Algorithms for Fundamental Invariants and Equivariants of a finite group

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Symmetry : groups and their representations

$\mathfrak{S}_4$: the symmetric group on 4 elements.

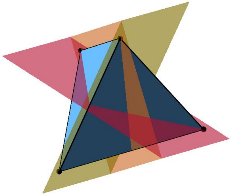
$$\sigma(s_1) = (12), \quad \sigma(s_2) = (23), \quad \sigma(s_3) = (34).$$

Represented in $\mathbb{R}^4$ by 4×4 permutation matrices

$$\rho(s_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(s_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathfrak{T}_h$: the group of symmetry of the tetrahedron

$$\tau(s_1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \tau(s_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau(s_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



A_3 : the order 24 group with generators s_1, s_2, s_3 and relationships

$$s_1^2 = s_2^2 = s_3^2 = (s_1 s_3)^2 = (s_1 s_2)^3 = (s_2 s_3)^2 = 1.$$

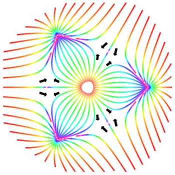
The invariants for $\mathfrak{S}_3$ are

$$s_1 = x + y + z, \quad s_2 = yz + zx + xy, \quad s_3 = xyz$$

Ring of invariants: $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3} = \mathbb{R}[s_1, s_2, s_3]$

Dynamical systems with symmetry

[Gatermann 01]



$$\dot{\mathrm{x}} = \mathrm{p}(\mathrm{x})$$

$$\mathrm{p}\,(\,\tau(g)\,\mathrm{x}\,) = \tau(g)\,\mathrm{p}(\,\mathrm{x}\,)$$

Invariants and equivariants $\mathfrak{S}_3$

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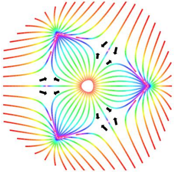
Dynamical systems with symmetry [Gatermann 01]

$$\dot{\mathrm{x}} = \mathrm{p}(\mathrm{x})$$

$$\mathrm{p}(\tau(g)\,\mathrm{x}) = \tau(g)\,\mathrm{p}(\mathrm{x})$$

p is a τ-equivariant

$\mathbb{R}[\mathrm{x}]_{\tau}^{\mathfrak{S}_3}$ is a $\mathbb{R}[\mathrm{x}]^{\mathfrak{S}_3}$-module



Sum of squares: [Gatermann & Parillo 06]

$$x^2 + y^2 + z^2 - (yz + zx + xy) = \frac{3}{4}\,(y-z)^2 + \frac{1}{4}\,(2x - y - z)^2$$

Symmetry adapted bases of the polynomial ring : main motivation.

Symmetry adapted bases : the key to symmetry reduction

Symmetry is expressed by the equivariance of a map

$$\phi : U^{\mu} \to V^{\nu}, \qquad \phi(\mu(g)\,u) = \nu(g)\,\phi(u)$$

The matrix of an equivariant map is block diagonal in a **s.a.b**

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$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$

$$\varrho(g) = R \begin{bmatrix} \varrho_1(g) & 0 \\ 0 & \varrho_2(g) \end{bmatrix} R^{-1} = \ldots = Q \begin{bmatrix} \mathrm{I}_{m_1} \otimes \mathfrak{r}^{(1)}(g) & & \\ & \ddots & \\ & & \mathrm{I}_{m_\mathfrak{t}} \otimes \mathfrak{r}^{(\mathfrak{t})}(g) \end{bmatrix} Q^{-1}$$

$\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(\mathfrak{t})}$ the irreducible representations of $\mathfrak{G}$:

$$\mathfrak{r}^{(1)}(g) = [1] \qquad \mathfrak{r}^{(\ell)} : \mathfrak{G} \to \mathrm{GL}_{\mathfrak{n}_\ell}(\mathbb{C})$$

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P provides a symmetry adapted basis for ϱ

It is computed thanks to the projections $\pi^{(\ell)} = \sum_{g\in\mathfrak{G}} \mathfrak{r}^{(\ell)}(g^{-1})\, \varrho(g)$

Symmetry adapted bases of $\mathbb{C}[\mathrm{x}]$ and basic equivariants

Global optimization [Gatermann Parillo], [Riener *et al.*] Approximation theory [Rodriguez & H.],[Collowald & H.] Cryptography, combinatorics, and other areas of mathematics... Physics, chemistry [Fässler Stiefels], [Muggli], [Cassam Chennai *et al.*], ...

$$\mathbb{C}[\mathrm{x}] = \mathbb{C}[x_1, \ldots, x_n] \qquad \rho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C}[\mathrm{x}]_d) \qquad \rho(g)(f) = f \circ \varrho(g^{-1})$$

$$\mathbb{C}[\mathrm{x}]_d = \mathbb{C}[\mathrm{x}]_d^{(1)} \oplus \ldots \oplus \mathbb{C}[\mathrm{x}]_d^{(\mathfrak{t})}$$

$\mathbb{C}[\mathrm{x}]_d^{(\ell)}$ spanned by $\mathrm{q}_1^{(\ell)}, \ldots, \mathrm{q}_{m_\ell}^{(\ell)}$ where $\mathrm{q}_k^{(\ell)} = \begin{bmatrix} q_1 & \ldots & q_{\mathfrak{n}_\ell} \end{bmatrix} \in \mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$

The $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-modules $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(1)}}^{\mathfrak{G}}, \ldots, \mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(1)}}^{\mathfrak{G}}$ provide **s.a.b.** for $\mathbb{C}[\mathrm{x}]$

Our contributions: Fundamental invariants and equivariants

From a s.a.b of $\mathbb{C}[\mathrm{x}]_{\leq d}$ compute minimal generators for $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$ and $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$

and these provide generators of $\mathbb{C}[\mathrm{x}]_{\tau}^{\mathfrak{G}}$ for any representation τ.

Algorithms for fundamental invariants and equivariants

finite groups

- Reflection groups : ideal interpolation along an orbit [Rodriguez Bazan, H. 20]

  The invariants are read on a H-basis of the ideal J of a generic orbit.
  The equivariants on the s.a.b. of the orthogonal complement of J^0.

- Free module generators over primary invariants

  from the s.a.b. of an invariant complement of the ideal generated by the primary invariants [Rodriguez Bazan, H. 19]

- Minimal set of generating invariants and equivariants (Molien free)

  Computing invariants and equivariants degree by degree.
  Constructing the Nullcone ideal N and the covariant algebra $\mathbb{C}[\mathrm{x}]/N$.

Symmetry preserving interpolation [Rodriguez Bazan & H. 19-22]

In:

- $\lambda_1, \ldots, \lambda_r : \mathbb{C}[x] \to \mathbb{C}$ f.i. $\lambda_i(p) = p(\xi_i)$

- Bases P_d of $\mathbb{C}[x]_d$, $1 \leq d \leq r$

Out :

- $q_1, \ldots, q_r \in \mathbb{C}[x]$ span an interpolation space Q *i.e.,*
  $\lambda_1(q) = \eta_1, \ldots, \lambda_r(q) = \eta_r$ has a unique solution $q \in Q$
  $\Leftrightarrow \det[\lambda_i(q_j)]_{ij} \neq 0$
- $H = \{h_1, \ldots, h_k\}$ a basis of the ideal $J = \cap_i \ker \lambda_i$
  Then $\mathbb{C}[x] = J \oplus Q$

H a Gröbner basis and $q_1, \ldots, q_r$ monomials [Marineri Möller Mora 91]...

$\rightsquigarrow$ Not canonical, breaks symmetry

Symmetry preserving interpolation [Rodriguez Bazan & H. 19-22]

In:

- $\lambda_1, \ldots, \lambda_r : \mathbb{C}[\mathrm{x}] \to \mathbb{C}$ f.i. $\lambda_i(p) = p(\xi_i)$

  $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{C}}$
- Bases P_d of $\mathbb{C}[\mathrm{x}]_d$, $1 \leq d \leq r$

Out :

- $q_1, \ldots, q_r \in \mathbb{C}[\mathrm{x}]$ span the least interpolation space $Q = \Lambda_{\downarrow}$

  [de Boor, Ron 90s]
- $H = \{h_1, \ldots, h_k\}$ a H-basis of the ideal $J = \cap_i \ker \lambda_i$

  Then $\mathbb{C}[\mathrm{x}] = J \oplus Q$ and $J^0 \perp Q$

$H = \{h_1, \ldots, h_k\}$ is a H-basis of J iff $H^0 = \{h_1^0, \ldots, h_k^0\}$ is a basis of J^0, where h^0 is the leading form of $h \in \mathbb{C}[\mathrm{x}]$.

[Macaulay 16]

Symmetry preserving interpolation [Rodriguez Bazan & H. 19-22]

In:

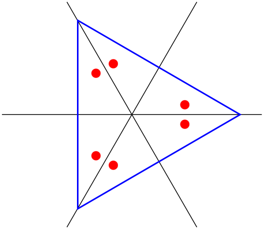
- $\lambda_1, \ldots, \lambda_r : \mathbb{C}[\mathrm{x}] \to \mathbb{C}$ f.i. $\lambda_i(p) = p(\xi_i)$
  $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{C}}$ is $\mathfrak{G}$-invariant f.i. $\xi_1, \ldots, \xi_r$ forms an orbit of $\mathfrak{G}$
- Symmetry adapted bases $P_d = \bigcup_\ell P_d^{(\ell)}$ of $\mathbb{C}[\mathrm{x}]_d$, $1 \le d \le r$

Out : Then J, J^0, Q are invariant

- $Q = \bigcup_\ell Q^{(\ell)}$ a s.a.b of the least interpolation space $Q = \Lambda_\downarrow$
- $H = \bigcup_\ell H^{(\ell)}$ a symmetry adapted H-basis of $J = \cap_i \ker \lambda_i$
  Then $\mathbb{C}[\mathrm{x}] = J \oplus Q$ and $J^0 \perp Q$

Relies on linear algebra in $\mathbb{C}[\mathrm{x}]_d$, for d increasing ($d \le r$).
The symmetry allows to block diagonalize the matrices involved.

Interpolation along an orbit of $\mathfrak{D}_3$

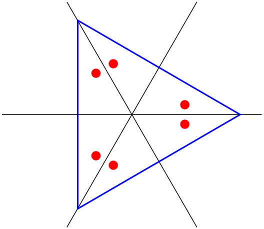


	s_1	s_2
$\mathfrak{r}^{(1)}$	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$
$\mathfrak{r}^{(2)}$	$\begin{bmatrix}-1\end{bmatrix}$	$\begin{bmatrix}-1\end{bmatrix}$
$\mathfrak{r}^{(3)}$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$

$$Q^{(1)} = \{1\},\ Q^{(2)} = \left\{y(y^2 - 3x^2)\right\},\ Q^{(3)} = \left\{[x, y], \left[y^2 - x^2, 2xy\right]\right\}$$

$$H^{(1)} = \left\{x^2+y^2 - 5,\ x(x^2-3y^2) - 2\right\}, \quad H^{(2)} = \emptyset, \quad H^{(3)} = \emptyset$$

Interpolation along an orbit of $\mathfrak{D}_3$



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$$H^{(1)} = \left\{x^2+y^2-5,\ x(x^2-3y^2)-2\right\},\quad H^{(2)} = \emptyset,\quad H^{(3)} = \emptyset$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3} = \mathbb{R}[x^2+y^2, x(x^2-3y^2)],\qquad \mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(2)}} = y(y^2-3x^2)\mathbb{R}[x,y]^{\mathfrak{D}_3}$$

$$\mathbb{R}[x,y]^{\mathfrak{D}_3}_{\mathfrak{r}^{(3)}} = [x,y]\mathbb{R}[x,y]^{\mathfrak{D}_3} \oplus \left[y^2-x^2, 2xy\right]\mathbb{R}[x,y]^{\mathfrak{D}_3}$$

Interpolation along a generic orbit for reflection groups

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$ $\qquad$ $\xi \in \mathbb{C}^n$ s.t. $\varrho(g)(\xi) = \xi \;\Rightarrow\; g = 1$

- $J = \bigcap_{g \in \mathfrak{G}} \ker \mathrm{e}_{\varrho(g)(\xi)}$ the ideal of the orbit of ξ : $\quad \dim \mathbb{C}[\mathrm{x}]/J = |\mathfrak{G}|$
- $h \in \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} \quad \Rightarrow \quad h - h(\xi) \in J \qquad N = \langle\, h \mid h \in \mathbb{C}[\mathrm{x}]^{\mathfrak{G}} \setminus \mathbb{C} \rangle \subset J^0$

$\varrho : \mathfrak{G} \to \mathrm{GL}_n(\mathbb{C})$ a *reflection group* [Chevalley 55]

- $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}} = \mathbb{C}[h_1, \ldots, h_n]$ $\qquad N = \langle h_1, \ldots, h_n \rangle$
- $\mathbb{C}[\mathrm{x}]/N$, the covariant algebra, has dimension $|\mathfrak{G}|$
- $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}(\ell)}$ is a free module of rank $\mathfrak{n}_\ell$ over $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$

Hence $J^0 = N$. [H. & Rodriguez Bazan]

- The computed H-basis of J is $H = \{ h_1 - h_1(\xi), \ldots, h_n - h_n(\xi) \}$
- The computed s.a.b $Q = \bigcup_\ell Q^{(\ell)}$ satisfies $\mathbb{C}[\mathrm{x}] = N \overset{\perp}{\oplus} \langle Q \rangle_{\mathbb{C}}$

  $\overset{\text{Nakayama}}{\Rightarrow}$ $Q^{(\ell)}$ is a free basis for $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}(\ell)}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module

Free bases of $\mathbb{C}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as $\mathbb{C}[h]$-modules [H. & Rodriguez Bazan]

$\varrho : \mathfrak{G} \to \mathrm{GL}(\mathbb{C}^n)$ $J = \langle h_1, \ldots, h_n \rangle$

$h_1, \ldots, h_n$ primary invariants if $\mathbb{C}[x]^{\mathfrak{G}}$ is a free module over $\mathbb{C}[h]$

- $\mathbb{C}[x]/J$ has dimension $m|\mathfrak{G}|$ where $m = |\mathfrak{G}|/\prod_i \deg(h_i)$
- $\mathbb{C}[x]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ is a free $\mathbb{C}[h]$-module of rank $m\,\mathfrak{n}_\ell$ [Stanley 79]

Construction: Dade's algo. Degrees from Molien's series. Invariants of a supgroup.

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- $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}(\ell)}$ is a free $\mathbb{C}[h]$-module of rank $m\,\mathfrak{n}_\ell$ [Stanley 79]

- $J = \langle h_1, \ldots, h_n \rangle$ and, from a G̈-basis, complement $\langle x^{\alpha_1}, \ldots, x^{\alpha_r} \rangle_{\mathbb{C}}$
- Define $\lambda_1, \ldots, \lambda_r$ by $p \equiv \lambda_1(p)\, x^{\alpha_1} + \ldots + \lambda_r(p)\, x^{\alpha_r} \mod J$
- $\bigcup_\ell Q^{(\ell)}$ a s.a.b. of the least interpolation space for $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{C}}$

Free bases of $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as $\mathbb{C}[h]$-modules [H. & Rodriguez Bazan]

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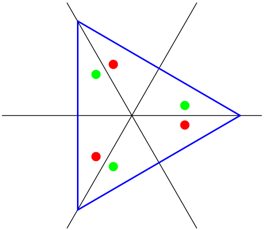
Then

- $Q^{(1)} = \{s_1, \ldots, s_m\}$ is a set of secondary invariants, i.e.,
$$\mathbb{C}[\mathrm{x}]^{\mathfrak{G}} = s_1\mathbb{C}[h] \oplus \ldots \oplus s_r\mathbb{C}[h]$$
- $Q^{(\ell)} = \left\{ \mathrm{q}_1^{(\ell)}, \ldots, \mathrm{q}_{m\mathfrak{n}_\ell}^{(\ell)} \right\}$ is a free basis of $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as $\mathbb{C}[h]$-module



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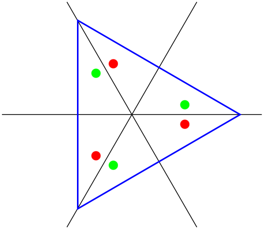
- $h_1 = x^2+y^2,\ h_2 = x(x^2-3y^2)$ are primary invariants (invariants of $\mathfrak{D}_3$)
- $\ddot{G} = \{x^2+y^2, xy^2, x^4\}$ and normal set $\{1, x, y, xy, y^2, y^3\}$
- $p \longrightarrow \lambda_1(p) + \lambda_2(p)\,x + \lambda_3(p)\,y + \lambda_4(p)\,xy + \lambda_5(p)\,y^2 + \lambda_6(p)\,y^3$
- Least interpolation s.a.b. for $\Lambda = \langle \lambda_1, \ldots, \lambda_6 \rangle_{\mathbb{R}}$ [Rodriguez Bazan & H. 19]

$$Q^{(1)} = \{1,\ y(y^2-3x^2)\},\ Q^{(2+3)} = \{[x, y], [y^2-x^2, 2xy]\}$$

Rotation symmetry of the triangle $\mathfrak{C}_3 \subset \mathfrak{D}_3$



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$$\mathbb{R}[x,y]^{\mathfrak{C}_3} = \mathbb{R}[h_1,h_2] \oplus y(y^2-3x^2)\mathbb{R}[h_1,h_2]$$

$$\mathbb{R}[x,y]^{\mathfrak{C}_3}_{\mathfrak{r}^{(2+3)}} = [x,y]\,\mathbb{R}[h_1,h_2] \oplus \left[y^2-x^2, 2xy\right]\mathbb{R}[h_1,h_2]$$

Basic ideas

Compute degree by degree

- an minimal H-basis H of $N = \langle\, h \,|\, h \in \mathbb{C}[\mathrm{x}] \setminus \mathbb{C} \rangle$
- a s.a.b. $Q = \bigcup_\ell Q^{(\ell)}$ of the orthogonal complement of N in $\mathbb{C}[x]$

Then

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a basis of $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}_{\mathfrak{r}^{(\ell)}}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module .

Basically

- $\mathbb{C}[\mathrm{x}]_d = \Psi_d(H_{d-1}) \overset{\perp}{\oplus} \langle K_d \rangle_{\mathbb{C}} \overset{\perp}{\oplus} \langle R_d \rangle_{\mathbb{C}}$

  $$\Psi_d(H) = \sum_{h \in H} \langle\, p\, h \mid \deg(p) + \deg(h) = d \,\rangle$$
- $H_d \leftarrow H_{d-1} \cup K_d; \qquad Q_d \leftarrow Q_{d-1} \cup R_d.$

taking into account the $\rho - \tau_d$ equivariance of Ψ

Fundamental invariants and equivariants

Algorithm [H. & Rodriguez Bazan]

$d := 0;\ R_0^{(1)} = \{1\};$

do $d \leftarrow d + 1$

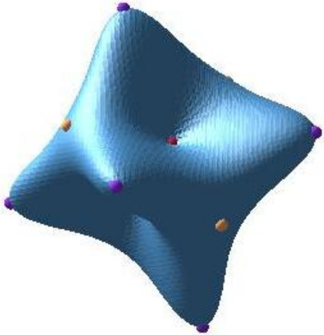
- $\mathbb{C}[\mathrm{x}]_d^{(1)} = \psi_d^{(1)}(H_{d-1}) \overset{\perp}{\oplus} \langle K_d \rangle_{\mathbb{C}}$
- $\mathbb{C}[\mathrm{x}]_d^{(\ell,1)} = \psi_d^{(\ell,1)}(H_{d-1}) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{C}} \overset{\text{lemma}}{=} \psi_d^{(1)}\left(Q_{d-1}^{(\ell,1)}\right) \overset{\perp}{\oplus} \langle R_d^{(\ell,1)} \rangle_{\mathbb{C}}$
- $H_d \leftarrow H_{d-1} \cup K_d, \quad Q_d^{(\ell)} \leftarrow Q_{d-1}^{(\ell)} \cup R_d^{(\ell)}$

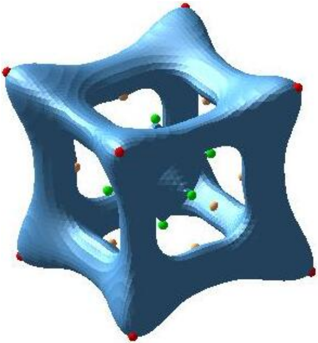
until $R_d^{(\ell)} = \emptyset$

Output:

- $H = \{h_1, \ldots, h_k\}$ is a minimal generating set of invariants
- $Q^{(\ell)}$ is a minimal basis of $\mathbb{C}[\mathrm{x}]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ as a $\mathbb{C}[\mathrm{x}]^{\mathfrak{G}}$-module

Thanks





E. Hubert, E. Rodriguez Bazan.
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