Algebraic and Local Solutions of Systems of Autonomous AODEs of Dimension One

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References and Acknowledgments

J.M. Aroca, J. Cano, R. Feng, X.S. Gao, *Algebraic general solutions of algebraic ordinary differential equations*. In Proceedings of ISSAC'05, 29–36, 2005.

J. Cano and S. Falkensteiner and J.R. Sendra, *Existence and Convergence of Puiseux Series Solutions for First Order Autonomous Differential Equations*. Journal of Symbolic Computation (in print), 2020. doi.org/10.1016/j.jsc.2020.06.010

J. Cano and S. Falkensteiner and J.R. Sendra, *Algebraic, Rational and Puiseux Series Solutions of Systems of Autonomous Algebraic ODEs of Dimension One*. Mathematics in Computer Science, 15:189–198, 2021.

D. Robertz, *Formal algorithmic elimination for PDEs*. Springer, 2014.

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Overview

1. Preliminaries
   - Formal Puiseux Series
   - Systems of Dimension One
2. (Differential) Elimination
   - Simple Systems
3. First Order Autonomous AODEs
4. Results on Systems of Dimension One
   - Convergence
   - Algebraic Solutions

Formal Puiseux series

K ... field of characteristic zero

$K[[x]]$ ... formal power series

$K((x)) = K[[x]][x^{-1}]$

$K\langle\langle x\rangle\rangle = \bigcup_{n\in\mathbb{N}^*} K((x^{1/n}))$ ... field of formal Puiseux series (expanded around zero)

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$y(x) \in K\langle\langle x\rangle\rangle$ and $Q(x,y) \in K[x,y] \setminus K[x]$ such that $Q(x, y(x)) = 0$ ... algebraic Puiseux series

Let $\tilde{y} \in K((x^{1/n}))$ such that there is no $m \mid n$ and $\tilde{y} \in K((x^{1/m}))$. Then n is called the ramification number of $\tilde{y}$.

Systems of Dimension One

Let

$$K\{y_1, \ldots, y_n\} = K[y_1, y_1', y_1'', \ldots, y_n, y_n', y_n'', \ldots]$$

be the ring of differential polynomials in the differential indeterminates $y_1, \ldots, y_n$ with coefficients in the field of constants K.

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be the ring of differential polynomials in the differential indeterminates $y_1, \ldots, y_n$ with coefficients in the field of constants K. Let

$$\mathcal{S} = \{F_1 = 0, \ldots, F_M = 0\}. \tag{1}$$

be a finite set of differential polynomials in $K\{y_1, \ldots, y_n\}$ whose sum of orders if equal to m. For an algebraically closed field $\mathbb{K} \supseteq K$, viewing (1) as algebraic set

$$\mathbb{V}_{\mathbb{K}}(\mathcal{S}) = \{a \in \mathbb{K}^{m+n} \mid F_1(a) = \cdots = F_M(a) = 0\},$$

we assume that $\mathbb{V}_{\mathbb{K}}(\mathcal{S})$ has (algebraic) dimension one, i.e. it is a union of space curves and, maybe, points.

Systems of Dimension One

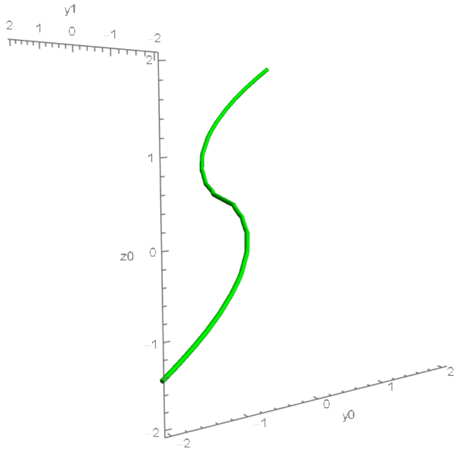
Given: $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ of dimension one as in (1).

Goal:

- Analyze the formal Puiseux series solution vectors $(y_1(x), \ldots, y_n(x)) \in \mathbb{C}\langle\langle x \rangle\rangle^n$ of $\mathcal{S}$.
- Compute the algebraic Puiseux series solution vectors of $\mathcal{S}$.

The following system defines a space curve as in (1).

$$\mathcal{S} = \{-8y'^3 + 27y = 0, z^5 - y^3 = 0, -5z^4z' + 3y^2y' = 0\}.$$



Simple Systems

By using algebraic and differential reduction (here we use the Thomas decomposition [4]), differential systems $\mathcal{S} \subset K\{y_1, \ldots, y_n\}$ can be decomposed into a finite collection of simple subsystems $(\mathcal{S}_k, \mathcal{U}_k)$ representing a set of equalities

$$\mathcal{S} = \{G_1 = 0, \ldots, G_M = 0\} \subset K\{y_1, \ldots, y_n\}$$

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The simple subsystems have as algebraic equations the same zeros as the given system. In particular, the decompositions has the same solutions set, i.e.

$$\mathrm{Sol}_{\bar{K}\langle\langle x\rangle\rangle}(\mathcal{S}) = \dot{\bigcup} \, \mathrm{Sol}_{\bar{K}\langle\langle x\rangle\rangle}(\mathcal{S}_k, \mathcal{U}_k).$$

Simple Systems

Simple systems have in particular the following properties:

- $G_1, \ldots, G_M, U_1, \ldots, U_N$ have pairwise distinct leading variables (they are in triangular form);
- $G_1, \ldots, G_M$ are pairwise differentially reduced and $U_1, \ldots, U_N$ are reduced with respect to the G_i's.

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System of dimension one $\mathcal{S}$ as in (1) can be decomposed into simple subsystems leading to constant solution components and to simple subsystems of the form

$$\left\{ \begin{array}{rcl} G_1(y_1, y_1') & = & 0, \\ G_s(y_1, y_1', y_2, \ldots, y_s) & = & 0, \qquad s \in \{2, \ldots, n\}, \\ U(y_1) & \neq & 0, \end{array} \right. \tag{I}$$

where the leading variables (w.r.t. the ordering $y_1 < y_1' < \cdots < y_n < y_n' < \cdots$) are $\mathrm{lv}(G_1) = y_1', \mathrm{lv}(G_s) = y_s$ and $U \in K[y_1] \setminus \{0\}$.

First Order Autonomous AODEs

In [1, 2], there are considered first order algebraic ordinary differential equations (AODEs) with constant coefficients, i.e.

$$F(y, y') = 0, \tag{2}$$

with $F \in K[y, y']$.

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$$F(y, y') = 0, \qquad (2)$$

with $F \in K[y, y']$.
By using local parametrizations of the plane curve $\mathbb{V}_{\bar{K}}(F)$, the Puiseux series solutions of $F = 0$ can be computed. In particular, existence, uniqueness and convergence (for $K \subseteq \mathbb{C}$) of the solutions can be ensured and their ramification index can be computed.

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Theorem [Algebraic Solutions]

Let $F \in K[y, y']$ be irreducible with an algebraic solution $y(x) \in \bar{K}\langle\langle x \rangle\rangle \setminus \bar{K}$. Then all formal Puiseux series solutions of $F = 0$ are algebraic over $\bar{K}(x)$.

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Moreover, if $Q \in \bar{K}[x, y]$ is the minimal polynomial of $y(x)$, then all non-constant formal Puiseux series solutions are given by $Q(x + c, y)$, where $c \in \bar{K}$.

Results

To conclude, systems of dimension one (1) can essentially be decomposed into simple systems of the type (I) and Puiseux series solutions of first order autonomous AODEs are convergent. Combining these two observations leads to the following result.

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Theorem [Convergence]

Let $K \subseteq \mathbb{C}$. All components of a formal Puiseux series solution vector of system (1), expanded around a finite point or at infinity, are convergent.

Algebraic Solutions

Computations with Puiseux series vectors are an algorithmically intricate problem. For algebraic series, however, computations simplify.

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Computations with Puiseux series vectors are an algorithmically intricate problem. For algebraic series, however, computations simplify. For a system of the type (I)

$$\left\{ \begin{array}{rcll} G_1(y_1, y_1') & = & 0, & \\ G_s(y_1, y_1', y_2, \ldots, y_s) & = & 0, & s \in \{2, \ldots, n\}, \\ U(y_1) & \neq & 0, & \end{array} \right.$$

and a polynomial relation $P_1(x, y_1) = 0$ with $P_1 \in K[x, y_1]$ and $\mathrm{lv}(P_1) = y_1$, we can again compute a decomposition into finitely many algebraic simple subsystems of the type

$$\left\{ \quad G_s(x, y_1, \ldots, y_s) \;=\; 0, \qquad s \in \{1, \ldots, n\}, \right. \qquad \text{(II)}$$

where $G_s \in K[x, y_1, \ldots, y_s]$ with $\mathrm{lv}(G_s) = y_s$.

Algebraic Solutions

Combining this observation with Theorem [Algebraic Solutions] we observe:

Corollary

Let $(\mathcal{S},\mathcal{U})$ be a simple system of the form (I) such that $G_1 \in K[y_1, y_1']$ is irreducible with an algebraic solution

$$y_1(x) \in \bar{K}\langle\langle x\rangle\rangle \setminus \bar{K}.$$

Then all formal Puiseux series solutions of $(\mathcal{S},\mathcal{U})$ are algebraic over $\bar{K}(x)$.

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Then all formal Puiseux series solutions of $(\mathcal{S}, \mathcal{U})$ are algebraic over $\bar{K}(x)$.

The description of the algebraic solutions can be done either as

- algebraic simple subsystems of the form (II), namely $\{G_1(x, y_1) = 0, \ldots, G_n(x, y_1, \ldots, y_n) = 0\}$; or
- the minimal polynomials $\{Q_1(x, y_1) = 0, \ldots, Q_n(x, y_n) = 0\}$.

Algebraic Solutions

By recursively computing the resultant with respect to the biggest leading variables in

$$P_i = \mathrm{Res}(G_i, \{Q_1, G_1, \ldots, G_{i-1}\}) \in K[x, y_i],$$

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the minimal polynomials of the solution components are the irreducible factors of P_i.
There are two main disadvantages of simplifying algebraic simple systems to the vector of minimal polynomials

- full factorization has to be performed;
- not every combination of roots has to be a solution of the given system, i.e.

$$\mathrm{Sol}_{K\langle\langle x\rangle\rangle}(\{G_1, \ldots, G_n\}) \subseteq \bigcup \mathrm{Sol}_{K\langle\langle x\rangle\rangle}(\{Q_1, \ldots, Q_n\}).$$

Summary

Given $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ of dimension one as in (1).

1) Compute a Thomas decomposition of $\mathcal{S}$.
2) For every simple subsystem involving no derivatives, there are only constant solutions. For the simple subsystems $(\tilde{\mathcal{S}}, \tilde{\mathcal{U}})$ of the type (I), check whether $G_1(y_1, y_1')$ has an algebraic solution $y_1(x) \in \mathbb{C}\langle\langle x \rangle\rangle$.
3) In the affirmative case, compute a Thomas decomposition of $(\tilde{\mathcal{S}} \cup \{Q_1\}, \tilde{\mathcal{U}})$ where Q_1 is the minimal polynomial of $y_1(x)$.
4) The algebraic solutions are then given as algebraic simple systems (or can be expressed as a vector of minimal polynomials).

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$$Q_1(x, y) = y^2 - x^3.$$

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The Thomas decomposition finds the algebraic simply subsystem

$$\{Q_1(x, y) = y^2 - x^3,\ G_2(x, y, z) = z^5 - x^3\, y\}. \quad (4)$$

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The solutions of (3) and (4) are the same: Let $z_1(x) = \zeta\, x^{9/10},\ z_2(x) = -\zeta\, x^{9/10}$ with $\zeta^5 = 1$. Then $(y_i(x), z_i(x))$ is a solution (but neither $(y_1(x), z_2(x))$ nor $(y_2(x), z_1(x))$).

The algebraic simple system (4),

$$\{Q_1(x,y) = y^2 - x^3,\ G_2(x,y,z) = z^5 - x^3\,y\},$$

leads to the vector of minimal polynomials

$$\{Q_1(x,y) = y^2 - x^3,\ Q_2(x,z) = z^{10} - x^9\}. \qquad (5)$$

The system (5), however, has $(y_1(x), z_2(x))$ and $(y_2(x), z_1(x))$ as solutions.