Cellular Structure of the Pommaret-Seiler Resolution for Quasi-Stable Ideals

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Basic concepts

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- The *class* of μ or x^μ, denoted by $\mathrm{cls}(\mu)$ or $\mathrm{cls}(x^\mu)$, is equal to $\min\{i | \mu_i \neq 0\}$.

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- The set of *multiplicative variables* of x^μ is $\mathcal{X}_P(x^\mu) = \{x_1, \ldots, x_{\mathrm{cls}(\mu)}\}$, and the set of *non-multiplicative variables* of x^μ is $\overline{\mathcal{X}}_P(x^\mu) = \mathcal{X} \setminus \mathcal{X}_P(x^\mu)$.

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- x^μ is an *involutive divisor* of x^ν if $x^\mu | x^\nu$ and $x^{\nu-\mu} \in \mathsf{k}[\mathcal{X}_P(x^\mu)]$.

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Quasi-stable Monomial Ideals

Definition

Let $\mathcal{H}$ be a finite collection of monomials, $\mathcal{H} \subseteq R$. We say that $\mathcal{H}$ is a *Pommaret basis* of the monomial ideal $I = \langle \mathcal{H} \rangle$ if $I = \bigoplus_{h \in \mathcal{H}} h \cdot \mathrm{k}[\mathcal{X}_P(h)]$ as vector spaces.

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Definition

We call a monomial ideal I *quasi-stable*, if it possesses a finite monomial Pommaret basis.

Quasi-stable Ideals

The name quasi-stable is due to the fact that these ideals are a generalization of the important class of *stable ideals*:

[cf. [Seiler, 2010], Proposition 5.5.6]

A monomial ideal I is stable if and only if its minimal monomial generating set is also a Pommaret basis for I.

Stable Ideals

Definition

A monomial ideal I is stable if for every $x^\mu \in I$ it satisfies that for each index $i > \min(x^\mu)$ we have that $x^\mu \frac{x_i}{x_{\min(x^\mu)}} \in I$, where $\min(x^\mu)$ denotes the index of the first variable that divides x^μ.

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The Eliahou-Kervaire minimal resolution for stable ideals

Eliahou and Kervaire in [Eliahou and Kervaire, 1990] give a closed form of the minimal free resolution of any stable monomial ideal. The Eliahou-Kervaire resolution raises as an iterated mapping cone [Charalambous and Evans, 1995, Herzog and Takayama, 2002], in fact, this is a way to prove its minimality [Peeva and Stillman, 2008].

The Eliahou-Kervaire Resolution

Proposition

Let I be a monomial ideal and $x^\mu \in I$. Then there exists a unique generator g and monomial h such that $x^\mu = gh$ and for every x_i dividing h we have that $i \leq \min(g)$.
We denote g by $\mathrm{beg}(x^\mu)$ and h by $\mathrm{end}(x^\mu)$

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Definition

Let I be a stable monomial ideal. An EK-symbol for I is a pair of the form $[f, u]$ where f is a minimal generator of I and u is a square-free monomial satisfying $\min(u) \gneq \min(f)$.

The Eliahou-Kervaire Resolution

The Eliahou-Kervaire Resolution

$$0 \longrightarrow \cdots \longrightarrow E_l \longrightarrow E_{l-1} \longrightarrow \cdots E_0 \longrightarrow I \longrightarrow 0,$$

The differential of the resolution is given by

$$d([f,u]) = \sum_{x_i|u} \operatorname{sgn}(x_i,u) x_i [f, \frac{u}{x_i}] - \sum_{x_i|u} \operatorname{sgn}(x_i,u) \operatorname{end}(x_i f)[\operatorname{beg}(x_i f), \frac{u}{x_i}],$$

where $\operatorname{sgn}(x_i,u) = 1$ if the cardinality of the set $\{x_j \text{ s.t. } x_j \text{ divides } u \text{ and } j \geq i\}$ is odd, and -1 otherwise.

The Pommaret-Seiler Resolution

For any h_α in $\mathcal{H}$ and any non-multiplicative variable $x_k \in \overline{\mathcal{X}}_P(h_\alpha)$ there exists a unique index $\Delta(\alpha, k)$ and a unique monomial $t_{\alpha;k} \in k[\mathcal{X}_P(h_{\Delta(\alpha,k)})]$ such that $x_k h_\alpha = t_{\alpha;k} h_{\Delta(\alpha,k)}$.

The Pommaret-Seiler Resolution

The Pommaret-Seiler Resolution

Let now denote by $\beta_0^{(k)}$ the number of generators in $\mathcal{H}$ of class k, and let $d = \min\{k|\beta_0^{(k)} > 0\}$. Then the Pommaret-Seiler resolution has the form

$$0 \longrightarrow R^{r_{n-d}} \longrightarrow \cdots \longrightarrow R^{r_1} \longrightarrow \cdots R^{r_0} \longrightarrow I \longrightarrow 0,$$

where the ranks of the free modules in the resolution are given by

$$r_i = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_0^{(k)}.$$

The Pommaret-Seiler Resolution

Definition

The generators of the i-th free module in the Pommaret-Seiler resolution are given by pairs of the form $[h_\alpha, u]$ where $h_\alpha \in \mathcal{H}$ and u is a is a degree i square-free monomial satisfying $\min(u) \gneq \mathrm{cls}(h_\alpha)$.

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The differential of the resolution is given by

$$\partial([h_\alpha, u]) = \sum_{j=1}^{i} (-1)^{i-j} \left(x_{u_j}[h_\alpha, \frac{u}{x_{u_j}}] - t_{\alpha,u_j}[h_{\Delta(\alpha,u_j)}, \frac{u}{x_{u_j}}] \right).$$

Mapping Cone Property

The Pommaret-Seiler resolution of a quasi-stable ideal I can be obtained as an iterated mapping cone for an adequate sorting of the generators of the Pommaret basis $\mathcal{H}$ of I [Albert et al., 2015]. This sorting is known as a P-ordering, and sorts the elements of $\mathcal{H}$ first by class and within each class lexicographically.

Cellular Property

Definition

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J. Mermin proves in [Mermin, 2010] that the Eliahou-Kervaire resolution is cellular and gives an explicit cellular structure for it.

Cellular Property

In [Dochtermann and Mohammadi, 2014], Dochtermann and Mohammadi give a sufficient condition (possession of a regular decomposition function) for an iterated mapping cone resolution to be cellular and prove that the Eliahou-Kervaire resolution satisifies this condition.

Linear Quotients Property

Definition

Let I be a monomial ideal. We say that I has *linear quotients* if there exists an ordering of the generators of I, $(m_1, \dots, m_r)$ such that for each $j \leq r$ the colon ideal $I_{j-1} : \langle m_j \rangle$ is generated by a subset of the variables, where $I_{j-1} = \langle m_1, \dots, m_{j-1} \rangle$ is the ideal generated by the first $j-1$ generators of I.
In such case, we denote by $\mathrm{set}(m_j)$ the set of variables that generate $I_{j-1} : \langle m_j \rangle$.

Decomposition Function

Definition

For any monomial ideal, let $M(I)$ be the set of monomials in I and $G(I)$ a set of generators of I. A *decomposition function* for I is an assignment $b : M(I) \to G(I)$. We say that the decomposition function b is regular if for each $m \in G(I)$ and every $x_t \in \mathrm{set}(m)$ we have that $\mathrm{set}(b(x_t m)) \subseteq \mathrm{set}(m)$.

Minimal Resolution

[Dochtermann and Mohammadi, 2014], Theorem 3.11

Suppose I has linear quotients with respect to some ordering $(m_1, \ldots, m_r)$ of the minimal generators, and furthermore suppose that I has a regular decomposition function. Then the minimal resolution of I obtained as an iterated mapping cone is cellular and supported on a regular CW-complex.

Linear Quotiens in Quasi-stable Ideals

[Albert et al., 2015], Proposition 7.2 and [Hashemi et al., 2012], Proposition 26

Let $\mathcal{H} = \{h_1, \ldots, h_r\}$ be a P-ordered monomial Pommaret basis of the quasi-stable monomial ideal I. Then I possesses linear quotients with respect to the basis $\mathcal{H}$ and $\langle h_{\alpha+1}, \ldots, h_s \rangle : h_\alpha = = \langle \overline{\mathcal{X}}_p(h_\alpha) \rangle$ for all $\alpha = 1, \ldots, s-1$.

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Note:

Observe that in this case, since $\mathcal{H}$ is in general a non-minimal generating set of I, then the iterated mapping cone resolution obtained (i.e. the Pommaret-Seiler resolution of I) is not the minimal free resolution. Minimality is only obtained if I is stable (in which case $\mathcal{H}$ is the minimal generating set of I).

The Pommaret-Seiler Resolution is Cellular

Theorem

The Pommaret-Seiler resolution of a quasi-stable monomial ideal is cellular.

Proof:

Let I be a quasi-stable ideal and $M(I)$ be the set of monomials in I. For every $h \in \mathcal{H}$ we define the involutive monomial cone of h with respect to the Pommaret division $\mathcal{C}(h)$ as the set of all monomials in $h \cdot \mathrm{k}[\mathcal{X}_P(h)]$. The fact that $\mathcal{H}$ is an involutive basis for I means that $M(I) = \bigcup_{h \in \mathcal{H}} \mathcal{C}(h)$ where the union is disjoint, hence every monomial $M(I)$ has a unique involutive divisor in $\mathcal{H}$...

The Pommaret-Seiler Resolution is Cellular

Proof:

We now define the decomposition function b

$$b : M(I) \to \mathcal{H} \text{ as } b(x^\mu) = h_\alpha,$$

where h_α is the unique element of $\mathcal{H}$ that is an involutive divisor of x^μ.

To see that b is regular observe that $\mathrm{set}(h_\alpha) = \overline{\mathcal{X}}_P(h_\alpha)$. Now, for each $x_t \in \overline{\mathcal{X}}_P(h_\alpha)$ we have that $b(x_t h_\alpha) = h_{\Delta(\alpha,t)}$ and $\mathrm{cls}(h_{\Delta(\alpha,t)}) \geq \mathrm{cls}(h_\alpha)$, hence $\mathrm{set}(h_{\Delta(\alpha,t)}) \subseteq \mathrm{set}(h_\alpha)$ and b is regular.

The rest of the proof follows the lines of the proof of Theorem 3.11 in [Dochtermann and Mohammadi, 2014] except for the minimality of the resolution.

Iterated Mapping Cone Construction

The iterated mapping cone property in a resolution implies that this resolution is cellular. This iterated mapping cone construction provides also with an explicit cellular structure for them, as seen in [Mermin, 2010] and [Dochtermann and Mohammadi, 2014].
We extend this construction to the Pommaret-Seiler resolution by means of the P-graph of the Pommaret basis, which is defined in [Seiler, 2009b], see also [Plesken and Robertz, 2005].

P-Graph of $\mathcal{H}$

Let I be a quasi-stable monomial ideal and $\mathcal{H}$ its Pommaret basis. We associate to it a directed graph, the P-graph of $\mathcal{H}$, which consists of a vertex for each $h_\alpha \in \mathcal{H}$ and a directed edge from h_α to $h_{\Delta(\alpha,t)}$ for each h_α in $\mathcal{H}$ and each $x_t \in \overline{\mathcal{X}}_P(h_\alpha)$.
The 0-cells of the CW-structure for the Pommaret-Seiler resolution of I are the vertices of the P-graph of $\mathcal{H}$ considered as points in $\mathbb{R}^n$.

Cellular Structure for the Pommaret-Seiler Resolution

Let now $h \in \mathcal{H}$, $\alpha = \{j_1, \ldots, j_p\} \subseteq \overline{\mathcal{X}}_P(h)$ with $j_1 < \cdots < j_p$ and σ a permutation of α. We define $\mathrm{ch}(h, \alpha, \sigma)$ to be the subset of $\mathbb{R}^n$ obtained as the convex hull of the elements of $\mathcal{H}$ that we reach by applying the decomposition function $b(h_i, t) = h_{\Delta(h_i,t)}$ in the order prescribed by σ.

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If there are no repetitions of elements of $\mathcal{H}$ involved in the description of $\mathrm{ch}(h, \alpha, \sigma)$ then $\mathrm{ch}(h, \alpha, \sigma)$ is a p-dimensional simplex, and we say that $\mathrm{ch}(h, \alpha, \sigma)$ is non-degenerate.

Cellular Structure for the Pommaret-Seiler Resolution

We define the cell $U(h, \alpha)$ as the union of the $\mathrm{ch}(h, \alpha, \sigma)$ over all permutations σ of α. For these cells we have a topological differential map

$$d(U(h,\alpha)) = \sum_i (-1)^i U(h, \alpha - j_i) - \sum_i (-1)^i U(h_{\Delta(h,j_i)}, \alpha - j_i).$$

Finally, by adding the monomials in these differentials we obtain the differential from before and have that the described structure is indeed the CW-structure that supports the Pommaret-Seiler resolution of I.

Reduction to the minimal

The Pommaret-Seiler resolution for quasi-stable ideals is known to be non-minimal, nevertheless, we know that it is a resolution on minimal length [Seiler, 2009b].
Let $\mathfrak{C}$ denote the CW-complex described in the previous section, then we have that $\dim(\mathfrak{C}) = \mathrm{pd}(I)$. In this section we show that $\mathfrak{C}$ can be reduced using Discrete Morse Theory [Forman, 2001], and equivalently reducing the Pommaret-Seiler resolution which suggest an algorithm for reduction to the minimal one.

The annoted P-Graph

The first step consists on building an *annotated* P-graph of the quasi-stable ideal I at the same time as we compute the Pommaret basis of I. Let us denote by G_I the P-graph of I, and by $\mathbb{P}$ the Pommaret-Seiler resolution of I. As we build the Pommaret basis $\mathcal{H}$ we can store the information of G_I assigning two pieces of information to each edge $e_{i,j}$, namely $k(e_{i,j})$ is the variable $k \in \overline{\mathcal{X}}_P(h_i)$ used to reach h_j from h_i and $t(e_{i,j})$ is the term $t_{i,k}$. This annotated graph allows us to directly read all the information in $\mathbb{P}$ from G_I.

Morse Matching

The second step consists on building a Morse matching in G_I or equivalently in $\Gamma_{\mathbb{P}}$, the graph associated to the Pommaret-Seiler resolution, see [Sköldberg, 2005].

For each directed path $p = (p_1, \ldots, p_l) = (e_{i_1,j_1}, \ldots, e_{i_l,j_l})$ in G_I we say that the multidegree of the path is $\mathrm{md}(p) = \prod_{k=1}^{l} t(e_{i_k,j_k})$.

We say that a path between nodes i and j is a *valid path* if $\mathrm{md}(p) = 1$.

For any multidegree μ consider then the following set of vertices in $\Gamma_{\mathbb{P}}$: $V_\mu = \{\alpha : \{u\} \mid \mathrm{md}(\alpha : \{u\}) = \mu\}$ where $\alpha : \{u\}$ indicates the vertex in $\Gamma_{\mathbb{P}}$ corresponding to the generator indexed by $[h_\alpha, u]$ in the Pommaret-Seiler resolution $\mathbb{P}$.

Reduction using Morse Matching

Consider now the following partial matching in $V_\mu : E_\mu = \{\alpha : \{u\} \to \beta : \{u/u_j\} \mid j = max(u)\}$.

Proposition

$\bigcup_{\mu \in \mathbb{N}^n} E_\mu$ is a Morse matching in $\Gamma_{\mathbb{P}}$.

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Proceeding iteratively by multidegree, we obtain a reduction of $\mathbb{P}$. If this reduction is already the minimal free resolution of I then we stop the algorithm. Otherwise, we can proceed by further use of Morse matchings using those pairs of generators in the reduced resolution such that the term in the differential is a nonzero scalar.

Reduction in the Cellular Complex $\mathfrak{C}$

The cellular structure $\mathfrak{C}$ of $\mathbb{P}$ allows us to read this reduction in terms of the geometrical differential of $\mathfrak{C}$.
We define the *skeleton P-graph of I* as the graph that has a vertex for each minimal generator of I and there is an edge from m_i to $m_j \in G(I)$ only if the following conditions hold for $\mu = \mathrm{lcm}(m_i, m_j)$:

- $\mu / m_i \in \mathsf{k}[\overline{\mathcal{X}}_P(m_i)]$,
- $\mu \in \mathsf{k}[\mathcal{X}_P(m_j)]$,
- $m_k \nmid \mu$ for all other $m_k \in G(I)$.

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Proposition

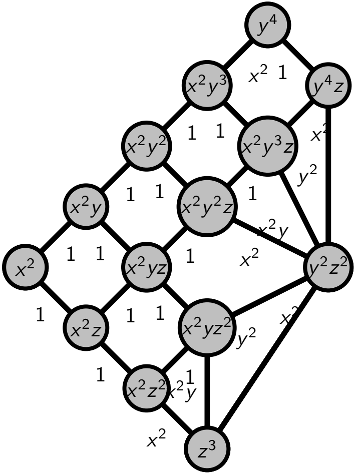
There is a Morse reduction from $\mathfrak{C}$ to a subcomplex of it whose 1-skeleton is the skeleton P-graph of I.

Example

Let $I = \langle x^2, y^4, y^2z^2, z^3 \rangle$ a quasi-stable ideal whose Pommaret basis is:

$$\mathcal{H} = \{x^2,\ x^2y,\ x^2y^2,\ x^2y^3,\ x^2z,\ x^2yz,\ x^2y^2z,\ x^2y^3z,\ x^2z^2,\ x^2yz^2,\ y^4,\ y^4z,\ y^2z^2,\ z^3\}.$$

Example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x^2z^2
z^3
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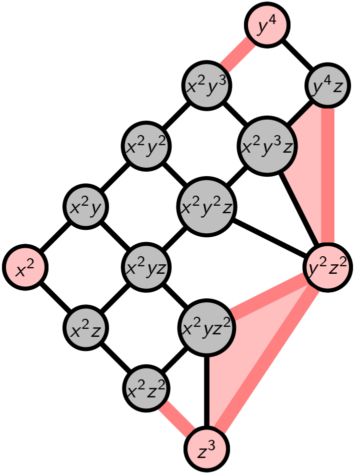
y^4
x^2y^3
y^4z
x^2y^2
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x^2y
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x^2yz
y^2z^2
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x^2yz^2
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z^3

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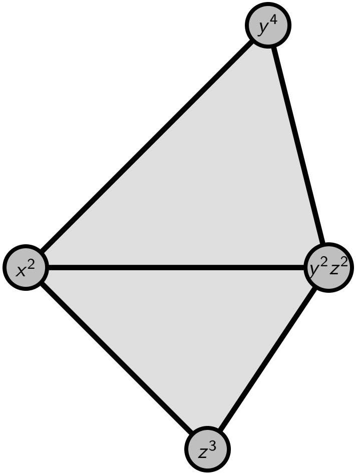

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