Characterizing infinite graphs allowing flexible frameworks

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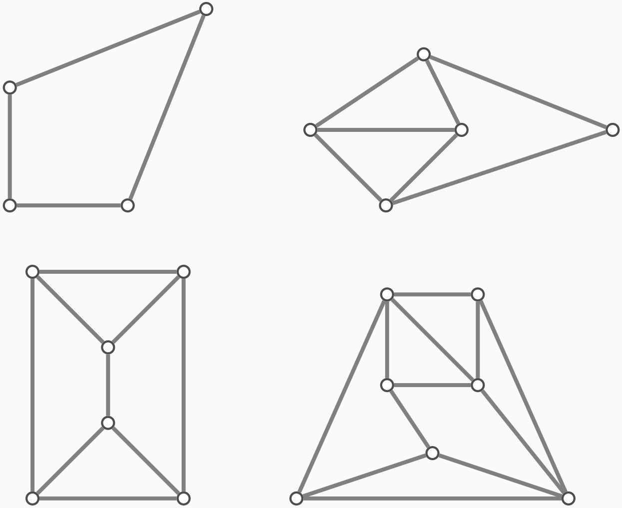
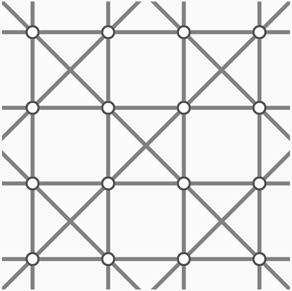




When does an infinite graph admit a flexible framework?

Definition

A map $\rho : V_G \to \mathbb{R}^2$ for a graph $G = (V_G, E_G)$ such that $\rho(u) \neq \rho(v)$ for every edge $uv \in E_G$ is called a *realization*.

A *flex* of the *framework* (G, ρ) is a continuous path $t \mapsto \rho_t$, $t \in [0, 1)$, in the space of realizations of G such that $\rho_0 = \rho$ and for all $t \in [0, 1)$ and all edges $uv \in E_G$

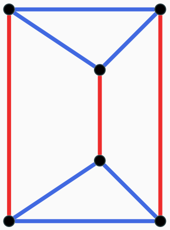
$$\|\rho_t(u) - \rho_t(v)\| = \|\rho(u) - \rho(v)\| .$$

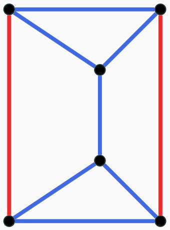
The flex is called *trivial* if the equation above holds for all $u, v \in V_G$ and all $t \in [0, 1)$.

The framework (G, ρ) is *flexible* if it has a non-trivial flex. Otherwise it is called *rigid*.

Definition

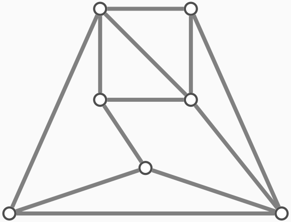
A coloring of edges $\delta : E_G \rightarrow \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.





Theorem (Grasegger, L., Schicho, 2019)

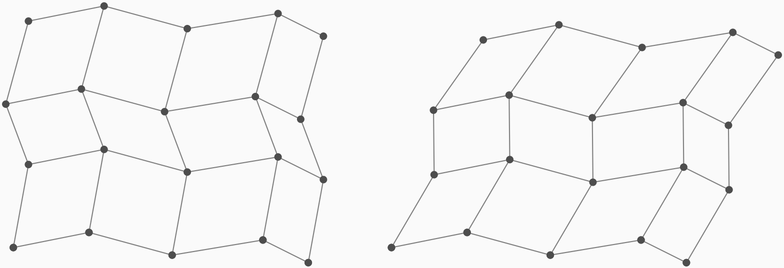
A connected finite graph admits a flexible framework in the plane if and only if it has a NAC-coloring.

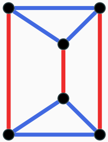


Theorem (Gallet, L., Schicho/Dewar)

A connected countably infinite graph admits a flexible framework in the plane if and only if it has a NAC-coloring.

NAC-coloring $\implies$ flex





Flex $\implies$ NAC-coloring

Let (G, ρ) be a finite flexible framework and $\lambda_{uv} = \|\rho(u) - \rho(v)\|$ for $uv \in E_G$. Fix $\bar{u}\bar{v} \in E_G$.

$$\begin{aligned}(x_{\bar{u}}, y_{\bar{u}}) &= (0, 0)\\ (x_{\bar{v}}, y_{\bar{v}}) &= (\lambda_{\bar{u}\bar{v}}, 0)\\ (x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda_{uv}^2 \quad \forall\, uv \in E_G\end{aligned}$$

An irreducible curve in the zero set is called an *algebraic motion*.

$$\begin{aligned}\lambda_{uv}^2 &= (x_v - x_u)^2 + (y_v - y_u)^2 \\ &= \underbrace{\big((x_v - x_u) + i(y_v - y_u)\big)}_{W_{u,v}} \cdot \underbrace{\big((x_v - x_u) - i(y_v - y_u)\big)}_{Z_{u,v}}\end{aligned}$$

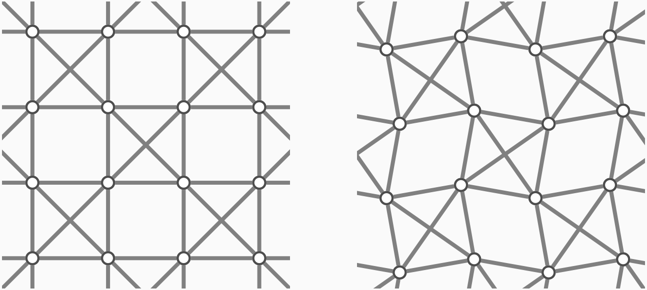
For every cycle $(u_0, u_1, \ldots, u_n, u_{n+1} = u_0)$:

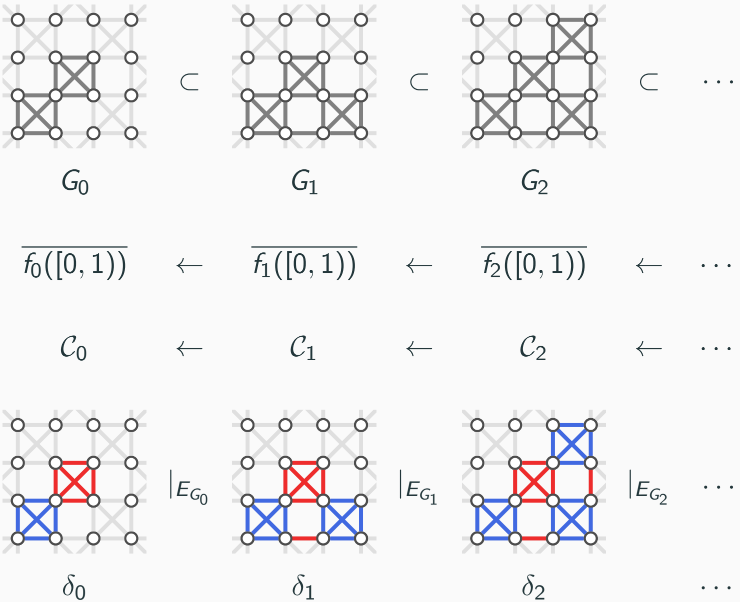
$$\sum_{i=0}^{n} W_{u_i,u_{i+1}} = 0 \quad \text{and} \quad \sum_{i=0}^{n} Z_{u_i,u_{i+1}} = 0$$

Lemma (Grasegger, L., Schicho, 2019)
Let $\mathcal{C}$ be an algebraic motion of (G, ρ). If $\alpha \in \mathbb{Q}$ and ν is a valuation of the complex function field of $\mathcal{C}$ such that there exists edges $\bar{u}\bar{v}, \hat{u}\hat{v}$ with $\nu(W_{\bar{u},\bar{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{red, blue\}$ given by

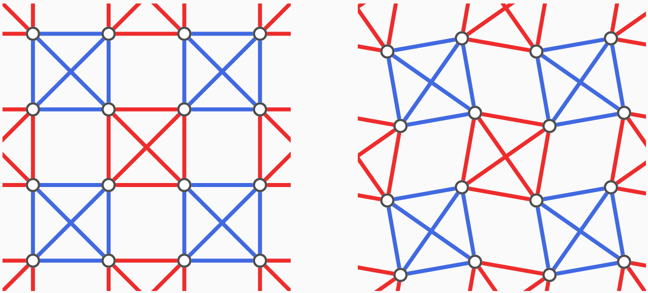
$$\delta(uv) = red \iff \nu(W_{u,v}) > \alpha\,,$$
$$\delta(uv) = blue \iff \nu(W_{u,v}) \leq \alpha\,.$$

is a NAC-coloring.





$G_0 \subset G_1 \subset G_2 \subset \cdots$
$\overline{f_0([0,1))} \leftarrow \overline{f_1([0,1))} \leftarrow \overline{f_2([0,1))} \leftarrow \cdots$
$\mathcal{C}_0 \leftarrow \mathcal{C}_1 \leftarrow \mathcal{C}_2 \leftarrow \cdots$
$|_{E_{G_0}}$ $|_{E_{G_1}}$ $|_{E_{G_2}}$ $\cdots$
δ_0 δ_1 δ_2 $\cdots$



Final remarks

- alternative proofs using König's lemma or by contradiction (Sean Dewar)
- n-fold symmetric case
- bracing of frameworks with 4-cycles being parallelograms

Thank you

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