Dimension of Tensor Network Varieties

A. Bernardi, C. De Lazzari, F. Gesmundo

Claudia De Lazzari

University of Trento

MEGA 2021, June 7-11

Motivations

Quantum Physics:

- Huge ambient (tensor) space: $|\Psi\rangle \in V = V_1 \otimes \cdots \otimes V_d$, with $\dim V_i = n_i$

$$\dim V = \prod_{i=1}^{d} n_i$$

- encoding entanglement

Modeling the state via a Tensor Network approach

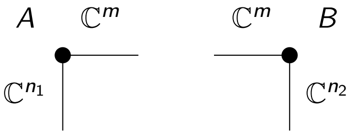
- reduces the number of parameters,
- encodes information about the correlation between components of the system.

Matrix Multiplication

$A \in \mathbb{C}^{n_1 \times m}$, $B \in \mathbb{C}^{m \times n_2}$, $C = AB \in \mathbb{C}^{n_1 \times n_2}$

Matrix Multiplication

$c_{ik} = (AB)_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$

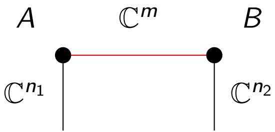


Matrix Multiplication

$A \in \mathbb{C}^{n_1 \times m}$, $B \in \mathbb{C}^{m \times n_2}$, $C = AB \in \mathbb{C}^{n_1 \times n_2}$

Matrix Multiplication

$C_{ik} = (AB)_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$

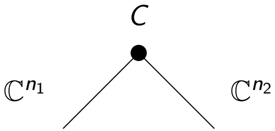


Matrix Multiplication

$A \in \mathbb{C}^{n_1 \times m}$, $B \in \mathbb{C}^{m \times n_2}$, $C = AB \in \mathbb{C}^{n_1 \times n_2}$

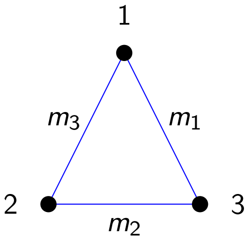
Matrix Multiplication

$C_{ik} = (AB)_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$



Graph Tensor

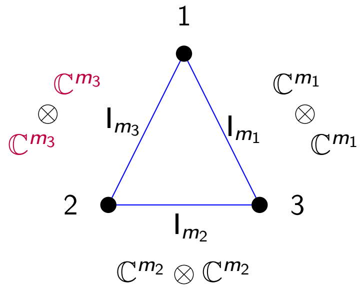
Consider a graph $\Gamma = (\mathrm{e}(\Gamma), \mathrm{v}(\Gamma))$, with d vertices $v(\Gamma) = \{1, \ldots, d\}$.



Fix weights $\mathrm{m} = (m_e)_{e \in \mathrm{e}(\Gamma)}$, one for each edge.

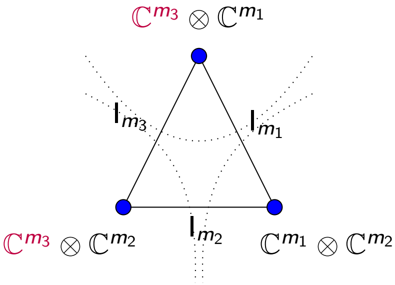
Graph Tensor

Associate $\mathsf{I}_{m_e} = \sum_{i=1}^{m_e} e_i \otimes e_i \in \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$ to each edge.



$$T(\Gamma, \mathrm{m}) = \bigotimes_{e \in \mathrm{e}(\Gamma)} \mathsf{I}_{m_e} \in \bigotimes_{e \in \mathrm{e}(\Gamma)} \mathbb{C}^{m_e} \otimes \mathbb{C}^{m_e}$$

For every $v \in \mathrm{v}(\Gamma)$: $W_v = \bigotimes_{e \in \mathrm{e}(v)} \mathbb{C}^{m_e}$

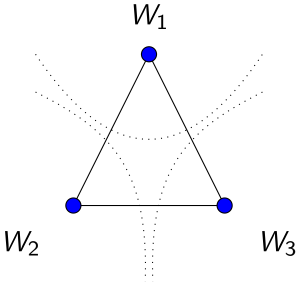


Definition (Graph Tensor)

$$T(\Gamma, \mathrm{m}) = \bigotimes_{e \in \mathrm{e}(\Gamma)} \mathrm{I}_{m_e} \in \bigotimes_{v \in \mathrm{v}(\Gamma)} W_v.$$

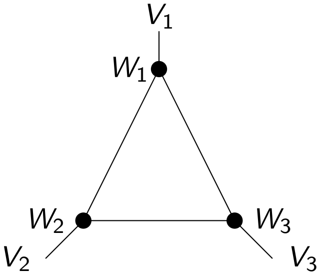
Graph Tensor

For every $v \in \mathrm{v}(\Gamma)$: $W_v = \bigotimes_{e \in \mathrm{e}(v)} \mathbb{C}^{m_e}$



$$T(\Gamma, \mathrm{m}) \bigotimes_{e \in \mathrm{e}(\Gamma)} \mathrm{I}_{m_e} \in \bigotimes_{v \in \mathrm{v}(\Gamma)} W_v .$$

Fix $V_1, \ldots, V_d$ of dimensions $\mathrm{n} = (n_v)_{v \in \mathrm{v}(\Gamma)}$



$$X_v \in \mathrm{Hom}(W_v, V_v), \quad \text{for } v \in \mathrm{v}(\Gamma).$$

$$\bar{\Phi} : \mathrm{Hom}(W_1, V_1) \times \cdots \times \mathrm{Hom}(W_d, V_d) \to V_1 \otimes \cdots \otimes V_d,$$
$$(X_1, \ldots, X_d) \mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathrm{m})$$

gives a parametrisation of $\mathcal{TNS}^{\Gamma\circ}_{\mathrm{m},\mathrm{n}} = \mathrm{Im}\,(\bar{\Phi}) \subseteq V_1 \otimes \cdots \otimes V_d$.

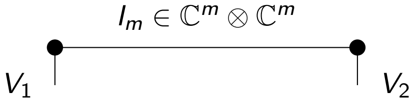
Definition (Tensor Network Variety)

$$\mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} = \overline{\mathrm{Im}\,(\bar{\Phi})}$$
$$= \overline{\left\{ T \in V_1 \otimes \cdots \otimes V_d : T = (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathrm{m}), X_j \in \mathrm{Hom}(W_j, V_j) \right\}}$$

Example

$T(\Gamma, \mathrm{m}) = \sum_{i=1}^{m} e_i \otimes e_i = \mathrm{I}_m \in \mathbb{C}^m \otimes \mathbb{C}^m$ the graph tensor.
Consider $(X_1, X_2) \in \mathrm{Hom}(\mathbb{C}^m, V_1) \times \mathrm{Hom}(\mathbb{C}^m, V_2)$ applied to it

$$\begin{aligned}(X_1 \otimes X_2) \cdot T(\Gamma, \mathrm{m}) &= \sum_{i=1}^{m} (X_1 e_i) \otimes (X_2 e_i) \\ &= \sum_{i=1}^{m} (X_1 e_i)(X_2 e_i)^t = X_1 \cdot \mathrm{I}_m \cdot X_2^t.\end{aligned}$$



$$\mathcal{TNS}^{P_2}_{m,(n_1,n_2)} = \{M \in \mathbb{C}^{n_1 \times n_2} : \mathrm{rank}(M) \leq m\}$$

.

Properties

- Assume $m_e > 1$ for every edge $e \in \mathrm{e}(\Gamma)$, otherwise it can be removed.
- if m' and m are such that $m'_e \leq m_e$ for every $e \in \mathrm{e}(\Gamma)$ then $\mathcal{TNS}^{\Gamma}_{\mathrm{m}',\mathrm{n}} \subseteq \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}}$.
- If Γ is connected

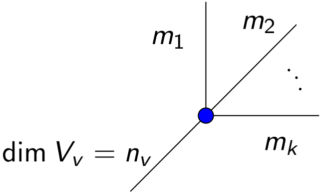
$$\mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} = V_1 \otimes \cdots \otimes V_d,$$

  for m_e big enough for every $e \in \mathrm{e}(\Gamma)$.
- If Γ has no loops, then $\mathcal{TNS}^{\Gamma\circ}_{\mathrm{m},\mathrm{n}} = \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}}$, otherwise $\mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} \setminus \mathcal{TNS}^{\Gamma\circ}_{\mathrm{m},\mathrm{n}} \neq \emptyset$. [Landsberg,Qi,Ye]

Definitions

Let $(\Gamma, \mathrm{m}, \mathrm{n})$ be a tensor network. A vertex $v \in \mathrm{v}(\Gamma)$ is called

- *subcritical* if $\dim W_v = \prod_{e \ni v} m_e \geq n_v$
- *supercritical* if $\dim W_v = \prod_{e \ni v} m_e \leq n_v$
- *critical* if v is both subcritical and supercritical.



Balanced bond

Theorem (Balanced bonds [BDG'21])

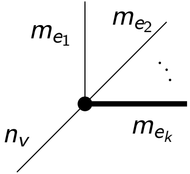
Assume $m_{e_1} \leq \cdots \leq m_{e_k}$. *If*

$$m_{e_k} > n_v \cdot m_{e_1} \cdots m_{e_{k-1}}, \quad m_{e_k} \text{ overabundant}$$

then

$$\mathcal{TNS}_{\mathrm{m},\mathrm{n}} = \mathcal{TNS}_{\bar{\mathrm{m}},\mathrm{n}}$$

where $\bar{\mathrm{m}}$ *is defined by* $\bar{m}_e = m_e$ *if* $e \neq e_k$ *and* $\bar{m}_{e_k} = n_v \cdot m_1 \cdots m_{e_{k-1}}$.



Supercritical Case

Theorem (Landsberg,Qi,Ye)

Suppose that the vertex $d \in \mathrm{v}(\Gamma)$ *is* *supercritical* *and write* $N = \dim W_d = \prod_{e \ni d} m_e (\leq n_d)$*. Let* $\mathrm{n}' = (n'_v : v \in \mathrm{v}(\Gamma))$ *be the d-tuple of local dimensions defined by* $n'_v = n_v$ *if* $v \neq d$ *and* $n'_d = N$*. Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} = N(n_d - N) + \dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}'}.$$

Parametrisation

$$\bar{\Phi} : \mathrm{Hom}(W_1, V_1) \times \cdots \times \mathrm{Hom}(W_d, V_d) \to V_1 \otimes \cdots \otimes V_d$$
$$(X_1, \ldots, X_d) \mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathrm{m})$$

$$\mathcal{TNS}^{\Gamma}_{\mathrm{m,n}} = \overline{\mathrm{Im}\,(\bar{\Phi})} \subseteq V_1 \otimes \cdots \otimes V_N$$

Parametrisation

$$\begin{array}{ccc} \bigoplus_{i=1}^{d} \mathrm{Hom}(W_i, V_i) & \xrightarrow{\ \mu\ } & \mathrm{Hom}(W_1 \otimes \cdots \otimes W_d, V_1 \otimes \cdots \otimes V_d) \\ (X_1, \ldots, X_d) & \longrightarrow & (X_1 \otimes \cdots \otimes X_d) \\ & \searrow^{\bar{\Phi}} & \downarrow \Phi \\ & & V_1 \otimes \cdots \otimes V_d \end{array}$$

$$\mathrm{Im}\,(\mu) := \mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$$

$$\begin{aligned} \Phi : \mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) &\to V_1 \otimes \cdots \otimes V_d \\ (X_1 \otimes \cdots \otimes X_d) &\mapsto (X_1 \otimes \cdots \otimes X_d) \cdot T(\Gamma, \mathrm{m}) \end{aligned}$$

gives a parametrisation of $\mathcal{TNS}^{\Gamma\circ}_{\mathrm{m},\mathrm{n}}$.

Dimension

$\Phi : \mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d) \to V_1 \otimes \cdots \otimes V_d$
parametrisation: $\mathcal{TNS}^{\Gamma}_{\mathrm{m,n}} = \overline{\mathrm{Im}\ (\Phi)}$.

For the Theorem of Dimension of the Fiber

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m,n}} = \dim\left[\mathrm{Hom}(W_1, \dots, W_d; V_1, \dots, V_d)\right] - \dim \Phi^{-1}(T),$$

where T is a generic tensor in the image of Φ.

Dimension

$\Phi : \mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d) \to V_1 \otimes \cdots \otimes V_d$
parametrisation: $\mathcal{TNS}^{\Gamma}_{\mathrm{m,n}} = \overline{\mathrm{Im}\,(\Phi)}$.

For the Theorem of Dimension of the Fiber

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m,n}} = \dim\left[\mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)\right] - \dim \Phi^{-1}(T),$$

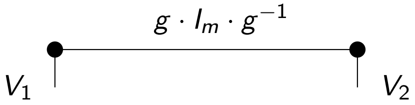
where T is a generic tensor in the image of Φ.

In the Fiber: Gauge subgroup

$$\bar{\Phi} : \mathrm{Hom}(\mathbb{C}^m, V_1) \times \mathrm{Hom}(\mathbb{C}^m, V_2) \to V_1 \otimes V_2$$
$$(X_1, X_2) \mapsto X_1 \cdot \mathrm{I}_m \cdot X_2^t$$

If $g \in PGL_m$ then

$$\bar{\Phi}(X_1, X_2) = X_1 \cdot (gg^{-1}) \cdot X_2^t = (X_1 g)(g^{-1} X_2^t) = \bar{\Phi}(X_1 g, X_2 (g^{-1})^t)$$



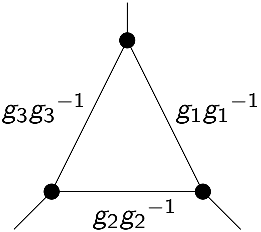
The fiber contains all the orbit of (X_1, X_2) under this action

In the Fiber: Gauge subgroup

The fiber which contains $X = (X_1 \otimes \cdots \otimes X_d) \in \mathrm{Hom}(W_1, \ldots, W_d; V_1, \ldots, V_d)$, contains all its $\mathcal{G}_{\Gamma,\mathrm{m}}$-orbit, where

Definition (Gauge Subgroup)

$$\mathcal{G}_{\Gamma,\mathrm{m}} \simeq \bigtimes_{e \in \mathrm{e}(\Gamma)} PGL_{m_e}.$$



$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} = \dim\left[\mathrm{Hom}(W_1,\ldots,W_d;V_1,\ldots,V_d)\right] - \dim\Phi^{-1}(T)$$
where T is a generic tensor in the image of Φ.

$$\dim\Phi^{-1}(T) \geq \dim(\mathcal{G}_{\Gamma,\mathrm{m}}\cdot X).$$

Lower bound given by the dimension of the $\mathcal{G}_{\Gamma,\mathrm{m}}$-orbit of a generic element of $X \in \mathrm{Hom}(W_1,\ldots,W_d;V_1,\ldots,V_d)$.

Main Theorem

Theorem (BDG'21)

Let $(\Gamma, \mathrm{m}, \mathrm{n})$ be a subcritical tensor network with no overabundant bond dimensions. Denote by $N_v = \prod_{e \ni v} m_e$ and $X = X_1 \otimes \cdots \otimes X_d$ with $X_v \in \mathrm{Hom}(W_v, V_v)$ generic. Then

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} \leq$$
$$\min \left\{ \prod_{v \in \mathrm{v}(\Gamma)} n_v, \left[\sum_{v \in \mathrm{v}(\Gamma)} N_v n_v - d + 1 \right] - \sum_{e \in \mathrm{e}(\Gamma)} (m_e^2 - 1) + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X) \right\}$$

Main Theorem

$$\begin{aligned}\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} &= \dim\left[\mathrm{Hom}(W_1,\dots,W_d;V_1,\dots,V_d)\right] - \dim \Phi^{-1}(T)\\ &\leq \dim \mathrm{Hom}(W_1,\dots,W_d;V_1,\dots,V_d) - \dim(\mathcal{G}_{\Gamma,\mathrm{m}}\cdot X)\\ &= \underbrace{\left(\sum_{v\in \mathrm{v}(\Gamma)} N_v n_v - d + 1\right)}_{\dim \mathrm{Hom}(W_1,\dots,W_d;V_1,\dots,V_d)} - \underbrace{\sum_{e\in \mathrm{e}(\Gamma)} (m_e^2-1)}_{\dim \mathcal{G}_{\Gamma,\mathrm{m}}} + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X)\end{aligned}$$

Main Theorem: Critical and Supercritical Case

Theorem

Let $(\Gamma, \mathrm{m}, \mathrm{n})$ *be a supercritical tensor network. Write* $N_v = \prod_{e \ni v} m_e$. *Then*

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} = \sum_{v \in \mathrm{v}(\Gamma)} n_v N_v - d + 1 - \sum_{e \in \mathrm{e}(\Gamma)} (m_e^2 - 1).$$

Main Theorem: Subcritical Case

- Is the dimension of the stabilizer $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X) = 0$?
- Is there something else in the fiber?

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} \leq \left(\sum_{v \in \mathrm{v}(\Gamma)} N_v n_v - d + 1 \right) - \sum_{e \in \mathrm{e}(\Gamma)} (m_e^2 - 1) + \dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X)$$

Main Theorem: Subcritical Case

- Is the dimension of the stabilizer $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X) = 0$?

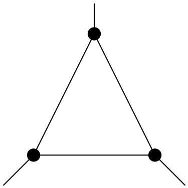
It is expected that in "most" cases the value of the dimension is

$$\min\left\{ \prod_{v\in \mathrm{v}(\Gamma)} n_v, \sum_{v\in \mathrm{v}(\Gamma)} (n_v \cdot \prod_{e\ni v} m_e) - d + 1 - \sum_{e\in \mathrm{e}(\Gamma)} (m_e^2 - 1) \right\}.$$

- MPS: associated to (C_d, m, n),
- PEPS; grid with at least nodes of degree 3 [Derksen,Makam].

“Defective” Cases

- MPS: the stabilizer is finite $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X) = 0$.

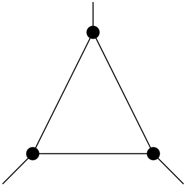


But there are cases in which the bound is not sharp

$$\dim \mathcal{TNS}^{\Gamma}_{\mathrm{m},\mathrm{n}} < \Big(\sum_{v \in \mathrm{v}(\Gamma)} N_v n_v - d + 1 \Big) - \sum_{e \in \mathrm{e}(\Gamma)} (m_e^2 - 1)$$

"Defective" Cases: 3 nodes

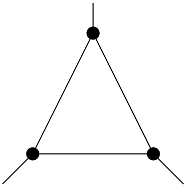
$$\mathcal{TNS}^{C_3}_{\mathrm{m},\mathrm{n}} \subseteq V_1 \otimes V_2 \otimes V_3, \qquad \mathrm{m} = (2,2,2)$$



	n	lower bound	upper bound
	(2, 2, 2)	8	8
	(2, 2, 3)	12	12
	(2, 2, 4)	16	16
	(2, 3, 3)	18	18
*	(2, 3, 4)	22	24
*	(2, 4, 4)	26	29
	(3, 3, 3)	25	25
	(3, 3, 4)	29	29
	(3, 4, 4)	31	31
	(4, 4, 4)	37	37

“Defective” Cases: 3 nodes

$$\mathcal{TNS}^{C_3}_{\mathrm{m},\mathrm{n}} \subseteq V_1 \otimes V_2 \otimes V_3, \qquad \mathrm{m} = (2,2,2)$$



	n	lower bound	upper bound
	(2, 2, 2)	8	8
	(2, 2, 3)	12	12
	(2, 2, 4)	16	16
	(2, 3, 3)	18	18
*	(2, 3, 4)	22	24
*	(2, 4, 4)	26	29
	(3, 3, 3)	25	25
	(3, 3, 4)	29	29
	(3, 4, 4)	31	31
	(4, 4, 4)	37	37

"Defective" Cases: 3 nodes

- $\mathrm{m} = (2,2,2)$, then $T(\Gamma, \mathrm{m}) \in \mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2} \otimes \mathbb{C}^{2\times 2}$,
- $\mathrm{n} = (2,3,4)$, $\quad \mathcal{TNS}^{C_3}_{\mathrm{m},\mathrm{n}} \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4)$.

Take $T \in \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ and consider the flattening

$$T_{\{1\}} : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^4$$

Then $L_T = \mathbb{P}(\mathrm{Im}\,(T_{\{1\}}))$ is a line in $\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^4)$, or a point.

Theorem (BDG'21)

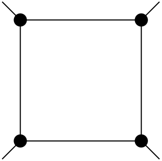
$T \in \mathcal{TNS}^{C_3}_{\mathrm{m},\mathrm{n}}$ *if and only if*

- *either* $\mathrm{rank}(L_T) = 1$
- *or* L_T *intersects* $\{M : \mathrm{rank}(M) \leq 2\}$ *in at least* 2 *points, counted with multiplicity.*

Therefore $\dim \mathcal{TNS}^{C_3}_{\mathrm{m},\mathrm{n}} \leq (=) 22 < 24$

“Defective” Cases: 4 nodes

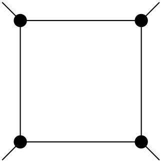
$$\mathcal{TNS}^{C_4}_{\mathrm{m},\mathrm{n}} \subseteq V_1 \otimes V_2 \otimes V_3 \otimes V_4, \qquad \mathrm{m} = (2,2,2,2).$$



	n	lower bound	upper bound
*	(2, 2, 2, 2)	15	16
*	(2, 2, 2, 3)	20	21
*	(2, 2, 2, 4)	24	25
*	(2, 3, 2, 3)	24	25
*	(2, 3, 2, 4)	28	29
*	(2, 4, 2, 4)	32	33

"Defective" Cases: 4 nodes

$$\mathcal{TNS}^{C_4}_{\mathrm{m,n}} \subseteq V_1 \otimes V_2 \otimes V_3 \otimes V_4, \qquad \mathrm{m} = (2,2,2,2).$$



	n	lower bound	upper bound
*	$(2,2,2,2)$	15	16
*	$(2,2,2,3)$	20	21
*	$(2,2,2,4)$	24	25
*	$(2,3,2,3)$	24	25
*	$(2,3,2,4)$	28	29
*	$(2,4,2,4)$	32	33

"Defective" Cases: 4 nodes

Theorem

Let $\mathbf{m} = (2,2,2,2)$ *and* $\mathbf{n} = (2,2,2,2)$. *Then*

$$\dim \mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}} = 15 \quad (< 16);$$

more precisely $\mathcal{TNS}^{C_4}_{\mathbf{m},\mathbf{n}}$ *is a hypersurface of degree* 6.

The equation is a degree 6 invariant for the action of $GL(V_1) \times \cdots \times GL(V_4)$ on $V_1 \otimes \cdots \otimes V_4$. Costruction in [Holweck,Luque,Thibon].

Conclusions

Open questions:

- Is the dimension of the stabilizer $\dim \mathrm{Stab}_{\mathcal{G}_{\Gamma,\mathrm{m}}}(X) = 0$?
- What else in the fiber?
- Geometric characterization of the defective cases.
  We expect a small defect.

Thank you!

References

- J.M.Landsberg, Y.Qi, K.Ye. *On the geometry of Tensor Network States.* Quantum Inf.Comput., 12(3-4):346–254, 2012.
- H.Derksen and V.Makam. *Maximum likelihood estimation for matrix normal models via quiver representations.* arXiv:2007.10206, 2020.
- F.Holweck, J.-G.Luque, J.-Y.Thibon. *Geometric descriptions of entangled states by auxiliary varieties.* J. Math. Phys., 53(10):102203, 2012.
- A.Bernardi, C.De Lazzari, F. Gesmundo. *Dimension of Tensor Network Varieties.* arXiv: 2101.03148, 2021.