# Local effectivity in projective spaces

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Local positivity is defined by means of Seshadri constants introduced by Demailly around 1990.

## Definition

Let $X$ be a smooth projective variety and let $L$ be an ample line bundle on $X$. Let $P \in X$ be a fixed point and let $f : \mathrm{Bl}_P X \to X$ be the blow up of $X$ at $P$ with the exceptional divisor $E$. The real number

$$\varepsilon(X; L, P) = \sup\{t \in \mathbb{R} : \ f^*L - tE \ \text{ is nef}\}$$

is the *Seshadri constant* of $L$ at $P$.

The term local effectivity is coined by us in analogy to the local positivity.

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$$\mu(X; L, P) = \sup \{ t \in \mathbb{R} : \; f^* L - tE \; \text{ is effective} \}$$

is the *$\mu$-invariant* of $L$ at $P$.

### Remark

*The reciprocal of* $\mu(X; L, P)$ *is known as the Waldschmidt constant and was studied in complex analysis before it attracted attention in algebraic geometry.*

Let $R = \mathbb{C}[x_0, \ldots, x_N]$ be the graded ring of complex polynomials.

**Definition**

The *initial degree* of a homogeneous ideal $I \subset R$ is

$$\alpha(\mathbb{P}^N; I) = \min\{d : (I)_d \neq 0\},$$

where $(I)_d$ denotes the degree $d$ part of $I$.

We are interested in finite sets $Z$ of points in $\mathbb{P}^N$. The saturated homogeneous ideal $I(Z) \subset R$ of $Z$ is defined by

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More generally, for a positive integer $m > 0$ we consider the ideal $I(mZ)$ of all polynomials vanishing to order at least $m$ in all points of $Z$:

$$\begin{aligned} I(mZ) = \{ & f \in R : \ Df(P) = 0 \text{ for all } P \in Z \\ & \text{and all differential operators } D \text{ of order } \leq m\}. \end{aligned}$$

# Waldschmidt constant

For a finite set of points $Z$ we consider the sequence of initial degrees:

$$\alpha(I(Z)),\ \alpha(I(2Z)),\ \alpha(I(3Z)), \ldots, \alpha(I(mZ)), \ldots$$

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*Clearly there is*

$$\alpha(I(mZ)) \leq m \cdot \alpha(I(Z))$$

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## Definition

The *Waldschmidt constant* of $I(Z)$ is

$$\widehat{\alpha}(I(Z)) = \inf_{m \geq 1} \frac{\alpha(I(mZ))}{m}.$$

### Chudnovsky Conjecture $\sim$ 1980

For an **arbitrary** finite set of points $Z \subseteq \mathbb{P}^N$, there is

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Moreover, for a set $Z$ of $r$ **very general** points in $\mathbb{P}^N$, there is

$$\widehat{\alpha}(I(Z)) = \sqrt[N]{r}$$

for $r$ sufficiently big.

**Demailly Conjecture $\sim$ 1982**

For an **arbitrary** finite set of points $Z \subseteq \mathbb{P}^N$, there is

$$\widehat{\alpha}(I(Z)) \geq \frac{\alpha\left(I(mZ)\right) + N - 1}{m + N - 1}.$$

**Theorem (Esnault, Viehweg 1983)**

*Both Conjectures hold for* $\mathbb{P}^2$.

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**Theorem (Dumnicki, Tutaj-Gasińska and Fouli, Mantero, Xie)**

*Chudnovsky Conjecture holds for very general sets of points in* $\mathbb{P}^N$. *Moreover, in this situation*

$$\widehat{\alpha}(I(Z)) \geq \lfloor \sqrt[N]{r} \rfloor,$$

*where* $r$ *is the number of points in* $Z$.

# Main results 1

**Theorem (Dumnicki, Tutaj-Gasińska and Fouli, Mantero, Xie)**

*Chudnovsky Conjecture holds for very general sets of points in* $\mathbb{P}^N$.

**Our result**

*Demailly's Conjecture holds for* $r \geq m^N$ *very general points in* $\mathbb{P}^N$.

**Theorem**

*For a set $Z$ of $r$ very general points*

$$\widehat{\alpha}(I(Z)) \geq \lfloor \sqrt[N]{r} \rfloor .$$

**Our result**

*Let $k$ be a positive integer and let $s$ be an integer in the range $1 \leq s \leq k$. Let $Z$ be a set of*

$$r \geq s(k+1)^{N-1} + (k+1-s)k^{N-1}$$

*very general points in $\mathbb{P}^N$. Then*

$$\widehat{\alpha}(I(Z)) \geq k + \frac{s}{k+1}.$$

## Definition

Let $H \cong \mathbb{P}^{N-1}$ be a hyperplane in $\mathbb{P}^N$ and let $Z \subseteq H$ be a subscheme in $H$. Let $D$ be a divisor of degree $d$ in $\mathbb{P}^N$. The Waldschmidt decomposition of $D$ with respect to $H$ and $Z$ is the sum of $\mathbb{R}$-divisors

$$D = D' + \lambda \cdot H$$

such that $\deg(D') = d - \lambda$,

$$\frac{d - \lambda}{\mathrm{mult}_Z D'} \geq \widehat{\alpha}(H \cong \mathbb{P}^{N-1}, Z)$$

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## Remark

*Note that $\lambda$ is the least multiplicity such that $H$ is numerically forced to be contained in $D$ with this multiplicity. It may well happen that the divisor $D'$ still contains $H$ as a component.*

## Theorem

*Let $H_1, \dots, H_s$ be $s \geq 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1, \dots, a_s \geq 1$ be real numbers such that*

$$\sum_{j=1}^{s-1} \frac{1}{a_j} < 1 \leq \sum_{j=1}^{s} \frac{1}{a_j} \text{ and } q := \left(1 - \sum_{j=1}^{s-1} \frac{1}{a_j}\right) \cdot a_s + s - 1.$$

*Let*

$$Z_i = \{P_{i,1}, \dots, P_{i,r_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$$

*be the set of $r_i$ points such that $\widehat{\alpha}(H_i; Z_i) \geq a_i$ and let $Z = \bigcup_{i=1}^{s} Z_i$. Then*

$$\widehat{\alpha}(\mathbb{P}^N; Z) \geq q.$$

## Notation

Let $\widehat{\alpha}(\mathbb{P}^N; r)$ denote the Waldschmidt constant of $r$ very general points in $\mathbb{P}^N$.

## Theorem

*Let $N \geq 2$, let $k \geq 1$ be an integer. Assume that for some integers $r_1, \ldots, r_{k+1}$ and rational numbers $a_1, \ldots, a_{k+1}$ we have*

$$\widehat{\alpha}(\mathbb{P}^{N-1}; r_j) \geq a_j \text{ for } j = 1, \ldots, k+1,$$

$$k \leq a_j \leq k+1 \text{ for } j = 1, \ldots, k, \quad a_1 > k, \quad a_{k+1} \leq k+1.$$

*Then*

$$\widehat{\alpha}(\mathbb{P}^N; r_1 + \ldots + r_{k+1}) \geq \left(1 - \sum_{j=1}^{k} \frac{1}{a_j}\right) a_{k+1} + k.$$

**Example**

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get:

$$r_1 = r_2 = 8, \; r_3 = 4$$

$$a_1 = a_2 = 48/17, \; a_3 = 2$$

$$\widehat{\alpha}(\mathbb{P}^3, 20) \geq 31/12$$

Figure: Upper and lower bounds for $\widehat{\alpha}(\mathbb{P}^3; r)$