Effective Algorithm for Computing Quotients of Semi-algebraic Equivalence Relations

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Outline

Quotients

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- Taking quotients of topological spaces is natural and practical:
- Let $\sim$ be the equivalence relation on the unit circle S^n generated by $x \sim -x$ for all $x \in S^n$.
- $S^n / \sim \cong \mathrm{RP}^\mathrm{n}$

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- Let $\sim$ be the equivalence relation on the unit circle S^n generated by $x \sim -x$ for all $x \in S^n$.
- $S^n/\sim \cong \mathrm{RP^n}$
- Given a function $f : X \to Y$, let $\sim$ be the equivalence relation on X where $x \sim x'$ if and only if $f(x) = f(x')$ and x and x' are in the same connected component of $f^{-1}(f(x))$.
- $X/\sim = \mathrm{Reeb}(f)$, the Reeb Space of X

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- We can restrict to topological spaces that are "tame" and equivalences relations that are proper in the hopes of preserving more properties under quotienting.

Theorem (van den Dries, "Tame Topology and O-minimal Structures" (1998))

Let a $E \subset X \times X$ be a definably proper equivalence relation on the definable set X. Then X/E exists as a definably proper quotient of X.

$$\begin{tikzcd} X \arrow[d, "\varphi"] \arrow[dr, "f"] & \\ X/E \arrow[r, leftrightarrow, "h"] & Y \end{tikzcd}$$

Examples

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- Basu, C., and Percival showed in "On the Reeb Spaces of Definable Maps" (2020) that if X is a closed and bounded definable set, then the Reeb Space equivalence relation is definably proper. Hence definability is preserved when passing to the Reeb Space.

- van den Dries's proof is an abstract existence proof, so it doesn't provide a means of computing quotient spaces.
- The ubiquity of quotient spaces leads naturally to the problem to design a general quotient algorithm.
- We can restrict to the semi-algebraic category to make effective the steps of van den Dries's proof.

Outline

Algorithmic Problem & Complexity

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- INPUT$(\mathcal{P}_1, \mathcal{P}_2, \Phi_X, \Phi_E)$
- OUTPUT:$(\mathcal{Q}_1, \mathcal{Q}_2, \Phi_f, \Phi_Y)$
- Complexity: The number of arithmetic operations performed by the algorithm on the inputs to obtain the outputs. Measured in terms of the number of input polynomials, their degree bound, and the number of variables.

Algorithmic Tools

Techniques in algorithmic semi-algebraic geometry:

1. Quantifier Elimination
2. Semi-Algebraic Triangulation
3. Computing Semi-Algebraic description of connected components
4. Compute the dimension of a semi-algebraic set
   See “Algorithms in Real Algebraic Geometry” by Basu, Pollack, and Roy (2006) for a more comprehensive list of algorithms used in real algebraic geometry.

Algorithmic Tools

Theorem (Quantifier Elimination - "Algorithms in Real Algebraic Geometry")

Let $\mathcal{P}$ be a set of at most k polynomials each of degree at most d in $m+n$ variables with coefficients in R. Let Π denote a partition of the list of variables $(X_1, \ldots, X_m)$ into blocks, $X_{[1]}, \ldots, X_{[\omega]}$, where block $X_{[i]}$ has size m_i for $1 \leq i \leq \omega$. Given $\Phi(Y)$, a $(\mathcal{P}, \Pi)$-formula, there exists an equivalent quantifier free formula, $\Psi(Y)$. Moreover, there is an algorithm to compute $\Psi(Y)$ with complexity

$$k^{(m_\omega+1)\cdots(m_1+1)(n+1)} d^{\mathcal{O}(m_\omega)\ldots\mathcal{O}(m_1)\mathcal{O}(n)}.$$

- Complexity: Doubly exponential in the number of quantifier alternations. Singly exponential if this number is fixed.

Algorithmic Tools

Theorem (Semi-Algebraic Triangulation - "Algorithms in Real Algebraic Geometry")

Let $S \subset \mathrm{R}^m$ be a closed and bounded semi-algebraic set, and let $S_1, \ldots, S_q$ be semi-algebraic subset of S. There exists a simplicial complex K in R^m with rational coordinates and a semi-algebraic homeomorphism $h : |K| \to S$ such that each S_j is the union of images by h of open simplices of K. Moreover, let $\mathcal{P}$ be a set of at most k polynomials each of degree at most d in m variables with coefficients in R. If S and each S_i are $\mathcal{P}$-semi-algebraic sets, then the semi-algebraic triangulation (K, h) can be computed in time

$$(kd)^{2^{\mathcal{O}(m)}}.$$

- Complexity: Doubly Exponential

Theorem (C.)

Let $\mathcal{P}_1 \subset \mathrm{R}[X_1, \ldots, X_m]$ containing k_1 polynomials of degree bounded by d_1 and $\mathcal{P}_2 \subset \mathrm{R}[X_1, \ldots, X_{2m}]$ containing k_2 polynomials of degree bounded by d_2. Given a $\mathcal{P}_1$-formula Φ_X and a $\mathcal{P}_2$-formula Φ_E, there exists an algorithm which returns formulas Φ_Y and Φ_f whose realizations are semi-algebraic sets Y, homeomorphic to X/E, and $\Gamma(f)$, the graph of a semi-algebraic function $f : X \to Y$. Moreover, an upper bound on the complexity of the algorithm is

$$\begin{cases} (mkd)^{2^{\mathcal{O}(m^3)}} & \text{if } k_1^{2^{\mathcal{O}(m)}} \approx k_2 (\approx k) \text{ and } d_1^{2^{\mathcal{O}(m)}} \approx d_2 (\approx d) \\ (mk_2d_2)^{2^{\mathcal{O}(m^3)}} & \text{if } k_2 >> k_1^{2^{\mathcal{O}(m^3)}} \text{ and } d_2 >> d_1^{2^{\mathcal{O}(m)}} \\ (mk_1d_1)^{2^{\mathcal{O}(m^4)}} & \text{if } k_1^{2^{\mathcal{O}(m)}} >> k_2 \text{ and } d_1^{2^{\mathcal{O}(m)}} >> d_2 \end{cases}$$

Complexity: Doubly exponential

Overview of Algorithm

Our algorithm consists of making effective the steps in van den Dries's proof. We use the effective tools of quantifier elimination and semi-algebraic triangulation.

1. First we determine the dimension, D, of X, represented by Φ_X
2. If $D > 0$, obtain a subset X' of X of lower dimension (Triangulation, Elimination)
3. Call the main algorithm again, but with $\Phi_{X'}$ as the input. Repeat until $D = 0$.
4. When $D = 0$, return a space Y' that contains exactly the least element of each equivalence class, and a map, f' that sends the elements of X to the lowest element equivalent to it (Quantifier Elimination)

Overview of Algorithm

5. To obtain the map and space from spaces $0 < \delta \leq D$, we set $Y = Y' \amalg_{f''} c(S_\delta)$, where $c(S_\delta)$ is the closure of the union of all δ-dimensional sets of the triangulation of X and f'' is f' restricted to $c(S_\delta) \cap X'$. (Quantifier Elimination)
6. We can then obtain a map $p : c(S_\delta) \amalg Y' \to Y$. (Triangulation, Elimination)
7. There is a commutative diagram:

$$\begin{tikzcd} c(S_\delta) \amalg X' \arrow[r, "j"] \arrow[d, "h"] & c(S_\delta) \amalg Y' \arrow[d, "p"] \\ c(S) \arrow[r, "g"] & Y \end{tikzcd}$$

We set $f = g \circ \sigma : X \to S \to Y$, where $\sigma : X \to S$ is the map that sends each element of X to the lowest element in X equivalent to it. (Quantifier Elimination)

Outline

Lower Complexity?

- Let $f : X \to f(X)$. Gabrielov, Vorobjov, and Zell, "Betti Numbers of Semialgebraic and Sub-Pfaffian Sets" (2004) proved that under certain assumptions on f,

$$b_k(f(X)) \leq \sum_{p+q=k} b_p(\underbrace{X \times_{f(X)} \cdots \times_{f(X)} X}_{q \text{ times}})$$

- For our purposes, we let $f : X \to X/E$ be the quotient map on X for an equivalence relation E on X.

Lower Complexity?

- If $X \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^{2n}$ are defined by $\mathcal{P}$- and $\mathcal{Q}$-formulae $\phi(\cdot)$ and $\psi(\cdot)$, respectively, $\underbrace{X \times_{X/E} \cdots \times_{X/E} X}_{p+1 \text{ times}}$ can be described by the formula

$$\bigwedge_{0 \leq i < p} \psi(x^{(i)}, x^{(i+1)}) \wedge \bigwedge_{i=0}^{p} \phi(x^{(i)})$$

- Using this we obtain an upper bound for the betti numbers of the fiber product, and hence of X/E as

$$b(X/E) \leq \mathcal{O}(nsd)^{(n+1)n}.$$

Conclusion and Future Work

- There is a “meta-theorem” in algorithmic semi-algebraic geometry that relates worst case algorithmic complexity with upper bounds on topological complexity.
- Future work in this area would be to design an algorithm that computes the quotient space in singly exponential time.

Thank you!