Combinatorial Differential Algebra of $x^p$

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Combinatorial Differential Algebra of $x^p$

- at the interface of geometric combinatorics and differential algebra

Linking...  

1. differential ideals  
2. lattice polytopes arising from graphs  
3. regular triangulations
Differential algebra

- study of polynomial O/PDEs with methods from Commutative Algebra

Differential rings and ideals

- \( \mathbb{C}[x^{(\infty)}] \) the ring of differential polynomials in \( x \) over \( \mathbb{C} \), i.e.,
  \((\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots], \partial), \partial(x^{(k)}) = x^{(k+1)}, \partial|_\mathbb{C} \equiv 0, \) Leibniz’ rule
- \( I \triangleleft \mathbb{C}[x^{(\infty)}] \) is a differential ideal if \( \partial(I) \subseteq I \)
- For \( S \subseteq \mathbb{C}[x^{(\infty)}] \), \( \langle S \rangle^\infty \) denotes the differential ideal generated by \( S \).

Bivariate case

\( \mathbb{C}[x^{(\infty, \infty)}] := \mathbb{C}[\{x^{(k, \ell)}\}, \{\partial_s, \partial_t\}] \) with

\[
\partial_s(x^{(k, \ell)}) = x^{(k+1, \ell)}, \quad \partial_t(x^{(k, \ell)}) = x^{(k, \ell+1)}, \quad \partial_s|_\mathbb{C} \equiv 0, \quad \partial_t|_\mathbb{C} \equiv 0
\]

the ring of partial differential polynomials in \( x \) over \( \mathbb{C} \) in the two independent variables \( s \) and \( t \).
Definition

$G \subseteq \mathbb{C}[x^{(\infty)}]$ is a differential Gröbner basis of $\langle G \rangle^{(\infty)}$ if

$\{\partial^k(g) \mid k \in \mathbb{N}, g \in G\}$ is an algebraic Gröbner basis of $\langle G \rangle^{(\infty)}$ w.r.t. $\prec$.

Theorem (Zobnin, 2009)

The singleton $\{x^p\}$ is a differential Gröbner basis of $\langle x^p \rangle^{(\infty)}$ with respect to the reverse lexicographical ordering.
Jets of the fat point $x^p$ on the affine line

$\begin{align*}
R_n & \quad \text{the polynomial ring } \mathbb{C}[x_0, \ldots, x_n] \\
f_{p,n} \in R_n[t] & \quad \text{the polynomial } (x_0 + x_1 t + \cdots + x_n t^n)^p \text{ in } t \\
C_{p,n} \triangleleft R_n & \quad \text{the ideal generated by the coefficients of } f_{p,n} \\
I_{p,n} \triangleleft \mathbb{C}[x(\infty)] & \quad \text{the differential ideal generated by } x^p \text{ and } x^{(n)}
\end{align*}$

Truncating Taylor series

$C_{p,n}$ encodes certain $n$-jets of the fat point $x^p$ on the affine line

Linking $C_{p,n}$ and $I_{p,n}$

$$R_n/C_{p,n} \cong \mathbb{C}[x(\infty)]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}.$$

Question

For fixed $n$, is $\dim_\mathbb{C}(R_n/C_{p,n})$ a polynomial in $p$ of degree $n + 1$?
Example: \( \dim_{\mathbb{C}}(R_6/C_{p,6})_{p \in \mathbb{N}} \)

The first 13 entries of the sequence \( \dim_{\mathbb{C}}(R_6/C_{p,6})_{p \in \mathbb{N}} \) are\(^1\)

\[ 0, 1, 34, 353, 2037, 8272, 26585, 72302, 173502, 377739, 760804, 1437799, 2576795, \]

coinciding with the sequence https://oeis.org/A244881.

Interpolating polynomial (computed on the values for \( p = 1, \ldots, 20 \)):

\[
\frac{17}{315} p^7 + \frac{17}{90} p^6 + \frac{53}{180} p^5 + \frac{19}{72} p^4 + \frac{13}{90} p^3 + \frac{17}{360} p^2 + \frac{1}{140} p,
\]

of degree \( 7 = 6 + 1 \).

\(^1\)computed with Singular
Counting lattice points of polytopes

\( C \) an integral \( d \)-dimensional polytope
\( tC \) the polytope dilated by \( t \in \mathbb{N} \)

Then: \( |tC \cap \mathbb{Z}^n| \) is a polynomial in \( t \) of degree \( d \), the \textbf{Ehrhart polynomial} of \( C \).

\textbf{Theorem (Ait El Manssour–S., 2021)}

The number \( \dim_C(R_n/C_{p,n}) \) is the Ehrhart polynomial of the polytope

\[ P_n := \{(w_0, \ldots, w_n) \in (\mathbb{R}_{\geq 0})^{n+1} | w_i + w_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n - 1 \} \]
evaluated at \( p - 1 \).

\textbf{Proof: Results from graph theory +}

\textbf{Proposition (Bruschek–Mourtada–Schepers, 2013)}

\( \text{in}_{\text{revlex}}(C_{p,n}) \) is generated by \( \{x_i^{u_i}x_{i+1}^{u_{i+1}} | u_i + u_{i+1} = p, 0 \leq i \leq n - 1 \} \).

\( \triangleright \) graph \( G \) with \( V = \{0, 1, \ldots, n\} \) and \( E = \{[i, i + 1]\}_{i=0,\ldots,n-1} \)
Fractional stable set polytope of a graph

\( G \) an undirected graph with vertices \( V \) and edges \( E \)
\( C(G) \) the cliques of \( G \)

Two polytopes

- **Stab\((G)\) := \( \text{conv}\{\chi^S \in \mathbb{R}^V | S \subseteq V \text{ stable}\} \)** the stable set polytope of \( G \), with \( \chi^S = (\chi^S_v)_{v \in V} \in \mathbb{R}^V \) incidence vectors
- **QStab\((G)\) := \{x \in \mathbb{R}^V | 0 \leq x(v) \forall v \in V, \sum_{v \in Q} x(v) \leq 1 \forall Q \in C(G)\} \)** the fractional stable set polytope of \( G \)

Then: \( \text{Stab}(G) = \text{conv}\{\{0, 1\}^V \cap \text{QStab}(G)\} \).

**Theorem (Chvátal, 1975)**

A graph \( G \) is perfect iff \( \text{Stab}(G) = \text{QStab}(G) \)
Bivariate case

\[ R_{m,n} \]

\[ f_{p,(m,n)} \in R_{m,n}[s, t] \]

\[ C_{p,(m,n)} \triangleleft R_{m,n} \]

\[ I_{p,(m,n)} \triangleleft \mathbb{C}[x^{(\infty, \infty)}] \]

the polynomial ring \( \mathbb{C}[\{x_k, \ell\}_{0 \leq k \leq m, 0 \leq \ell \leq n}] \)

the polynomial \( (x_{00} + x_{10}s + \cdots + x_{mn}s^m t^n)^p \) in \( s \) and \( t \)

the ideal generated by the coefficients of \( f_{p,(m,n)} \)

the differential ideal generated by \( x^p, x^{(m,0)}, \) and \( x^{(0,n)} \)

Linking \( C_{p,(m,n)} \) and \( I_{p,(m,n)} \)

\[ \frac{R_{m,n}}{C_{p,(m,n)}} \cong \frac{\mathbb{C}[x^{(\infty, \infty)}]}{I_{p,(m+1,n+1)}}, \quad x_k, \ell \mapsto \frac{1}{k! \ell!} x^{(k, \ell)}. \]

Looking for monomial orderings...

...for which the coefficients of \( f_{p,(m,n)} \) are a Gröbner basis of \( C_{p,(m,n)}. \)
The regular triangulation $T_{m,2}$

$T_{m,2}$ is the placing triangulation of the $m \times 2$-rectangle for the point configuration $[(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), \ldots, (m,0), (m,1), (m,2)]$ induced by the vector $(1, 2, \ldots, 2^{3m+2})$ (lower hull convention).

Figure: The regular triangulation $T_{m,2}$
The regular triangulation $T_{m,n}$

$T_{m,n}$ the placing triangulation of the $m \times n$-rectangle for the point configuration

$[(0,0), (0,1), \ldots, (0,n), (1,0), \ldots, (1,n), \ldots, (m,0), \ldots, (m,n)]$

**Figure:** The regular triangulation $T_{1,n}$
T-orderings

$T$ a triangulation of the $m \times n$-rectangle

**Definition**

A monomial ordering $\prec$ on $\mathbb{C}[\{x^{(k,\ell)}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}]$ is called **T-ordering** if each of the leading monomials of $(x^p)^{(k,\ell)}$ is supported on a triangle of $T$.

**Proposition (Ait El Manssour–S., 2021)**

For all $k, \ell$, $(x^p)^{(k,\ell)} \in \mathbb{C}[x^{(\leq m, \leq n)}]$ has a unique monomial supported on a triangle of $T_{m,n}$. The reverse lexicographical ordering $\prec$ on $\mathbb{C}[x^{(\leq m, \leq n)}]$ is a $T_{m,n}$-ordering for all $p$. 
A higher-dimensional analog of Zobnin’s result

≺ a \( T_{m,2} \)-ordering

**Theorem (Ait El Manssour–S., 2021)**

For all \( m, p \in \mathbb{N} \), \( \{(x^p)^{(k,\ell)}\}_{0 \leq k \leq mp, 0 \leq \ell \leq 2p} \) is a Gröbner basis of \( \langle x^p \rangle^{(\infty,\infty)} \) in \( \mathbb{C}[x^{(\leq m,\leq 2)}] \) with respect to any \( T_{m,2} \)-ordering.

**Theorem (Ait El Manssour–S., 2021)**

For all \( m \in \mathbb{N} \), \( \dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)}) \) is the Ehrhart polynomial of the \( 3(m+1) \)-dimensional lattice polytope

\[
P_{(m,2)} := \left\{ (u_{00}, u_{01}, u_{02}, \ldots, u_{m0}, u_{m1}, u_{m2}) \in (\mathbb{R}_{\geq 0})^{3(m+1)} \mid u_{k1,1} + u_{k2,2} + u_{k3,3} \leq 1 \right. \\
\text{for all indices s.t. } \{(k_1, \ell_1), (k_2, \ell_2), (k_3, \ell_3)\} \text{ is a triangle of } T_{m,2} \}
\]
evaluated at \( p - 1 \).
Example: $C_{3,(2,2)}$

$C_{(2,2)}$ the ideal in $R_{2,2} = \mathbb{C}[x_{00}, x_{10}, x_{01}, x_{02}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}]$ generated by the $(2p + 1)^2$ many coefficients of $f_{p,(2,2)} \in R_{2,2}[s, t]$ 

the **weighted reverse lexicographical ordering** on $R_{2,2}$ for $w_{2,2} := (2^8 + 1, \ldots, 2^8 + 1) - (2^0, 2^1, \ldots 2^8) \in \mathbb{N}^9$

In the leading monomials of the coefficients of $f_{3,(2,2)}$, the following triples of variables show up:

$$\{x_{00}, x_{01}, x_{10}\}, \{x_{01}, x_{02}, x_{10}\}, \{x_{02}, x_{10}, x_{11}\}, \{x_{02}, x_{11}, x_{12}\},$$
$$\{x_{10}, x_{11}, x_{20}\}, \{x_{11}, x_{12}, x_{20}\}, \{x_{12}, x_{20}, x_{21}\}, \{x_{12}, x_{21}, x_{22}\}.$$

The indices of those define the triangles of the regular triangulation $T_{2,2}$:
Figure: For height vectors inducing those four regular unimodular triangulations of the 3 × 2-rectangle, the weighted reverse lexicographical ordering turns the coefficients of $f_{p,(3,2)}$ into a Gröbner basis of $C_{p,(3,2)}$. 

$m = 3, n = 2$
Open problems

Question 1
For which $m, n, p \in \mathbb{N}$ does there exist a regular unimodular triangulation $T$ of the $m \times n$-rectangle such that the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of $C_{p,(m,n)}$ with respect to the weighted reverse lexicographical ordering for a vector inducing that triangulation in the upper hull convention?

Question 2
Are the four triangulations depicted on slide 15, continued to the $m \times 2$-rectangle, all regular unimodular triangulations that give rise to a Gröbner basis?

Question 3
As $p$ varies, is $\dim_C(R_{m,n}/C_{p,(m,n)})$ the Ehrhart polynomial of the (fractional) stable set polytope of the edge graph of $T$ and is this graph perfect?

Inc($\mathbb{N}$)-stable ideals [KLS16, HS09, NR17]
Parallels to be worked out
A “geometrical provocation” inspired by $T_{2,2}$.
Find more of them on www.alsattelberger.de!
Thank you very much for your attention!
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