Equations and multidegrees for inverse symmetric matrix pairs

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Let $\mathbb{S}^n = \mathrm{Sym}^2(\mathbb{C}^n)$ be the space of $n \times n$ symmetric matrices over $\mathbb{C}$.
Let $\mathbb{P}^{m-1}$ be the projectivization $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$, where $m = \binom{n+1}{2}$.

Main object of study

All possible pairs of an invertible symmetric matrix and its inverse:

$$\Gamma := \overline{\{(M, M^{-1}) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

Objectives

1. Equations of Γ.
2. Multidegrees of Γ.
3. Some applications in Algebraic Statistics.
4. "Show that Rees algebras are powerful tools to study rational maps".

Γ is the graph of a rational map

Since $M^{-1} = \frac{1}{\det(M)} M^+$, it follows that M^{-1} and M^+ coincide in $\mathbb{P}^{m-1} = \mathbb{P}(\mathbb{S}^n)$.
Therefore, we can write:

$$\Gamma = \overline{\left\{(M, M^+) \mid M \in \mathbb{P}^{m-1} \text{ and } \det(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

– $X = (X_{i,j})_{1 \leq i,j \leq n}$ and $Y = (Y_{i,j})_{1 \leq i,j \leq n}$ generic symmetric matrices over $\mathbb{C}$.
– $R = \mathbb{C}[X_{i,j}]$ polynomial ring in $m = \binom{n+1}{2}$ variables.
– $Z_{i,j} \in R$ is the signed (i,j)-minor of X.

Let $\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}$ be the rational map

$$\mathcal{F} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}, \quad (X_{1,1} : X_{1,2} : \cdots : X_{n,n}) \mapsto (Z_{1,1} : Z_{1,2} : \cdots : Z_{n,n}).$$

Then, it follows that

$$\Gamma = \overline{\text{graph}(\mathcal{F})} = \overline{\left\{(M, \mathcal{F}(M)) \mid M \in \mathbb{P}^{m-1}, \mathcal{F}(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$$

The Rees algebra

– $S = \mathbb{C}[X_{i,j}, Y_{i,j}]$ bigraded polynomial ring ($\mathrm{BiProj}(S) = \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$).
– $I = I_{n-1}(X) = (Z_{i,j})$ ideal of $(n-1) \times (n-1)$-minors of X.

Rees algebra (or blow-up algebra)

The Rees algebra of the ideal I is given by

$$\mathcal{R}(I) = R[It] = \bigoplus_{n=0}^{\infty} I^n t^n \subset R[t].$$

$\mathcal{R}(I)$ can be presented as a quotient of S by using the map

$$\Psi \;:\; S \longrightarrow \mathcal{R}(I), \quad Y_{i,j} \mapsto Z_{i,j}t,$$

Important fact

$\Gamma = \overline{\mathrm{graph}(\mathcal{F})} = \mathrm{BiProj}(\mathcal{R}(I))$. This implies that:

- Equations of $\Gamma = \mathrm{Ker}(\Psi)$.
- Multidegrees of Γ = multidegrees of $\mathcal{R}(I) \cong S/\mathrm{Ker}(\Psi)$.

Syzygies and the geometry of rational maps

Rational maps is a classical and important subject that goes back to the work of Cremona in 1865, at least!

In the last 20 years there has been an increasing interest in the study of rational maps from an algebraic point of view (i.e. analyze the syzygies of the base ideal).

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- Busé, Cid-Ruiz, D'Andrea 2020: **saturated special fiber ring**.

Equations of Γ

$$\Gamma := \overline{\left\{(M, M^{-1}) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

Theorem (–)

The ideal of defining equations of Γ is given by the prime ideal

$$\mathfrak{J} = \left(\begin{array}{ll} \sum_{k=1}^n X_{i,k} Y_{k,j}, & 1 \leq i \neq j \leq n \\ \sum_{k=1}^n X_{i,k} Y_{k,i} - \sum_{k=1}^n X_{j,k} Y_{k,j}, & 1 \leq i,j \leq n \end{array} \right).$$

Skecth of the proof

– $\mathfrak{J} = \mathrm{Ker}(\Psi)$.

– By Kotsev 1991, we have that $\mathcal{R}(I) = \mathrm{Sym}(I)$ (***I* is of linear type!**).

– The defining equations of $\mathrm{Sym}(I)$ are obtained from a presentation of I.

– A resolution of I was computed by Goto and Tachibana 1977 and by Józefiak 1978.

Multidegrees of Γ (van der Waerden 1928)

Geometrical intuition

Note $\dim(\Gamma) = m-1$. For $i+j=m-1$, we have the multidegree of type (i,j):

$$\deg^{(i,j)}(\Gamma) \;:=\; \#\big(\Gamma \,\cap\, (\mathcal{L}_1 \times \mathcal{L}_2)\big),$$

where $\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{P}^{m-1}$ are general linear spaces of codimension i and j, resp.

Hilbert polynomial

There is a polynomial $P_\Gamma(t_1,t_2) = \sum_{n_1,n_2\geq 0} e(n_1,n_2)\binom{t_1+n_1}{n_1}\binom{t_2+n_2}{n_2}$, where $e(n_1,n_2) \in \mathbb{Z}$ and $P_\Gamma(k_1,k_2) = \dim_{\mathbb{k}}\left([\mathcal{R}(I)]_{(k_1,k_2)}\right)$ when $k_i \gg 0$.

Then, we have $\deg^{(i,j)}(X) = e(i,j)$ when $i+j=m-1$.

Intersection theory

Chow ring $A^*(\mathbb{P}^{m-1}\times\mathbb{P}^{m-1}) = \mathbb{Z}[H_1,H_2]/\left(H_1^m, H_2^m\right)$. Then

$$[\Gamma] \;=\; \sum_{i+j=m-1} \deg^{(i,j)}(X)\; H_1^{m-1-i} H_2^{m-1-j} \quad \in A^*(\mathbb{P}^{m-1}\times\mathbb{P}^{m-1}).$$

Multidegrees can also be defined in terms of Hilbert series.

Let $\psi_i := 2^{i-1}, \psi_{i,j} := \sum_{k=i}^{j-1} \binom{i+j-2}{k}$ if $i < j$, and for $\alpha = (\alpha_1, \ldots, \alpha_r) \subset \{1, \ldots, n\}$ let

$$\psi_\alpha := \begin{cases} \mathrm{Pf}\left(\psi_{\alpha_k,\alpha_l}\right)_{1\le k<l\le n} & \text{if } r \text{ is even,} \\ \mathrm{Pf}\left(\psi_{\alpha_k,\alpha_l}\right)_{0\le k<l\le n} & \text{if } r \text{ is odd,} \end{cases}$$

where $\psi_{\alpha_0,\alpha_k} = \psi_{\alpha_k}$ and Pf denotes the Pfaffian. Let

$$\beta(n, d) := \sum_{\substack{\alpha\subset\{1,\ldots,n\} \\ ||\alpha||=d}} \psi_\alpha \psi_{\alpha^c},$$

where α runs over all strictly increasing subsequences of $\{1, \ldots, n\}$, including the case $\alpha = \emptyset$, and $||\alpha||$ denotes the sum of the entries of α.

Theorem (–)

For each $0 \le d \le m - 1$,

$$\deg^{m-1-d,d}(\Gamma) = \sum_{j=0}^{d} (-1)^j \beta(n, d - j).$$

Sketch of the proof

1. By our computation of the equations of Γ, we have
$$0 \to \mathcal{R}(I)(-1,-1) \xrightarrow{\cdot b} \mathcal{R}(I) \to S/I_1(XY) \to 0$$
where $b = (XY)_{1,1} = \sum_{l=1}^{n} X_{1,k} Y_{k,1} \in S$.

2. $\Sigma = V(I_1(XY)) \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ variety of pairs of symmetric matrices with product zero. Consider the **multidegree polynomials**:
$$\mathcal{C}(\Gamma; t_1, t_2) \ := \ \sum_{i+j=m-1} \deg^{i,j}(\Gamma)\, t_1^{m-1-i} t_2^{m-1-j} \ \in \ \mathbb{N}[t_1, t_2]$$
and
$$\mathcal{C}(\Sigma; t_1, t_2) \ := \ \sum_{i+j=m-2} \deg^{i,j}(\Sigma)\, t_1^{m-1-i} t_2^{m-1-j} \ \in \ \mathbb{N}[t_1, t_2].$$
Then, we obtain
$$t_1^m + t_2^m + \mathcal{C}(\Sigma; t_1, t_2) \ = \ (t_1 + t_2) \cdot \mathcal{C}(\Gamma; t_1, t_2).$$

3. A formula for the multidegrees of Σ can be directly computed using the work of Nie, Ranestad, and Sturmfels 2010 and of von Bothmer, and Ranestad 2009.

Example

Take $n = 3$. $\Gamma := \overline{\{(M, M^{-1}) \mid M \in \mathbb{P}^5 \text{ and } \det(M) \neq 0\}} \subset \mathbb{P}^5 \times \mathbb{P}^5$.
The generic matrices X and Y are given by

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{1,2} & X_{2,2} & X_{2,3} \\ X_{1,3} & X_{2,3} & X_{3,3} \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_{1,1} & Y_{1,2} & Y_{1,3} \\ Y_{1,2} & Y_{2,2} & Y_{2,3} \\ Y_{1,3} & Y_{2,3} & Y_{3,3} \end{pmatrix}.$$

The equations of Γ

$$\mathfrak{J} = \left(\begin{array}{ll} (X \cdot Y)_{i,j} = \sum_{k=1}^{3} X_{i,k} Y_{k,j}, & 1 \leq i \neq j \leq 3 \\ (X \cdot Y)_{i,i} - (X \cdot Y)_{j,j} = \sum_{k=1}^{3} X_{i,k} Y_{k,i} - \sum_{k=1}^{3} X_{j,k} Y_{k,j}, & 1 \leq i, j \leq 3 \end{array} \right).$$

Multidegrees of Γ

$$\left(\deg^{(0,5)}(\Gamma), \deg^{(1,4)}(\Gamma), \ldots, \deg^{(5,0)}(\Gamma) \right) = \left(1, 2, 4, 4, 2, 1\right).$$

Application in algebraic statistics

$$\Gamma := \overline{\left\{(M, M^{-1}) \mid M \in \mathbb{P}(\mathbb{S}^n) \text{ and } \det(M) \neq 0\right\}} \subset \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}.$$

The following four numbers coincide

- $\phi(n, d)$: the **maximum likelihood degree** of the linear concentration model defined by a generic d-dimensional linear subspace of $\mathrm{Sym}^2(\mathbb{R}^n)$ (Sturmfels and Uhler, 2010).
- the degree of the variety obtained by inverting all matrices in a general d-dimensional linear subspace of $\mathbb{S}^n = \mathrm{Sym}^2(\mathbb{C}^n)$.
- the number smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $m - d = \binom{n+1}{2} - d$ given points and are tangent to $d - 1$ given hyperplanes.
- $\deg^{(m-d,d-1)}(\Gamma)$.

Conjecture (Sturmfels - Uhler)

$\phi(n, d)$ is a polynomial in n of degree $d - 1$.

Theorem (Manivel, Michałek, Monin, Seynnaeve and Vodička)

Sturmfels-Uhler conjecture indeed holds: $\phi(n, d)$ is a polynomial in n of degree $d-1$.
(Their main tool was the **space of complete quadrics**.)

Theorem (–)

We can provide an alternative proof of this strong result.

Take away idea

Rees algebras can be a very powerful tool to study rational maps. There is a gigantic "algebraic literature" on Rees algebras that can be used for geometical purposes.

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Thanks!