Complete quadrics and algebraic statistics

1 3 9 17 21 21 17 9 3 1
2 6 18 34 42 34 18 6 2
4 12 36 68 68 36 12 4
8 24 72 **104** 72 24 8
16 48 112 112 48 16
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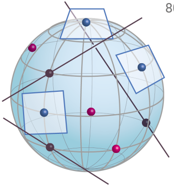

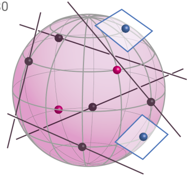

Laurent Manivel, Mateusz Michałek, Leonid Monin, Tim Seynnaeve, Martin Vodička

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Linear concentration models

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- Maximum likelihood estimation: given a sample covariance matrix S, find the concentration matrix $K \in L$ which maximizes the log likelihood function

$$K \mapsto \log(\det K) - \mathrm{tr}(S \cdot K).$$

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ML-degree of the linear concentration model

The *ML-degree* of $L \subset \mathrm{Sym}_2 \mathbb{R}^n$ is the number of (complex) critical points of the log likelihood function $K \mapsto \log(\det K) - \mathrm{tr}(S \cdot K)$, for S generic.

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ML-degree of the *generic* linear concentration model

If the d-dimensional linear space $L \subseteq \mathrm{Sym}_2 \mathbb{R}^n$ is chosen generically, the ML-degree only depends on n and d, not on L. We will denote this number by $\phi(n,d)$.

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Proposition [Sturmfels-Uhler 2010]

$\phi(n, d)$ is equal to the degree of the variety L^{-1}, where $L \subseteq \mathrm{Sym}_2 \mathbb{C}^n$ is a general d-dimensional linear subspace, and L^{-1} is the (Zariski or Euclidean) closure of $\{A^{-1} \mid A \in L \text{ and } A \text{ invertible}\}$.

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Remark

We can pass to projective space: let $\mathcal{L} = \mathbb{P}(L) \subseteq \mathbb{P}(\mathrm{Sym}_2 \mathbb{C}^n)$ be a general linear subspace of (projective) dimension $d - 1$, then the degree of $\mathcal{L}^{-1} := \mathbb{P}(L^{-1})$ is equal to $\phi(n, d)$.

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$\phi(n,d)$ is the number of pairs $(K,\Sigma) \in \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ with

$$\Sigma \cdot K = Id_n,\ K \in \mathcal{L},\ \Sigma \in \mathcal{M},$$

where $\mathcal{L} \subset \mathbb{P}(S^2V)$ and $\mathcal{M} \subset \mathbb{P}(S^2V^*)$ are general linear subspaces, of dimension $(d-1)$ respectively codimension $(d-1)$.

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Let $X \subset \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ variety parametrized by (K, K^{-1}). The points of X are all pairs $(K, \Sigma) \in \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ with $K \cdot \Sigma = Id_n$ or $K \cdot \Sigma = 0$.

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Technical difficulty: X is singular. We will replace it with a smooth variety: *the space of complete quadrics*.

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For $i = 1, \ldots, n-1$, we define $E_i \subset \mathcal{CQ}(V)$ to be the strict transform of the exceptional locus of $Bl_{\tilde{D}_i}$.

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$\mathcal{CQ}(V)$ is the closure of the image of the set of invertible matrices under the map

$$\varphi : \mathbb{P}(S^2V) \hookrightarrow \mathbb{P}\left(S^2V\right) \times \mathbb{P}\left(S^2(\bigwedge^2 V)\right) \times \cdots \times \mathbb{P}\left(S^2(\bigwedge^{n-1} V)\right),$$

sending a matrix A to $(A, \bigwedge^2 A, \ldots, \bigwedge^{n-1} A)$. Here $\bigwedge^k A \in S^2(\bigwedge^k V)$ is the k*-th compound matrix*, whose entries are the $k \times k$ minors of A.

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$$\begin{aligned} E_r =& \{(A_1, \ldots, A_{n-1}) \in \mathcal{CQ}(V) \mid \operatorname{rk} A_r = 1\} \\ =& \overline{\{(A_1, \ldots, A_{n-1}) \in \mathcal{CQ}(V) \mid \operatorname{rk} A_1 = r\}}. \end{aligned}$$

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Remark

$\phi(n,d)$ is the number of smooth (or complete) quadric hypersurfaces in $\mathbb{P}^{n-1}$ containing $\binom{n+1}{2} - d$ given (general) points and tangent to $d-1$ given (general) hyperplanes.

Computing $\phi(n, d)$ using intersection theory

$\mathcal{CQ}(V)$ is a smooth variety such that

$$\begin{array}{ccc} & \mathcal{CQ}(V) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_{n-1}} \\ \mathbb{P}(S^2V) & \overset{}{\underset{A \mapsto A^{-1}}{\dashrightarrow}} & \mathbb{P}(S^2V^*) \end{array}$$

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- $\phi(n,d) = |\pi_1^{-1}(\mathcal{L}) \cap \pi_{n-1}^{-1}(\mathcal{M})|$, where $\dim(\mathcal{L}) = d-1$ and $\operatorname{codim}(\mathcal{M}) = d-1$.

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- Let $L_i \in A^1(\mathcal{CQ}(V))$ be the pullback of the hyperplane class in $\mathbb{P}(S^2 \bigwedge^i V)$ under the map $\pi_i : \mathcal{CQ}(V) \to \mathbb{P}(S^2 \bigwedge^i V)$.

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- We have
$$\phi(n,d) = \int_{\mathcal{CQ}(V)} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1}.$$

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We have the following relations in $A(\mathcal{CQ}(V))$:

$$2L_k = L_{k-1} + S_k + L_{k+1}$$

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$$\begin{aligned}\phi(n, d) &= \int_{\mathcal{CQ}(V)} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1} \\ &= \frac{1}{n} \sum_{s=1}^{n-1} s \int_{\mathcal{CQ}(V)} L_1^{\binom{n+1}{2}-d-1} L_{n-1}^{d-1} S_{n-s}.\end{aligned}$$

The algebraic degree of SDP

We will write

$$\delta(d,n,r) := \int_{\mathcal{CQ}(V)} L_1^{\binom{n+1}{2}-d-1} L_{n-1}^{d-1} S_r.$$

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- $\delta(d,n,r)$ is the number of degenerate complete quadrics in S_r tangent to $(d-1)$ hyperplanes and passing through $\binom{n+1}{2}-d-1$ points.
- $\delta(d,n,r)$ is known as the *algebraic degree of semidefinite programming*.

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- Pushforward via $S_r \to Gr(r,V)$.
- We find that $\delta(d,n,r) =$

$$\int_{Gr(r,V)} \mathrm{Seg}_{(\binom{n+1}{2}-d-\binom{r+1}{2})}(S^2\mathcal{U})\,\mathrm{Seg}_{(d-\binom{n-r+1}{2})}(S^2\mathcal{Q}^*)$$

where Seg is the Segre class, and $\mathcal{U}$ and $\mathcal{Q}$ are the universal sub- and quotient bundles.

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- $A(Gr(r,V))$ is a quotient of the ring of symmetric polynomials in r variables.
- For $I \subset \mathbb{N}$, define the *Lascoux coefficient* $\psi_I \in \mathbb{Z}$ via

$$\mathrm{Seg}_d(S^2\mathcal{U}) = \sum_{\substack{\lambda(I) \vdash d \\ |I| = r}} \psi_I s_{\lambda(I)},$$

where $\lambda(I) := [i_r - (r-1), i_{r-1} - (r-2), \ldots, i_2 - 1, i_1]$ is the partition corresponding to $I = \{i_1 > \cdots > i_r\}$, and s_λ is the corresponding Schur polynomial.

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Theorem (Graf von Bothmer - Ranestad 2009)

$$\delta(d,n,r) = \sum_{\substack{I\subset[n] \\ |I|=n-r \\ \sum I=d-n+r}} \psi_I \psi_{[n]\setminus I}$$

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One can show that $\delta(d,n,n-s)=0$ for $\binom{s+1}{2} > d$. Hence

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This can be proven using the following recursive relations:

- For $j_1 > 0$:

$$\psi_{\{j_1,\dots,j_s\}} = (s+1)\psi_{\{0,j_1,\dots,j_s\}} - 2\sum_{\ell=1}^{s} \psi_{\{0,j_1,\dots,j_\ell - 1,\dots,j_s\}}.$$

- For $j_1 = 0$:

$$\psi_{\{j_1,j_2,\dots,j_s\}} = \sum_{j_\ell \le j'_\ell < j_{\ell+1}} \psi_{\{j'_1,\dots,j'_{s-1}\}}.$$

Polynomiality

Using the formula

$$\phi(n,d) = \frac{1}{n}\sum_{s=1}^{n} s \left(\sum_{\substack{|I|=s \\ \sum I = d-s}} \psi_I \psi_{[n]\setminus I} \right)$$

and the recursive relations for ψ_I and $\psi_{[n]\setminus I}$, we get a fast algorithm for computing $\phi(n,d)$.

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$$\phi(n,d) = \frac{1}{n}\sum_{s=1}^{n} s \left(\sum_{\substack{|I|=s \\ \sum I = d-s}} \psi_I \psi_{[n]\setminus I} \right)$$

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Until now, only the case $d \leq 5$ was known (Sturmfels-Uhler):

- $\phi(n,d) = (n-1)^{(d-1)}$ for $d = 1, 2, 3$.
- $\phi(n,4) = (n-1)^3 - \binom{n+1}{3} = \frac{1}{6}(n-1)(n-2)(5n-3)$.
- $\phi(n,5) = \frac{1}{12}(n-1)(n-2)(7n^2-19n+6)$.

Polynomiality

Using the formula

$$\phi(n,d) = \frac{1}{n}\sum_{s=1}^{n} s \left(\sum_{\substack{|I|=s \\ \sum I = d-s}} \psi_I \psi_{[n]\setminus I} \right)$$

and the recursive relations for ψ_I and $\psi_{[n]\setminus I}$, we get a fast algorithm for computing $\phi(n,d)$.

Our algorithm can compute at least up to $d = 50$. For instance:

$$\phi(n,18) = \frac{1}{355687428096000}(n-5)(n-4)(n-3)(n-2)(n-1)$$
$$(3024902557n^{12} - 111489409997n^{11} + 1862235028288n^{10} -$$
$$18676382506290n^9 + 125446336704681n^8 - 594987544526781n^7 +$$
$$2047718727437714n^6 - 5214795516381220n^5 + 10138037306327912n^4$$
$$- 15696938913831072n^3 + 18622763914779648n^2$$
$$- 12286614789872640n + 2964061900800)$$

Thank you!

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