathbb{P}^1 \\
2 & \implies \text{the } i\text{-th factor becomes } \mathbb{P}^2 
\end{cases}
\]

Therefore we reduce to work with multiprojective spaces given by products of projective lines and planes.
Examples 1 and 2

Let $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Consider the Segre embedding on the last two factors and take a hyperplane section which intersects $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ in a conic $C$.

Take a point $q \in \langle \nu(\mathbb{P}^2 \times C) \rangle$. We distinguish two different cases depending on whether $C$ is irreducible or not.
Example 1
Irreducible $\mathcal{C}$

The previous construction is equivalent to consider an irreducible divisor $G \in |\mathcal{O}_Y(0, 1, 1)|$.

- $G \cong \mathbb{P}^2 \times \mathbb{P}^1$ embedded via $\mathcal{O}(1, 2)$.
- Therefore $\dim \sigma_2(\nu(G)) = 7$ and thus $\sigma_2(\nu(G)) \subsetneq \langle \nu(G) \rangle \cong \mathbb{P}^8$.

As a direct consequence we get that a general point $q \in \langle \nu(G) \rangle$ has $\nu(G)$-rank 3 and it is not-identifiable because of the non-identifiability of the points on $\langle \mathcal{C} \rangle$.
Moreover we proved that $S(Y, q) = S(G, q)$. 
Example 2
Reducible \( C \)

Let \( Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \). Take \( G = G_1 \cup G_2 \) a reducible element of \( |\mathcal{O}_Y(0, 1, 1)| \), where

- \( G_1 \in |\mathcal{O}_Y(0, 0, 1)| \), i.e. \( G_1 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \{ \text{pt} \} \),
- \( G_2 \in |\mathcal{O}_Y(0, 1, 0)| \), i.e. \( G_2 \cong \mathbb{P}^2 \times \{ \text{pt} \} \times \mathbb{P}^1 \).

We proved that

\[
\langle \nu(G) \rangle = \text{Join}(\sigma_2(\nu(G_1)), \nu(G_2)) = \text{Join}(\sigma_2(\nu(G_2)), \nu(G_1)).
\]

A general \( q \in \langle \nu(G) \rangle = \text{Join}(\sigma_2(\nu(G_1)), \nu(G_2)) \) has rank 3 and for the subsets evincing its rank we have a 4-dimensional family of sets \( A \) such that \( \#(A) = 3, \#A \cap G_1 = 2, \#A \cap G_2 = 1 \)
\( A \cap G_1 \cap G_2 = \emptyset \) (analogously, by looking at \( q \) as an element of \( \text{Join}(\sigma_2(\nu(G_2)), \nu(G_1)) \)).

Also in this case \( S(Y, q) = S(G, q) \).
Example 3

Take $Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\} \subseteq Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \geq 2$, $n_1, n_2 \leq 2$, $n_3 = \cdots = n_k = 1$.

Take $q' \in \langle \nu(Y') \rangle \setminus \nu(Y')$, $A \in S(Y', q')$ and $p \in Y \setminus Y'$.

Assume that $Y$ is the minimal multiprojective space containing $A \cup \{p\}$ and take $q \in \langle \{q', \nu(p)\} \rangle \setminus \{q', \nu(p)\}$.

- If $k \geq 3$ and $\sum_{i=1}^{k} n_i \geq 4$ then $r_{\nu(Y)}(q) = 3$ and $S(Y, q) = \{\{p\} \cup A\}_{A \in S(Y', q')}$. 
Rank-3 tensors

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the multiprojective space of the concise Segre of a rank-3 tensor $q$. The rank-3 tensor $q$ is identifiable except in the following cases:

1. $q$ is a rank-3 matrix, in this case $\dim(S(Y, q)) = 6$;
2. $q$ belongs to a tangent space of the Segre embedding of $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(S(Y, q)) \geq 2$;
3. $q$ is an order-4 tensor of $\sigma_3^0(Y)$ with $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(S(Y, q)) \geq 1$. \(^3\)

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Rank-3 tensors

Main theorem

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the multiprojective space of the concise Segre of a rank-3 tensor $q$. The rank-3 tensor $q$ is identifiable except in the following cases:

1. $q$ is a rank-3 matrix, in this case $\dim(S(Y, q)) = 6$;

2. $q$ belongs to a tangent space of the Segre embedding of $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(S(Y, q)) \geq 2$;

3. $q$ is an order-4 tensor of $\sigma^0_3(Y)$ with $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(S(Y, q)) \geq 1$.

4. $q$ is as in Example 1 where $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $\dim(S(Y, q)) = 3$;

5. $q$ is as in Example 2 where $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, in this case $S(Y, q)$ contains two different 4-dimensional families;

6. $q$ is as in Example 3. In this case $\dim(S(Y, q)) \geq 2$ and if $n_1 + n_2 + k \geq 6$ then $\dim(S(Y, q)) = 2$. 
Outline of the proof

Let \( Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \), where all \( n_i \in \{1, 2\} \). Let \( q \in \mathbb{P}^N \), assume \( A, B \in S(Y, q) \) and call \( S := A \cup B \).

We proved that \( \#A \cap B \leq 1 \).

Main Tool

\( \varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) and \( \hat{\varepsilon}_i = (1, \ldots, 1, 0, 1, \ldots, 1) \).

Lemma (BBCG\(^3\))

Let \( k \geq 2 \) and consider \( Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \), where all \( n_i \geq 1 \). Let \( q \in \mathbb{P}^N \), \( A, B \in S(Y, q) \) be two different subsets evincing the rank of \( q \) and write \( S = A \cup B \). Let \( D \in |\mathcal{O}_Y(\varepsilon)| \) be an effective Cartier divisor such that \( A \cap B \subset D \), where \( \varepsilon = \sum_{i \in I} \varepsilon_i \) for some \( I \subset \{1, \ldots, k\} \). If \( h^1(\mathcal{I}_{S \setminus S \cap D}(\hat{\varepsilon})) = 0 \) then \( S \subset D \).

Thank you!


