Computing efficiently the non-properness set of polynomial maps on the plane

Joint work with Elias Tsigaridas

Boulos El Hilany

Dominant polynomial maps and properness

Definition

Let $\mathbb{k}$ be the field $\mathbb{C}$ or $\mathbb{R}$. A map $f = (f_1, \ldots, f_n) : \mathbb{k}^n \to \mathbb{k}^n$ is

- **dominant** if $|\operatorname{Jac}_x f| \neq 0$ for some $x \in \mathbb{k}^n$, and
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Example

The map $(u,\ v) \mapsto (uv,\ v)$ is non-proper at $\{(r,\ s) \in \mathbb{k}^2 \mid s = 0\}$ since $f^{-1}(r, s) = (r/s,\ s)$.

The set of non-properness

$f : \mathbb{k}^n \to \mathbb{k}^n$ – dominant polynomial map, $\mathbb{k} \in \{\mathbb{C}, \mathbb{R}\}$.
Let $\mathcal{J}_f$ denote the **non-properness set** of f, i.e.

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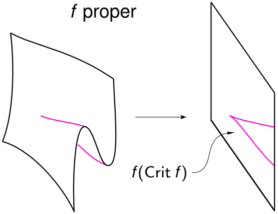

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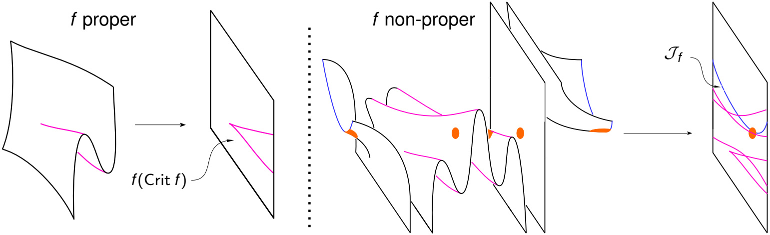

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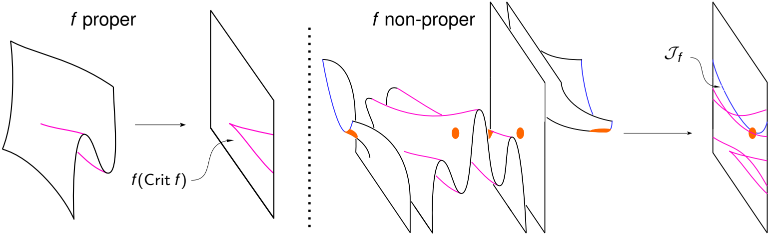

Question

How to compute/approximate $\mathcal{J}_f$?

Appearance in classical problems

- The **Jacobian conjecture** for polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$:

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- Maps $F : X \to Y$ model problems in **multiview geometry** and **robotics**.

$\mathcal{J}_F$ represents all inputs $y \in Y$ to which the solution $F^{-1}(y)$ is sub-optimal.

Geometry of the non-properness set

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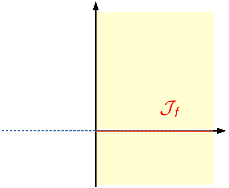
Example

$f : (u, v) \mapsto (u^2 v^2,\ v) \rightsquigarrow \mathcal{J}_f = \mathbb{R}_+ \times \{0\}$



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$$\deg \mathcal{J}_f \leq d^{n-1},$$

and introduced methods from **elimination theory** to compute $\mathcal{J}_f$ for $\mathbb{k} = \mathbb{C}$. Their bit complexity is

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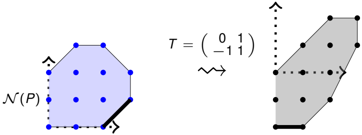
Problem

Design an efficient and complete method for computing $\mathcal{J}_f$.

Notations

P – bivariate polynomial $\sum_{w\in\mathbb{N}^2} c_w x^w$; $\quad x^w = x_1^{w_1} x_2^{w_2}$.

$\mathcal{N}(P)$ – **Newton polytope** of P, i.e. convex hull in $\mathbb{R}^2$ of $\{w \in \mathbb{N}^2 | \ c_w \neq 0\}$



Definition

Let Γ be an **edge** of $\mathcal{N}(P)$. A change of variables $z = \tau(x)$ is called **toric** if

$$\begin{aligned} \tau : (\mathbb{k}^*)^2 &\to (\mathbb{k}^*)^2 \\ x &\mapsto \left(x^{T_1},\ x^{T_2}\right) \end{aligned}$$

for some $T := (T_1,\ T_2) \in \mathrm{SL}(2,\mathbb{Z})$. It is **$\Gamma$-toric** if additionally $T \cdot \Gamma$ is horizontal.

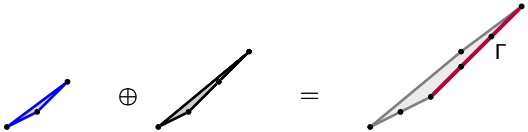
Pick the "smallest" $w \in \mathbb{N}^2$ such that P_τ has non-negative exponents, where

$$P_\tau := z^w P \circ \tau^{-1}$$

Γ-transformations for maps

$f = (f_1, f_2) : \mathbb{k}^2 \to \mathbb{k}^2$ – dominant polynomial map and y– generic point in $\mathbb{k}^2$

$$\mathcal{N}(f) := \{a + b \mid a \in \mathcal{N}(f_1 + y_1),\ b \in \mathcal{N}(f_2 + y_2)\}$$



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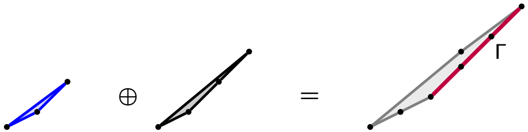

For any edge $\Gamma \subset \mathcal{N}(f)$ and a Γ-toric change of variables τ, we write

$$(f - y)_{\overline{\tau}} := \left((f_1 - y_1)_{\overline{\tau}},\ (f_2 - y_2)_{\overline{\tau}}\right)$$

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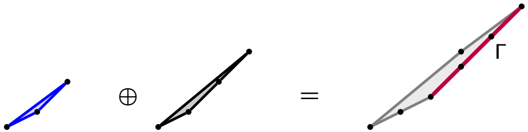

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Example

Let $f - y$ be defined as the pair

$$(-y_1 + 2uv - u^2v^3,\ \ -y_2 + 12uv - 10u^2v^3 + 2u^3v^5)$$

and let Γ be the *purple* edge and $T = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$. Then, $(f - y)_{\overline{\tau}}$ is

$$\left(- y_1 b + 2 - a,\ \ - y_2 b + 12 - 10a + 2a^2\right)$$

$f = (f_1, f_2) : \mathbb{k}^2 \to \mathbb{k}^2$ – dominant polynomial map
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The **Γ-multiplicity set** of f, $\mathcal{M}_f(\Gamma)$, is the set of all points $\boldsymbol{y} \in \mathbb{k}^2$ such that for any $q \longrightarrow \boldsymbol{y}$, there exists a point $p \in (\mathbb{k}^*)^2$ such that $p \longrightarrow \boldsymbol{\varrho} \in \mathbb{k}^* \times \{0\}$, and

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The multiplicity of the solution $\boldsymbol{\varrho}$ to $(f - y)_{\tau} = 0$ increases at points $\boldsymbol{y} \in \mathcal{M}_f(\Gamma)$.

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Example

Consider the pair $(f - y)_{\mp}(a, b)$ from before:

$$\big(2 - a - y_1 b,\ 12 - 10a + 2a^2 - y_2 b\big).$$

The system $(f - y)_{\mp} = 0$ has solutions $(2, 0)$ and $\left(\frac{6y_1 - y_2}{2y_1},\ \frac{y_2 - 2y_1}{2y_1^2}\right)$. We have

$$\big|\mathrm{Jac}(f - y)_{\mp}\big|_{|(2,0)} = y_2 - 2y_1.$$

The Γ-multiplicity set $\mathcal{M}_f(\Gamma)$ is given by

$$\{y \in \mathbb{k}^2 \mid y_2 - 2y_1 = 0\}.$$

Main result

Theorem (EH, Tsigaridas, 2020)

Let $\Bbbk$ be the field $\mathbb{C}$ or $\mathbb{R}$ and let $f : \Bbbk^2 \to \Bbbk^2$ be a dominant polynomial map. Then, for some choices of Γ-toric change of variables, we have

$$\mathcal{J}_f = \bigcup_{\Gamma} \mathcal{M}_f(\Gamma),$$

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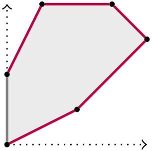
Output The union of sets $\mathcal{M}_f(\Gamma)$



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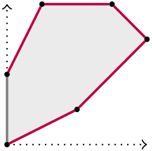

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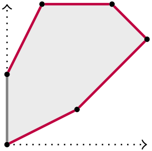

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Remark: Points $y \in \mathcal{M}_f(\Gamma)$ change the multiplicities of solutions in $\mathbb{k}^* \times \{0\}$ to

$$(f - y)_{\overline{\tau}} = 0$$

Computing multiplicities for bivariate systems

Setup $g_1, g_2 \in \mathbb{k}[y_1, y_2][a, b]$; $0 \in \mathbb{k}^2$ is a common root.

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Then, we have

$$\mathcal{M}_f(\Gamma) \subset \mathrm{Mult}_f(\Gamma)$$

Our contribution

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Flexible: Can be optimized and refined to some applications

Thank you!

ArXiv:2101.05245