Cluster Duality for Lagrangian and Orthogonal Grassmannians

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Overview

Cluster Algebras

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Fix $l \geq m \geq 0$ and a field $\mathcal{F} \cong \mathbb{C}(x_1, \ldots, x_l)$.

Definition

A *cluster seed* Σ in $\mathcal{F}$ consists of a pair $(\mathbf{x}_\Sigma, B_\Sigma)$ where $\mathbf{x}_\Sigma \in \mathcal{F}^l$ is such that $\mathcal{F} = \mathbb{C}(\mathbf{x}_\Sigma)$, and B_Σ is an $l \times m$ integer matrix whose upper $m \times m$ matrix is skew-symmetrizable.

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Definition

The *cluster algebra of rank m* A_Σ associated to an initial seed Σ is the ring $A_\Sigma = \mathbb{C}[X_\Sigma]$ generated by all polynomials in cluster variables. We say that any algebra A obtained in this way has a *cluster structure*

(A_Σ is usually not a polynomial ring because mutation introduces relations among cluster variables.)

Cluster Algebras

When the coordinate ring of a variety V has a cluster structure, we call V a *cluster variety*. Roughly speaking, this means that an open subset of V is described by unions of algebraic tori $(\mathbb{C}^*)^l$, each of which is indexed by one cluster seed S. Furthermore, these tori are identified along birational maps coming from mutations.

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There are two flavors of cluster varieties: $\mathcal{A}$-varieties and $\mathcal{X}$-varieties. They differ in the precise formulas that govern the mutation process (both on the level of cluster seeds and birational maps).

Grassmannians and Clusters

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In [RW19], Rietsch and Williams use these cluster structures on the coordinate rings of the Grassmannians $\check{\mathbb{X}} = \mathrm{Gr}(k, n)$ and $\mathbb{X} = \mathrm{Gr}(n-k, n)$. On $\check{\mathbb{X}}$, they use the cluster structure to study *superpotential polytopes* Γ_S, and on $\mathbb{X}$, they study the *Newton-Okounkov body* Δ_S, both associated to a choice of seed in the cluster structure.

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Theorem ([RW19])

For any cluster seed S, $\Gamma_S = \Delta_S$.

Grassmannians and Clusters

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Our $\check{\mathbb{X}}$ is the *Orthogonal Grassmannian* $\mathrm{OG}^{co}(n+1, 2n+1)$ of co-isotropic $(n+1)$-dimensional subspaces of $\mathbb{C}^{2n+1}$ with respect to a *quadratic form*. It is a homogeneous group for SO_{2n+1} (type B_n).

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Our $\mathbb{X}$ is the *Lagrangian Grassmannian* $\mathrm{LG}(n, 2n)$ of isotropic n-dimensional subspaces of $\mathbb{C}^{2n}$ with respect to a *symplectic form*. It is a homogeneous group for Sp_{2n} (type C_n).

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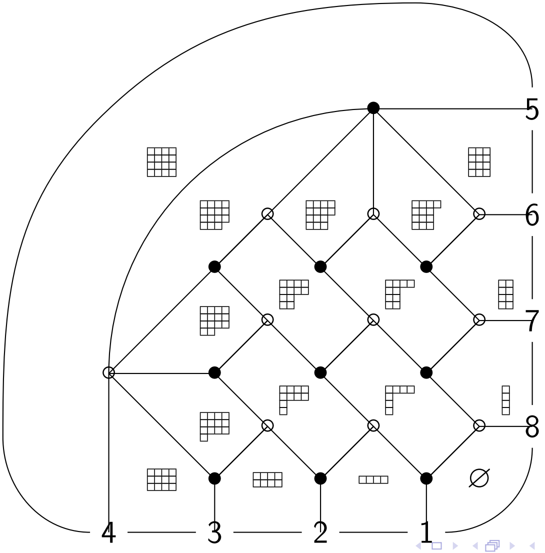
Our *co-rectangles seed* will be given by example for $n = 4$ on the next slide, and the generalization to arbitrary n is straightforward.

co-rectangles seed



Overview

1. Cluster Algebras
2. Newton-Okounkov Bodies
3. Superpotential Polytopes
4. Unimodular Equivalence

Newton-Okounkov Bodies

A Newton-Okounkov body is a convex set associated to a variety V and a divisor D on it. The convex geometry of a Newton-Okounkov body encodes geometric information about the pair (V, D), generalizing the way that a lattice polytope encodes information about its associated toric variety.

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Our variety will be $\mathbb{X} = \mathrm{LG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ in its Plücker embedding, and our divisor will be the vanishing locus of a particular Plücker coordinate on $\mathbb{X}$. In this situation, the Newton-Okounkov body turns out to be a polytope, and we give its construction now.

Newton-Okounkov Bodies

Recall that for a cluster variety, we have an embedded torus $\mathbb{T}_\Sigma \cong (\mathbb{C}^*)^N$ for each cluster seed Σ. This embedding gives a morphism

$$\mathbb{C}[\mathbb{X}] \hookrightarrow \mathbb{C}[\mathbb{T}_\Sigma]$$

of the coordinate ring of $\mathbb{C}[\mathbb{X}]$ into the Laurent polynomial ring $\mathbb{C}[\mathbb{T}_\Sigma]$, which is generated by the N torus parameters. This allows us to define a valuation associated to Σ on $\mathbb{C}[\mathbb{X}]$.

Newton-Okounkov Bodies

Fix a total ordering on the torus parameters N. (This choice will not play a crucial role.)

Definition

Define $\mathrm{val}_\Sigma : \mathbb{C}[\mathbb{X}] \setminus \{0\} \to \mathbb{Z}^N$ as follows. For $f \neq 0 \in \mathbb{C}[\mathbb{X}]$, first express f as an element of $\mathbb{C}[\mathbb{T}_\Sigma]$ using the inclusion $\mathbb{C}[\mathbb{X}] \hookrightarrow \mathbb{C}[\mathbb{T}_\Sigma]$.

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Newton-Okounkov Bodies

Now we can define the Newton-Okounkov Δ_n body associated to a cluster seed Σ.

Definition

$$\Delta_n = \overline{\mathrm{conv}\left(\bigcup_{r=1}^{\infty} \frac{1}{r}\mathrm{val}_{\Sigma}(H^0(\mathbb{X}, \mathcal{O}(rD)))\right)}$$

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Concretely, we think of $H^0(\mathbb{X}, \mathcal{O}(rD))$ as the space of degree r polynomials in $\mathbb{C}[\mathbb{X}]$.

Newton-Okounkov Bodies

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Proposition

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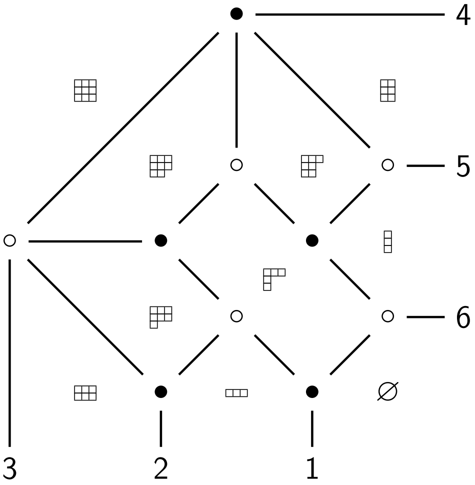
We now give an example for $n = 3$ of how the combinatorics of symmetric plabic graphs allows us to compute these valuations explicitly.

Newton-Okounkov bodies



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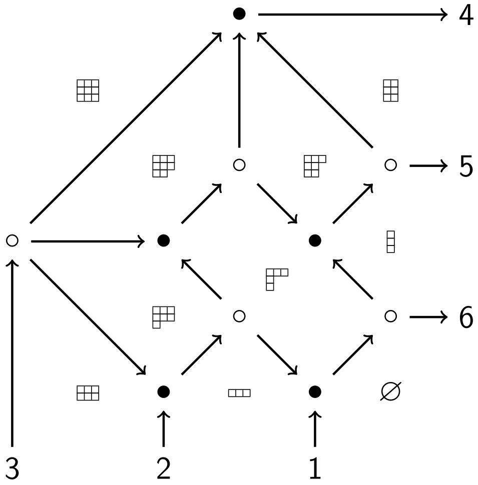

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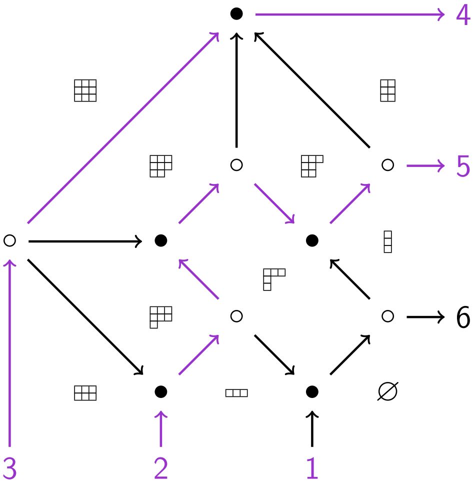

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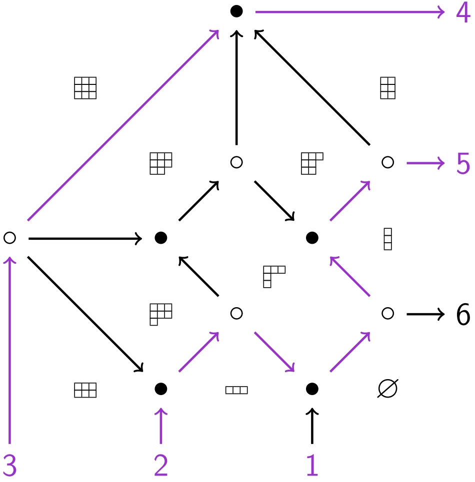

Newton-Okounkov Bodies

For the first flow, there are no face labels to the left of the path $1 \to 1$. The face labels to the left of $3 \to 4$ are $\yng(3,3,3)$. The face labels to the left of $2 \to 5$ are $\yng(3,3)$, $\yng(3,3,1)$, $\yng(3,3,2)$, $\yng(3,2,2)$, $\yng(2,2,2)$, and $\yng(3,3,3)$, contributing a monomial

$$x_{(3,3,3)}^2 x_{(3,3)}^2 x_{(3,3,2)} x_{(3,2,2)}^2.$$

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For the first flow, there are no face labels to the left of the path $1 \to 1$. The face labels to the left of $3 \to 4$ are . The face labels to the left of $2 \to 5$ are , , , , , and , contributing a monomial

$$x^2_{(3,3,3)} x^2_{(3,3)} x_{(3,3,2)} x^2_{(3,2,2)}.$$

For the second flow, there are no face labels to the left of the path $1 \to 1$. The face labels to the left of $3 \to 4$ are . The face labels to the left of $2 \to 5$ are , , , , , , , contributing a monomial

$$x^2_{(3,3,3)} x^2_{(3,3)} x_{(3,3,2)} x^2_{(3,2,2)} x_{(3,1,1)}.$$

Newton-Okounkov Bodies

Thus, the expression of the Plücker coordinate $P_{1,4,5}$ as a Laurent polynomial in $\mathbb{C}[\mathbb{T}_\Sigma]$ is

$$P^G_{\{1,4,5\}} = (x^2_{(3,3,3)} x^2_{(3,3)} x_{(3,3,2)} x^2_{(3,2,2)})(1 + x_{(3,1,1)})$$

giving a valuation of $(0, 2, 0, 2, 1, 2)$.

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giving a valuation of $(0,2,0,2,1,2)$. We list as a table the remaining valuations:

$I \in \binom{[6]}{3}$	$\mathrm{val}_\Sigma(p_I)$	$I \in \binom{[6]}{3}$	$\mathrm{val}_\Sigma(p_I)$
123	$(0,0,0,0,0,0)$	$146 = 245$	$(1,2,1,2,1,2)$
124	$(0,0,0,0,0,1)$	$156 = 345$	$(1,3,1,3,2,2)$
$125 = 134$	$(0,1,0,1,1,1)$	236	$(2,2,1,2,1,1)$
$126 = 234$	$(1,1,1,2,1,1)$	246	$(2,2,1,2,1,2)$
135	$(0,2,0,2,1,1)$	$256 = 346$	$(2,3,1,3,2,2)$
$136 = 235$	$(1,2,1,2,1,1)$	356	$(2,4,1,4,2,2)$
145	$(0,2,0,2,1,2)$	456	$(2,4,1,4,2,3)$

Overview

Superpotential Polytopes

The *superpotential*, W_q is a regular function on an open subset of $\check{\mathbb{X}} = \mathrm{OG}^{co}(n+1, 2n+1)$. This function encodes (e.g. via its Jacobi ring) enumerative information about $\mathbb{X}$. In particular, it recovers the small quantum cohomology of $\mathbb{X}$. The pair $(\check{\mathbb{X}}, W_q)$ is called a *Landau-Ginzburg model* for $\mathbb{X}$.

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To obtain a polytope from this information, we use an expression for the superpotential derived by Pech and Rietsch ([PR13]). The definition is a bit technical, so we give an example for $n = 3$.

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$$W_q = a_{11} + a_{12} + a_{13} + a_{22} + a_{23} + a_{33}$$
$$+\frac{q}{a_{11}a_{12}a_{13}} + \frac{q}{a_{11}a_{12}a_{23}} + \frac{q}{a_{11}a_{22}a_{23}} + \frac{q}{a_{11}a_{22}a_{33}}$$

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For our previous superpotential, the inequalities are

$$\begin{aligned} A_{ij} &\geq 0 \\ 1 - A_{11} - A_{12} - A_{13} &\geq 0 \\ 1 - A_{11} - A_{12} - A_{23} &\geq 0 \\ 1 - A_{11} - A_{22} - A_{23} &\geq 0 \\ 1 - A_{11} - A_{22} - A_{33} &\geq 0 \end{aligned}$$

and we set Γ_n to be the polytope defined by these inequalities.

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For $n = 3$, the isomorphism above (sending Γ_n to Δ_n) is given by the matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

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The proof strategy is as follows:

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2. Define a map $\Gamma_n \to \Delta_n$.
3. Check that the image of this map is actually all of Δ_n

Future Work

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2. Use this to give a change of coordinates for Γ_n to obtain a true equality.
3. Use this to study Δ_n and Γ_n for different cluster seeds.

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