Exact Moment Representation in Polynomial Optimization [1]

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This is a joint work with Bernard Mourrain

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Polynomial Optimization

Let $f, g_1, \ldots, g_r \in \mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \ldots, X_n]$, $\mathcal{S}(\mathbf{g}) = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \forall i\}$.
Goal of Polynomial Optimization: find $f^* = \inf\{f(x) \in \mathbb{R} \mid x \in \mathcal{S}(\mathbf{g})\}$.

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Real Algebraic Geometry

- Σ^2 **Sum of Squares polynomials** (SoS);
- $\mathcal{Q}(\mathbf{g}) = \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_r$ **quadratic module**;
- $\mathcal{O}(\mathbf{g})$ **preordering** (closure by multiplication of $\mathcal{Q}(\mathbf{g})$);
- $\operatorname{supp} Q = Q \cap -Q$ **support** of Q (ideal in $\mathbb{R}[\mathbf{X}]$);
- **Real Nullstellensatz**: $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt{\operatorname{supp} \mathcal{O}(\mathbf{g})}$, in particular $\mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt{\operatorname{supp}(I + \Sigma^2)} = \sqrt[\mathbb{R}]{I}$.

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Duality

$\mathbb{R}[\mathbf{X}]^* = \hom_{\mathbb{R}}(\mathbb{R}[\mathbf{X}], \mathbb{R})$ is an $\mathbb{R}[\mathbf{X}]$-module with $\star$ defined, for $f \in \mathbb{R}[\mathbf{X}]$ and $\sigma \in \mathbb{R}[\mathbf{X}]^*$, as: $f \star \sigma = \sigma \circ m_f$ (i.e. $\langle f \star \sigma | g \rangle = \langle \sigma | fg \rangle$).

Lasserre Relaxations

$$f^* = \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathrm{Pos}(S) \} = \inf \{ \int f \, \mathrm{d}\mu \in \mathbb{R} \mid \mu \in \mathcal{M}^{(1)}(S) \}$$

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Lasserre Relaxations

Let $Q = \mathcal{Q}(\mathbf{g})$ and $S = \mathcal{S}(\mathbf{g})$. Replace:

- $\mathrm{Pos}(S)$ by Q and
- $\mathcal{M}(S)$ by $\mathcal{L}(Q) = Q^{\vee} = \{\sigma \in \mathbb{R}[\mathbf{X}]^* \mid \forall f \in Q \ \langle \sigma | f \rangle \geq 0\}$,

and for increasing $d \in \mathbb{N}$, at order d restrict to polynomials of deg $\leq 2d$:

- $\mathcal{Q}_{2d}(\mathbf{g}) \subset \mathbb{R}[\mathbf{X}]_{2d}$ polynomials in Q generated in deg $\leq 2d$;
- $\mathcal{L}_{2d}(\mathbf{g}) = \mathcal{Q}_{2d}(\mathbf{g})^{\vee} \subset (\mathbb{R}[\mathbf{X}]_{2d})^*$.

SoS relaxation: $\mathcal{Q}_{2d}(\mathbf{g})$ and $f^*_{\mathrm{SoS},d} = \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \}$.

MoM relaxation: $\mathcal{L}^{(1)}_{2d}(\mathbf{g}) = \{ \sigma \in \mathcal{L}_{2d}(\mathbf{g}) \mid \langle \sigma | 1 \rangle = 1 \}$ and $f^*_{\mathrm{MoM},d} = \inf \{ \langle \sigma | f \rangle \in \mathbb{R} \mid \sigma \in \mathcal{L}^{(1)}_{2d}(\mathbf{g}) \}$.

Properties

- $f^*_{\mathrm{SoS},d} \leq f^*_{\mathrm{MoM},d} \leq f^*$ for all d;
- If $\mathcal{Q}(\mathbf{g})$ is Archimedean (i.e. $r^2 - \|\mathbf{X}\|^2 \in \mathcal{Q}(\mathbf{g})$ for some $r \in \mathbb{R}$) then $\lim_{d\to+\infty} f^*_{\mathrm{SoS},d} = \lim_{d\to+\infty} f^*_{\mathrm{MoM},d} = f^*$ [Lasserre, 2001];
- $\mathcal{Q}_{2d}(\mathbf{g})$ and $\mathcal{L}_{2d}(\mathbf{g})$ are convex sets in a finite dimensional vector space;
- λ and $\langle\sigma|f\rangle$ are linear objective functions.

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Thus solving a Lasserre Relaxation is a Convex Optimization Problem.

- In practice, efficient numerical solvers are used (MOSEK, CSDP, SDPA,..., based on Interior Point method).

Finite Convergence

We say that the MoM (resp. the SoS) relaxation has the finite convergence property if $f^*_{\text{MoM},d} = f^*_{\text{MoM},d+1} = \cdots = f^*$ (resp. $f^*_{\text{SoS},d} = f^*_{\text{SoS},d+1} = \cdots = f^*$) for some $d \in \mathbb{N}$.

[Laurent, 2007], [Nie, 2013], [Demmel, Nie, Sturmfels, 2006], [Demmel, Nie, Powers, 2007] studied finite convergence.

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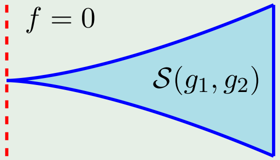
Example (Marshall, 2008)

Let $g_1 = X^3 - Y^2$ and $g_2 = 1 - X$. $\mathcal{O}(g_1, g_2) = \mathcal{Q}(g_1, g_2, g_1 g_2)$ is Archimedean, but for $f = X$ the SoS relaxation $\mathcal{Q}_{2d}(g_1, g_2, g_1 g_2)$ and the MoM relaxation $\mathcal{L}^{(1)}_{2d}(g_1, g_2, g_1 g_2)$ do not have finite convergence.

- $\mathcal{O}_{2d}(g_1, g_2) = \mathcal{Q}_{2d}(g_1, g_2, g_1 g_2)$ is closed;
- $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{MoM},d} < f^* = 0$;
- $f - f^* \notin \mathcal{O}(g_1, g_2)$.
- the min. of f on $\mathcal{S}(g_1, g_2)$ is singular;



SoS Exactness

We say that the SoS relaxation is exact if it has the f.c. property and $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ (in other words sup = max).

Example (Nie, Demmel and Sturmfels 2006)

Let $f = (X^4Y^2 + X^2Y^4 + Z^6 - 2X^2Y^2Z^2) + X^8 + Y^8 + Z^8 \in \mathbb{R}[X,Y,Z]$. We want to optimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z})$.

- The SoS relaxation $\mathcal{Q}_{2d}(\pm\frac{\partial f}{\partial X}, \pm\frac{\partial f}{\partial Y}, \pm\frac{\partial f}{\partial Z})$ has f.c. but is not exact;
- The MoM relaxation has f.c. and there exists $\sigma \in \mathcal{L}_{2d}(\pm\frac{\partial f}{\partial X}, \pm\frac{\partial f}{\partial Y}, \pm\frac{\partial f}{\partial Z})$ such that $\langle \sigma | f \rangle = f^*$.

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Question: Are all the min. linear functionals coming from measures? For $\sigma \in \mathcal{L}_{2d}^{(1)}(\mathbf{g})$ such that $\langle \sigma | f \rangle = f^*$, does there exist $\mu \in \mathcal{M}^{(1)}(\mathcal{S}(\mathbf{g}))$ such that $\langle \sigma | h \rangle = \int h \, \mathrm{d}\mu$ for all $h \in \mathbb{R}[\mathbf{X}]_k$?

MoM Exactness

- $\mathbf{e}_x$ denotes the **evaluation** at $x \in \mathbb{R}^n$: $\langle \mathbf{e}_x | f \rangle = \int f \, d\mathbf{e}_x = f(x)$;
- Richter–Tchakaloff theorem: $\mathcal{M}(S)^{[t]} = \mathrm{cone}(\mathbf{e}_x : x \in S)^{[t]}$.

We say that the MoM relaxation is exact if it has the f.c. property and the moment minimizers (i.e. $\langle \sigma | f \rangle = f^*$) are coming from measures:

$$\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[k]} = \mathrm{conv}(\mathbf{e}_\xi : \xi \in S, \ f(\xi) = f^*)^{[k]}.$$

MoM Exactness property will be the main topic of the talk.

- Flat extention criterion to detect MoM f.c. and exactness;
- Moment sequences yield the minimizers when the MoM is exact ($f^*_{\mathrm{MoM},d} = f^*$ not sufficient).

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	SoS f. c.	SoS ex.	MoM f.c.	MoM ex.	dim S
No f.c.	NO	NO	NO	NO	$1, 2, \geq 3$
Cylinder	YES	YES	YES	NO	≥ 3
S finite	YES	NO	YES	YES	0

Generic elements

We denote by $\mathrm{Ann}(\sigma)$ the annihilator of $\sigma \in \mathbb{R}[\mathbf{X}]^*$ w.r.t. $\star$ (and $\mathrm{Ann}_d(\sigma)$ for the truncated version): $\mathrm{Ann}(\sigma) = \{\, h \in \mathbb{R}[\mathbf{X}] \mid h \star \sigma = 0 \,\}$.

Example (Sum of Evaluations)

Let $x, y \in \mathbb{R}^n$. Then:

$$\begin{aligned}\mathrm{Ann}(\mathbf{e}_x + \mathbf{e}_y) &= \{h \in \mathbb{R}[\mathbf{X}] \mid h \star (\mathbf{e}_x + \mathbf{e}_y) = 0\} \\ &= \{h \in \mathbb{R}[\mathbf{X}] \mid f(x)h(x) + f(y)h(y) = 0 \quad \forall f \in \mathbb{R}[\mathbf{X}]\} \\ &= \mathrm{Ann}(\mathbf{e}_x) \cap \mathrm{Ann}(\mathbf{e}_y) = \mathcal{I}_{\mathbb{R}}(x) \cap \mathcal{I}_{\mathbb{R}}(y).\end{aligned}$$

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Genericity

$\sigma^* \in \mathcal{L}_{2d}(\mathbf{g})$ is generic if $\mathrm{Ann}_d(\sigma^*) = \bigcap_{\sigma \in \mathcal{L}_{2d}(\mathbf{g})} \mathrm{Ann}_d(\sigma)$.

- A generic σ^* is in the relative interior of $\mathcal{L}_{2d}(\mathbf{g})$.

Generic Annihilators

Theorem (_, Mourrain)

Let $Q = \mathcal{Q}(\mathbf{g})$ *and* $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$*. Then there exists* $d, k \in \mathbb{N}$ *such that for* $\sigma^* \in \mathcal{L}_d(\mathbf{g})$ *generic, we have* $J = (\operatorname{Ann}_k(\sigma^*))$.

Corollary

Let $I = (\mathbf{h})$ *and* $d \geq \deg \mathbf{h}$*. Then for* $\sigma^* \in \mathcal{L}_{2d}(\pm\mathbf{h})$ *generic* $I \subset (\operatorname{Ann}_d(\sigma^*)) \subset \sqrt[\mathbb{R}]{I}$*. Moreover for* d *big enough* $(\operatorname{Ann}_d(\sigma^*)) = \sqrt[\mathbb{R}]{I}$*.*

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Zero dimensional case: flat extention criterion.

Corollary

If $\sigma^* \in \mathcal{L}_d(\Pi\mathbf{g})$ *is generic then* $(\operatorname{Ann}_k(\sigma^*)) = \mathcal{I}_{\mathbb{R}}(\mathcal{S}(\mathbf{g}))$ *for* $d, k \in \mathbb{N}$ *big enough (where* $\Pi\mathbf{g}$ *are the generators of* $\mathcal{O}(\mathbf{g})$ *as a quadratic module).*

Finite Semialgebraic Sets and Exactness

Theorem (_, Mourrain)

Suppose that $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$*. Then,* $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \ldots, \xi_r\}$ *is non-empty and finite and there exists* $d \in \mathbb{N}$ *such that* $\forall k \in \mathbb{N}$*:*

$$\mathcal{L}_{d+k}(\mathbf{g})^{[2(\rho-1)+k]} = \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[2(\rho-1)+k]},$$

where $\rho = \rho(\xi_1, \ldots, \xi_r)$ *is the regularity. The MoM relaxation is exact* $\forall f$*.*

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where $\rho = \rho(\xi_1, \ldots, \xi_r)$ *is the regularity. The MoM relaxation is exact* $\forall f$.

- If $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic, then $\mathcal{V}_{\mathbb{C}}(\operatorname{Ann}_d(\sigma^*)) = \mathcal{V}_{\mathbb{R}}(\operatorname{Ann}_d(\sigma^*))$ is equal to the set of minimizers.
- By strong duality [Josz, Henrion, 2016] the SoS has f.c. property.
- These results generalize [Lasserre, Laurent, Rostalski, 2008], [Lasserre, Laurent, Mourrain, Rostalski, Trébuchet, 2013] and [Nie, 2013] for real radicals of zero dimensional ideals and finite real varieties.

Generic Exactness

- **Boundary Hessian Conditions** (BHC): local regularity conditions introducd in [Marshall, 2006] (see also [Scheiderer, 2005, 2009]): local, positive linear dependence of ∇f and $\nabla g_1, \ldots, \nabla g_r$ and positive definite Hessian;
- If $\mathcal{Q}(\mathbf{g})$ is Archimedean and BHC hold at every zero of f on $\mathcal{S}(\mathbf{g})$, then the zeros are smooth isolated points and $f \in \mathcal{Q}(\mathbf{g})$ (BHC at every minimizer $\Rightarrow f - f^* \in \mathcal{Q}(\mathbf{g})$: **SoS exactness**);
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Theorem (_, Mourrain)

Let $f \in \mathbb{R}[\mathbf{X}]$ and $Q = \mathcal{Q}(\mathbf{g})$ be an Arch. quad. module. If the BHC hold at every minimizer of f on $\mathcal{S}(\mathbf{g})$ then the MoM relaxation $\mathcal{L}_{2d}(\mathbf{g})$ is exact and, for $\sigma^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic, $\mathcal{V}_{\mathbb{R}}(\mathrm{Ann}_k(\sigma^*))$ is equal to the set of minimizers.

Corollary: MoM exactness holds generically.

Examples (MomentTools)

Example (Motzkin Polynomial)

We minimize globally (i.e. using $\mathcal{L}_d(1)$) the Motzkin polynomial:

$$f = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \qquad f^* = 0$$

```
v, M = minimize(f, [], [], X, 4, Mosek.Optimizer)
```

Here $f^*_{\text{MoM},4} = v = -1.23437 \cdot 10^{-10}$, but we can't recover the four min $(\pm 1, \pm 1)$. We add $f - f^*_{\text{MoM},4} = 0$ to find them, i.e.use $\mathcal{L}_d(\pm(f - f^*_{\text{MoM},4}))$

```
v, M = minimize(f, [f-v], [], X, 4, Mosek.Optimizer)
```

Here $f^*_{\text{MoM},4} = v = 1.849085 \cdot 10^{-10}$ and we can recover the minimizers:

```
w, Xi = get_measure(M)
```

x:	-1.000000944864053	-1.00000094519377	1.00000094519377	1.0000009448640537
y:	1.000000945033868	-1.000000945211865	1.000000945211865	-1.000000945033868

Examples (MomentTools)

Example (Robinson form)

We minimize on the unit sphere (i.e. using $\mathcal{L}_d(\pm(x^2+y^2+z^2-1))$) the Robinson form (BHC hold):

$$f = x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4(y^2+z^2) - y^4(x^2+z^2) - z^4(x^2+y^2)$$
$$h = x^2+y^2+z^2-1, \ f^* = 0$$

```
v, M = minimize(f, [h], [], X, 5, Mosek.Optimizer)
w, Xi = get_measure(M)
```

Here $f^*_{\mathrm{MoM},5} = v = -1.27211 \cdot 10^{-7}$ and we can recover the minimizers $\frac{\sqrt{3}}{3}(\pm1,\pm1,\pm1), \frac{\sqrt{2}}{2}(0,\pm1,\pm1), \frac{\sqrt{2}}{2}(\pm1,0,\pm1), \frac{\sqrt{2}}{2}(\pm1,\pm1,0)$:

x:	0.577351068999	8.812477930640 10^{-12}	0.707107158043	0.707107157553
y:	0.577351069076	0.707107158048	1.271729446125 10^{-13}	0.707107157555
z:	0.577351066102	0.707107158048	0.707107158042	2.478771201340 10^{-9}

Perspectives

Extentions:

- $(\mathrm{Ann}_d(\sigma^*)) = \sqrt[\mathbb{R}]{I}$ in positive dimension [_, Mourrain, Computing Real Radicals by Moment Optimization ISSAC 2021]
- How to achieve exactness: Polar Ideal (rank conditions for the Jacobian of $f, g_1, \ldots, g_r$)

Open questions:

- Order of convergence of positive linear functionals to measures (from Effective Putinar's Positivstellensatz)
- Structure and sparsity of $\mathcal{L}_d(\mathbf{g})$

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More details in [**Lorenzo Baldi, Bernard Mourrain. Exact Moment Representation in Polynomial Optimization**. (preprint, 2020) (hal-03082531v2)]

Thank you for your attention!