# Quantum Magic Squares

## (joint with G. de las Cuevas, T. Netzer)

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# Classical Magic Squares: Definition

## Definition

*A (classical) magic square of size $n$ is a* **real** *$n \times n$ matrix $A = (a_{i,j})_{i,j}$ such that*

- *all entries of $A$ are* **nonnegative**, *i.e. $a_{i,j} \geq 0$ for all $i, j$*
- *the entries in each row and in each column of $A$* **sum to 1**, *i.e.*

$$\sum_{k=1}^{n} a_{i,k} = \sum_{k=1}^{n} a_{k,j} = 1$$

*for all $i, j$*

*Notation:*

$$\mathcal{M}_1^{(n)} := \text{set of all (classical) magic squares of size } n$$

# Classical Magic Squares: Examples and Properties

Example: **permutation matrices** are magic squares

Notation:

$$\mathcal{P}_1^{(n)} := \text{set of all } n \times n \text{ permutation matrices}$$

**Proposition**

*$\mathcal{M}_1^{(n)}$ is a* **convex** *and* **compact** *subset of* $\mathrm{Mat}_n(\mathbb{R})$.

# The Birkhoff-von Neuman-Theorem

**Theorem (Birkhoff-von Neumann)**

*The* **extreme points** *of* $\mathcal{M}_1^{(n)}$ *are precisely the* $n \times n$ **permutation matrices**.

- $\mathcal{M}_1^{(n)}$ is a polytope (Birkhoff polytope)
- every magic square can be written as a convex combination of permutation matrices (Krein-Milman-Theorem)

Observation: The rows and columns of a magic square can be interpreted as a discrete **probability distribution**.

In Quantum Physics:

probability distribution $\rightarrow$ POVM (positive operator-valued measurement)

**POVM** $=$ tuple of psd matrices $(A_1, \ldots, A_n)$ with $A_1 + \cdots + A_n = I$

# Quantum Magic Squares

## Definition

*A quantum magic square over* $\mathrm{Mat}_s(\mathbb{C})$ *of size* $n$ *is a matrix* $A = (a_{i,j})_{i,j} \in \mathrm{Mat}_n(\mathrm{Mat}_s(\mathbb{C}))$ *with entries* $a_{i,j} \in \mathrm{Mat}_s(\mathbb{C})$ *such that*

- *all entries* $a_{i,j}$ *are* **hermitian** *and* **psd**
- *the entries in each row and in each column of* $A$ **sum to the identity matrix**, *i.e.*

$$\sum_{k=1}^{n} a_{i,k} = \sum_{k=1}^{n} a_{k,j} = I_s$$

*for all* $i, j$

*Notation:*

$$\mathcal{M}_s^{(n)} := \text{set of all quantum magic squares of size } n \text{ with entries in } \mathrm{Mat}_s(\mathbb{C})$$

# Quantum Permutation Matrices

Observation:

$$\begin{aligned} P \text{ permutation matrix} &\Leftrightarrow P \text{ magic square and all entries are 0 or 1} \\ &\Leftrightarrow P \text{ magic square and all entries are idempotent} \end{aligned}$$

**Definition**

*A quantum permutation matrix of size $n$ with entries in $\mathrm{Mat}_s(\mathbb{C})$ is a quantum magic square $P = (p_{i,j})_{i,j}$ such that all entries $p_{i,j}$ are* **projectors**.

*Notation:*

$$\mathcal{P}_s^{(n)} := \text{set of all quantum permutation matrices of size } n \text{ with entries in } \mathrm{Mat}_s(\mathbb{C})$$

# The Quantum Jump: Convexity

- Each of the sets $\mathcal{M}_s^{(n)}$ is obviously convex in the classical sense.
- The entire family of sets $\mathcal{M}^{(n)} := (\mathcal{M}_s^{(n)})_s$ has an even stronger property:

  For $A_k = (a_{i,j}^{(k)})_{i,j} \in \mathcal{M}_{s_k}^{(n)}$ and $v_k \in \mathrm{Mat}_{s_k,t}(\mathbb{C})$ with $\sum_k v_k^* v_k = I_t$ let

$$B := \left( \sum_k v_k^* a_{i,j}^{(k)} v_k \right)_{i,j} .$$

  Then $B \in \mathcal{M}_t^{(n)}$.

**Definition**

*A family of sets with the above property is called* **matrix-convex**. *$B$ is also called a* **compression** *of the $A_k$.*
*For a family $\mathcal{C} = (\mathcal{C}_s)_s$ the* **matrix-convex hull** *of $\mathcal{C}$, denoted*

$$\mathrm{mconv}(\mathcal{C}),$$

*is the smallest matrix-convex family containing $\mathcal{C}$.*

# Arveson Extreme Points

Example: For $A = (a_{i,j})_{i,j} \in \mathcal{M}_s^{(n)}$ and $B = (b_{i,j})_{i,j} \in \mathcal{M}_t^{(n)}$ and a unitary matrix $u \in U(s+t)$ let

$$C := \left( u^* \begin{pmatrix} a_{i,j} & 0 \\ 0 & b_{i,j} \end{pmatrix} u \right)_{i,j} \in \mathcal{M}_{s+t}^{(n)}.$$

Then $A$ is a **trivial compression** of $C$ (choose $v = u^* \begin{pmatrix} I \\ 0 \end{pmatrix}$).

## Definition

*An* ***Arveson extreme point*** *of $\mathcal{M}^{(n)}$ is a point that can only be obtained by trivial compressions from $\mathcal{M}^{(n)}$.*

## Corollary

*$\mathcal{M}^{(n)}$ is the matrix-convex hull of its Arveson extreme points.*

Proof: Follows from a more general result by Evert-Helton (arXiv:1806.09053). □

# Quantum Birkhoff-von Neumann?

**Theorem 1 (de las Cuevas, Drescher, Netzer)**

*For every $n$ every quantum permutation matrix of size $n$ is an Arveson extreme point of $\mathcal{M}^{(n)}$.*

$\rightarrow$ One direction of the Birkhoff-von Neumann-Theorem can be generalized to the quantum setup ...

**Theorem 2 (de las Cuevas, Drescher, Netzer)**

*For $n \geq 3$ there is an Arveson extreme point of $\mathcal{M}^{(n)}$ in $\mathcal{M}_2^{(n)}$ that is* **not** *a quantum permutation matrix.*

$\rightarrow$ ... but the other cannot

# Some Remarks on the Proof of Theorem 2

- the general idea is to show that $\mathrm{mconv}(\mathcal{P}^{(n)}) \subsetneq \mathcal{M}^{(n)}$
- for $n = 3$ one can give an explicit example for a quantum magic square $A \in \mathcal{M}_2^{(3)}$ that is not in the matrix-convex hull of the quantum permutation matrices
- for $n > 3$ this **cannot** be easily extended with a simple embedding argument of the form

$$A \to \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

- the condition $A = (a_{i,j})_{i,j} \in \mathrm{mconv}(\mathcal{P}^{(n)})$ can be relaxed with a semidefinite program (SDP), i.e. a problem of the form

$$\exists x_1, \dots, x_m : \qquad B \otimes I + \sum_{i,j} C_{i,j} \otimes a_{i,j} + \sum_k D_k \otimes x_k \qquad \text{is psd}$$

  with $B, C_{i,j}, D_k$ given
- this relaxation behaves well under embeddings

# Further Questions

Question: What is the matrix-convex hull of $\mathcal{P}^{(n)}$?

- SDP relaxation from above can be further tightened but does not seem to terminate in general
- question for good description still open

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- SDP relaxation from above can be further tightened but does not seem to terminate in general
- question for good description still open
- we considered the matrix-convex hull of the following subset instead

## Definition

*Let*

$$\mathcal{CP}_s^{(n)} := \{P = (p_{i,j})_{i,j} \in \mathcal{P}_s^{(n)} \mid \textit{all } p_{i,j} \textit{ commute}\}$$

*be the set of all quantum permutation matrices with commuting entries.*

# The Semiclassical Case

## Theorem 3 (de las Cuevas, Drescher, Netzer)

*Let $A \in \mathcal{M}_s^{(n)}$. Then the following are equivalent:*

1. $A \in \mathrm{mconv}(\mathcal{CP}^{(n)})$
2. *There are psd matrices $q_\pi$ for all permutations $\pi \in S_n$ such that $\sum_\pi q_\pi = I_s$ and*

$$A = \sum_\pi P_\pi \otimes q_\pi,$$

*where $P_\pi$ denotes the permutation matrix corresponding to $\pi$.*

- we call matrices that satisfy the equivalent conditions of Theorem 3 **semiclassical**
- both conditions are sort of a generalization of the Birkhoff-von Neumann-Theorem
- the second condition can be checked with and SDP
- the set of semiclassical matrices is a neighborhood of the matrix $(\frac{1}{n} I_s)_{i,j}$ (in the subspace topology of $\mathcal{M}_s^{(n)}$)

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# Conclusions

- give papers fancy titles and you will receive a lot of media attention
- the Birkhoff-von Neumann-Theorem can be partially extended to the quantum setup
- Open problem: How can we efficiently check membership in $\mathrm{mconv}(\mathcal{P}^{(n)})$ ?
- Open problem: What are the Arveson extreme points of $\mathcal{M}^{(n)}$ ?

Thank You!