Dimension of Tensor Network Varieties
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Motivations

Quantum Physics:

- Huge ambient (tensor) space: \( |\psi\rangle \in V = V_1 \otimes \cdots \otimes V_d \), with \( \dim V_i = n_i \)

\[
\dim V = \prod_{i=1}^{d} n_i
\]

- encoding entanglement

Modeling the state via a Tensor Network approach

- reduces the number of parameters,
- encodes information about the correlation between components of the system.
Matrix Multiplication

$A \in \mathbb{C}^{n_1 \times m}, \ B \in \mathbb{C}^{m \times n_2}, \ C = AB \in \mathbb{C}^{n_1 \times n_2}$

Matrix Multiplication

$c_{ik} = (AB)_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$
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Matrix Multiplication

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Consider a graph $\Gamma = (e(\Gamma), v(\Gamma))$, with $d$ vertices $v(\Gamma) = \{1, \ldots, d\}$.

Fix weights $m = (m_e)_{e \in e(\Gamma)}$, one for each edge.
Associate $I_{m_e} = \sum_{i=1}^{m_e} e_i \otimes e_i \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$ to each edge.

$$T(\Gamma, m) = \bigotimes_{e \in e(\Gamma)} I_{m_e} \in \bigotimes_{e \in e(\Gamma)} \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$$
Graph Tensor

For every $v \in v(\Gamma)$: $W_v = \bigotimes_{e \in e(v)} \mathbb{C}^{m_e}$

Definition (Graph Tensor)

$$T(\Gamma, m) = \bigotimes_{e \in e(\Gamma)} I_{m_e} \in \bigotimes_{v \in v(\Gamma)} W_v.$$
For every $v \in \nu(\Gamma)$: $W_v = \bigotimes_{e \in e(v)} \mathbb{C}^{m_e}$
Fix $V_1, \ldots, V_d$ of dimensions $n = (n_v)_{v \in v(\Gamma)}$

$$X_v \in \text{Hom}(W_v, V_v), \quad \text{for } v \in v(\Gamma).$$
Definition (Tensor Network Variety)

\[ \mathcal{TNS}_{m,n}^{\Gamma} = \text{Im} (\Phi) = \left\{ T \in V_1 \otimes \cdots \otimes V_d : T = (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, m), X_j \in \text{Hom}(W_j, V_j) \right\} \]
Example

\[ T(\Gamma, m) = \sum_{i=1}^{m} e_i \otimes e_i = I_m \in \mathbb{C}^m \otimes \mathbb{C}^m \] the graph tensor.

Consider \((X_1, X_2) \in \text{Hom}(\mathbb{C}^m, V_1) \times \text{Hom}(\mathbb{C}^m, V_2)\) applied to it

\[
(X_1 \otimes X_2) \cdot T(\Gamma, m) = \sum_{i=1}^{m} (X_1 e_i) \otimes (X_2 e_i) \\
= \sum_{i=1}^{m} (X_1 e_i)(X_2 e_i)^t = X_1 \cdot I_m \cdot X_2^t.
\]

\[ I_m \in \mathbb{C}^m \otimes \mathbb{C}^m \]

\[ V_1 \]
\[ V_2 \]

\[ TNS_{m,(n_1,n_2)}^{P_2} = \{ M \in \mathbb{C}^{n_1 \times n_2} : \text{rank}(M) \leq m \} \]
• Assume $m_e > 1$ for every edge $e \in e(\Gamma)$, otherwise it can be removed.

• if $m'$ and $m$ are such that $m'_e \leq m_e$ for every $e \in e(\Gamma)$ then $\mathcal{TNS}_{m',n}^{\Gamma} \subseteq \mathcal{TNS}_{m,n}^{\Gamma}$.

• If $\Gamma$ is connected

\[
\mathcal{TNS}_{m,n}^{\Gamma} = V_1 \otimes \cdots \otimes V_d,
\]

for $m_e$ big enough for every $e \in e(\Gamma)$.

• If $\Gamma$ has no loops, then $\mathcal{TNS}_{m,n}^{\Gamma \circ} = \mathcal{TNS}_{m,n}^{\Gamma}$, otherwise $\mathcal{TNS}_{m,n}^{\Gamma} \setminus \mathcal{TNS}_{m,n}^{\Gamma \circ} \neq \emptyset$. [Landsberg, Qi, Ye]
Let \((\Gamma, m, n)\) be a tensor network. A vertex \(v \in v(\Gamma)\) is called
- \textit{subcritical} if \(\dim W_v = \prod_{e \ni v} m_e \geq n_v\)
- \textit{supercritical} if \(\dim W_v = \prod_{e \ni v} m_e \leq n_v\)
- \textit{critical} if \(v\) is both subcritical and supercritical.
Theorem (Balanced bonds [BDG’21])

Assume $m_{e_1} \leq \cdots \leq m_{e_k}$. If

$$m_{e_k} > n_v \cdot m_{e_1} \cdots m_{e_{k-1}}, \quad m_{e_k} \text{ overabundant}$$

then

$$\mathcal{TNS}_{m,n} = \mathcal{TNS}_{\bar{m},n}$$

where $\bar{m}$ is defined by $\bar{m}_e = m_e$ if $e \neq e_k$ and

$\bar{m}_{e_k} = n_v \cdot m_1 \cdots m_{e_k-1}$. 

Claudia De Lazzari
Dimension of Tensor Network Varieties
Supercritical Case

**Theorem (Landsberg, Qi, Ye)**

Suppose that the vertex \( d \in \mathcal{v}(\Gamma) \) is supercritical and write
\[
N = \dim \mathcal{W}_d = \prod_{e \ni d} m_e(\leq n_d).
\]
Let \( n' = (n'_v : v \in \mathcal{v}(\Gamma)) \) be the \( d \)-tuple of local dimensions defined by \( n'_v = n_v \) if \( v \neq d \) and \( n'_d = N \). Then
\[
\dim \mathcal{TN}S^\Gamma_{m,n} = N(n_d - N) + \dim \mathcal{TN}S^\Gamma_{m,n'}.
\]
\[ \Phi : \text{Hom}(W_1, V_1) \times \cdots \times \text{Hom}(W_d, V_d) \rightarrow V_1 \otimes \cdots \otimes V_d \]

\[ (X_1, \ldots, X_d) \mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, m) \]

\[ \mathcal{TNS}_{m,n}^\Gamma = \text{Im} (\Phi) \subseteq V_1 \otimes \cdots \otimes V_N \]
\[ \bigoplus_{i=1}^{d} \text{Hom}(W_i, V_i) \xrightarrow{\mu} \text{Hom}(W_1 \otimes \cdots \otimes W_d, V_1 \otimes \cdots \otimes V_d) \]

\[
\begin{align*}
(X_1, \ldots, X_d) & \quad \xrightarrow{\Phi} \quad (X_1 \otimes \cdots \otimes X_d) \\
\Phi & \quad \downarrow \quad \Phi \\
V_1 \otimes \cdots \otimes V_d &
\end{align*}
\]

\[
\text{Im } (\mu) := \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)
\]

\[
\Phi : \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \rightarrow V_1 \otimes \cdots \otimes V_d
\]

\[
(X_1 \otimes \cdots \otimes X_d) \mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, m)
\]

gives a parametrisation of \( TNS_{m,n}^{\Gamma_{\circ}} \).
\[ \Phi : \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \to V_1 \otimes \cdots \otimes V_d \]

parametrisation: \( \mathcal{TNS}_{m,n}^\Gamma = \text{Im}(\Phi). \)

For the Theorem of Dimension of the Fiber

\[ \dim \mathcal{TNS}_{m,n}^\Gamma = \dim [\text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)] - \dim \Phi^{-1}(T), \]

where \( T \) is a generic tensor in the image of \( \Phi. \)
\( \Phi : \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \to V_1 \otimes \cdots \otimes V_d \)

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\]

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In the Fiber: Gauge subgroup

\[ \bar{\Phi} : \text{Hom}(\mathbb{C}^m, V_1) \times \text{Hom}(\mathbb{C}^m, V_2) \to V_1 \otimes V_2 \]
\[ (X_1, X_2) \mapsto X_1 \cdot I_m \cdot X_2^t \]

If \( g \in PGL_m \) then

\[ \bar{\Phi}(X_1, X_2) = X_1 \cdot (gg^{-1}) \cdot X_2^t = (X_1g)(g^{-1}X_2^t) = \bar{\Phi}(X_1g, X_2(g^{-1})^t) \]

The fiber contains all the orbit of \((X_1, X_2)\) under this action
The fiber which contains
\( X = (X_1 \otimes \cdots \otimes X_d) \in \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \), contains all its \( G_{\Gamma, m} \)-orbit, where

**Definition (Gauge Subgroup)**

\[
G_{\Gamma, m} \simeq \bigtimes_{e \in e(\Gamma)} PGL_{m_e}.
\]
\[ \dim \mathcal{TNS}_{m,n} = \dim [\text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)] - \dim \Phi^{-1}(T) \]

where \( T \) is a generic tensor in the image of \( \Phi \).

\[
\dim \Phi^{-1}(T) \geq \dim (G_{\Gamma,m} \cdot X).
\]

**Lower bound** given by the dimension of the \( G_{\Gamma,m} \)-orbit of a generic element of \( X \in \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \).
Theorem (BDG’21)

Let \((\Gamma, m, n)\) be a subcritical tensor network with no overabundant bond dimensions. Denote by \(N_v = \prod_{e \ni v} m_e\) and \(X = X_1 \otimes \cdots \otimes X_d\) with \(X_v \in \text{Hom}(W_v, V_v)\) generic. Then

\[
\dim \mathcal{TNS}_{m,n}^\Gamma \leq \min \left\{ \prod_{v \in v(\Gamma)} n_v, \left[ \sum_{v \in v(\Gamma)} N_v n_v - d + 1 \right] - \sum_{e \in e(\Gamma)} (m_e^2 - 1) + \dim \text{Stab}_{\mathcal{G}_{\Gamma, m}}(X) \right\}
\]
Main Theorem

$$\dim TNS_{m,n} = \dim \left[ \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \right] - \dim \Phi^{-1}(T)$$

$$\leq \dim \text{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) - \dim(\mathcal{G}_{\Gamma,m} \cdot X)$$

$$= \left( \sum_{v \in v(\Gamma)} N_v n_v - d + 1 \right) - \sum_{e \in e(\Gamma)} (m_e^2 - 1) + \dim \text{Stab}_{\mathcal{G}_{\Gamma,m}}(X)$$
Theorem

Let \((\Gamma, m, n)\) be a supercritical tensor network. Write \(N_v = \prod_{e \ni v} m_e\). Then

\[
\dim TNS_{m,n}^\Gamma = \sum_{v \in v(\Gamma)} n_v N_v - d + 1 - \sum_{e \in e(\Gamma)} (m_e^2 - 1).
\]
Main Theorem: Subcritical Case

- Is the dimension of the stabilizer $\dim \text{Stab}_{\Gamma,m}(X) = 0$?
- Is there something else in the fiber?

$$\dim \mathcal{TNS}_{m,n}^{\Gamma} \leq \left( \sum_{\nu \in \nu(\Gamma)} N_{\nu} n_{\nu} - d + 1 \right) - \sum_{e \in e(\Gamma)} (m_{e}^{2} - 1) + \dim \text{Stab}_{\Gamma,m}(X)$$
Main Theorem: Subcritical Case

Is the dimension of the stabilizer $\dim \text{Stab}_{G,\gamma, m}(X) = 0$?

It is expected that in “most” cases the value of the dimension is

$$
\min \left\{ \prod_{v \in v(\Gamma)} n_v, \sum_{v \in v(\Gamma)} (n_v \cdot \prod_{e \ni v} m_e) - d + 1 - \sum_{e \in e(\Gamma)} (m_e^2 - 1) \right\}.
$$

- MPS: associated to $(C_d, m, n)$,
- PEPS; grid with at least nodes of degree 3 [Derksen, Makam].
MPS: the stabilizer is finite \( \dim \text{Stab}_{\Gamma,m}(X) = 0 \).

**But** there are cases in which the bound is not sharp

\[
\dim \mathcal{TNS}_{m,n}^\Gamma < \left( \sum_{v \in v(\Gamma)} N_v n_v - d + 1 \right) - \sum_{e \in e(\Gamma)} (m_e^2 - 1)
\]
“Defective” Cases: 3 nodes

\[ \mathcal{TNS}_{m,n}^{c_3} \subseteq V_1 \otimes V_2 \otimes V_3, \quad m = (2, 2, 2) \]

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<tr>
<td>(4, 4, 4)</td>
<td>37</td>
<td>37</td>
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</table>
“Defective” Cases: 3 nodes

\[ \mathcal{TNS}^{C_3}_{m,n} \subseteq V_1 \otimes V_2 \otimes V_3, \quad m = (2, 2, 2) \]

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"Defective" Cases: 3 nodes

- \( m = (2, 2, 2) \), then \( T(\Gamma, m) \in \mathbb{C}^{2 \times 2} \otimes \mathbb{C}^{2 \times 2} \otimes \mathbb{C}^{2 \times 2} \),
- \( n = (2, 3, 4) \), \( \mathcal{TNS}^{C_3}_{m,n} \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4) \).

Take \( T \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4 \) and consider the flattening

\[
T_{\{1\}} : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^4
\]

Then \( L_T = \mathbb{P}(\text{Im} (T_{\{1\}})) \) is a line in \( \mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^4) \), or a point.

**Theorem (BDG’21)**

\( T \in \mathcal{TNS}^{C_3}_{m,n} \) if and only if

- either \( \text{rank}(L_T) = 1 \)
- or \( L_T \) intersects \( \{M : \text{rank}(M) \leq 2\} \) in at least 2 points, counted with multiplicity.

Therefore \( \text{dim} \mathcal{TNS}^{C_3}_{m,n} \leq (=) 22 < 24 \)
\( \mathcal{TNS}_{m,n}^C \subseteq V_1 \otimes V_2 \otimes V_3 \otimes V_4, \quad m = (2, 2, 2, 2). \)
"Defective" Cases: 4 nodes

\[ \mathcal{TNS}_{m,n}^{C_4} \subseteq V_1 \otimes V_2 \otimes V_3 \otimes V_4, \quad m = (2, 2, 2, 2). \]

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<td>* (2, 4, 2, 4)</td>
<td>32</td>
<td>33</td>
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</tbody>
</table>
Theorem

Let \( m = (2, 2, 2, 2) \) and \( n = (2, 2, 2, 2) \). Then

\[
\dim \mathcal{TNS}_{m,n}^{C_4} = 15 \quad (\leq 16);
\]

more precisely \( \mathcal{TNS}_{m,n}^{C_4} \) is a hypersurface of degree 6.

The equation is a degree 6 invariant for the action of \( GL(V_1) \times \cdots \times GL(V_4) \) on \( V_1 \otimes \cdots \otimes V_4 \). Construction in [Holweck, Luque, Thibon].
Open questions:

- Is the dimension of the stabilizer $\dim \text{Stab}_{\Gamma,m} (X) = 0$?
- What else in the fiber?
- Geometric characterization of the defective cases.
  We expect a small defect.
Thank you!

