Quantum Magic Squares
(joint with G. de las Cuevas, T. Netzer)

Tom Drescher

University of Innsbruck
Classical Magic Squares: Definition

**Definition**

A (classical) magic square of size $n$ is a **real** $n \times n$ matrix $A = (a_{i,j})_{i,j}$ such that

- all entries of $A$ are **nonnegative**, i.e. $a_{i,j} \geq 0$ for all $i,j$
- the entries in each row and in each column of $A$ sum to 1, i.e.

$$
\sum_{k=1}^{n} a_{i,k} = \sum_{k=1}^{n} a_{k,j} = 1
$$

for all $i,j$

**Notation:**

$$
\mathcal{M}_{1}^{(n)} := \text{set of all (classical) magic squares of size } n
$$
Example: permutation matrices are magic squares

Notation:

\[ \mathcal{P}_1^{(n)} := \text{set of all } n \times n \text{ permutation matrices} \]

Proposition

\[ M_1^{(n)} \text{ is a convex and compact subset of } \text{Mat}_n(\mathbb{R}). \]
The extreme points of $\mathcal{M}_1^{(n)}$ are precisely the $n \times n$ permutation matrices.

- $\mathcal{M}_1^{(n)}$ is a polytope (Birkhoff polytope)
- every magic square can be written as a convex combination of permutation matrices (Krein-Milman-Theorem)
Observation: The rows and columns of a magic square can be interpreted as a discrete probability distribution.

In Quantum Physics:

probability distribution $\rightarrow$ POVM (positive operator-valued measurement)

$$\text{POVM} = \text{tuple of psd matrices } (A_1, \ldots, A_n) \text{ with } A_1 + \cdots + A_n = I$$
A quantum magic square over $\text{Mat}_s(\mathbb{C})$ of size $n$ is a matrix $A = (a_{i,j})_{i,j} \in \text{Mat}_n(\text{Mat}_s(\mathbb{C}))$ with entries $a_{i,j} \in \text{Mat}_s(\mathbb{C})$ such that

- all entries $a_{i,j}$ are hermitian and psd
- the entries in each row and in each column of $A$ sum to the identity matrix, i.e.

$$\sum_{k=1}^{n} a_{i,k} = \sum_{k=1}^{n} a_{k,j} = I_s$$

for all $i,j$.

Notation:

$$\mathcal{M}_s^{(n)} := \text{set of all quantum magic squares of size } n \text{ with entries in } \text{Mat}_s(\mathbb{C})$$
Quantum Permutation Matrices

Observation:

\[ P \text{ permutation matrix} \iff P \text{ magic square and all entries are 0 or 1} \iff P \text{ magic square and all entries are idempotent} \]

Definition

A quantum permutation matrix of size \( n \) with entries in \( \text{Mat}_s(\mathbb{C}) \) is a quantum magic square \( P = (p_{i,j})_{i,j} \) such that all entries \( p_{i,j} \) are projectors.

Notation:

\[ \mathcal{P}_s^{(n)} := \text{set of all quantum permutation matrices of size } n \text{ with entries in } \text{Mat}_s(\mathbb{C}) \]
The Quantum Jump: Convexity

- Each of the sets $\mathcal{M}_s^{(n)}$ is obviously convex in the classical sense.
- The entire family of sets $\mathcal{M}^{(n)} := (\mathcal{M}_s^{(n)})_s$ has an even stronger property:
  
  For $A_k = (a_{i,j}^{(k)})_{i,j} \in \mathcal{M}_{s_k}^{(n)}$ and $v_k \in \text{Mat}_{s_k,t}(\mathbb{C})$ with $\sum_k v_k^* v_k = I_t$ let
  
  $$B := \left( \sum_k v_k^* a_{i,j}^{(k)} v_k \right)_{i,j}.$$

  Then $B \in \mathcal{M}_t^{(n)}$.

**Definition**

A family of sets with the above property is called matrix-convex. $B$ is also called a compression of the $A_k$.

For a family $C = (C_s)_s$ the matrix-convex hull of $C$, denoted

$$\text{mconv}(C),$$

is the smallest matrix-convex family containing $C$. 
Arveson Extreme Points

Example: For $A = (a_{i,j})_{i,j} \in \mathcal{M}_s^{(n)}$ and $B = (b_{i,j})_{i,j} \in \mathcal{M}_t^{(n)}$ and a unitary matrix $u \in U(s + t)$ let

$$C := \left( u^* \begin{pmatrix} a_{i,j} & 0 \\ 0 & b_{i,j} \end{pmatrix} u \right)_{i,j} \in \mathcal{M}_{s+t}^{(n)}.$$ 

Then $A$ is a trivial compression of $C$ (choose $\nu = u^* \begin{pmatrix} I \\ 0 \end{pmatrix}$).

Definition

An Arveson extreme point of $\mathcal{M}^{(n)}$ is a point that can only be obtained by trivial compressions from $\mathcal{M}^{(n)}$.

Corollary

$\mathcal{M}^{(n)}$ is the matrix-convex hull of its Arveson extreme points.

Theorem 1 (de las Cuevas, Drescher, Netzer)

For every $n$ every quantum permutation matrix of size $n$ is an Arveson extreme point of $\mathcal{M}^{(n)}$.

→ One direction of the Birkhoff-von Neumann-Theorem can be generalized to the quantum setup ...

Theorem 2 (de las Cuevas, Drescher, Netzer)

For $n \geq 3$ there is an Arveson extreme point of $\mathcal{M}^{(n)}$ in $\mathcal{M}_2^{(n)}$ that is not a quantum permutation matrix.

→ ... but the other cannot
Some Remarks on the Proof of Theorem 2

- the general idea is to show that \( \text{mconv}(\mathcal{P}(n)) \subsetneq \mathcal{M}(n) \)
- for \( n = 3 \) one can give an explicit example for a quantum magic square \( A \in \mathcal{M}_2(3) \) that is not in the matrix-convex hull of the quantum permutation matrices
- for \( n > 3 \) this cannot be easily extended with a simple embedding argument of the form
  \[ A \rightarrow \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \]
- the condition \( A = (a_{i,j})_{i,j} \in \text{mconv}(\mathcal{P}(n)) \) can be relaxed with a semidefinite program (SDP), i.e. a problem of the form
  \[ \exists x_1, \ldots, x_m : \quad B \otimes I + \sum_{i,j} C_{i,j} \otimes a_{i,j} + \sum_k D_k \otimes x_k \quad \text{is psd} \]
  with \( B, C_{i,j}, D_k \) given
  - this relaxation behaves well under embeddings
Question: What is the matrix-convex hull of $\mathcal{P}^n$?

- SDP relaxation from above can be further tightened but does not seem to terminate in general.
- Question for good description still open.
Further Questions

**Question:** What is the matrix-convex hull of $\mathcal{P}(n)$?

- SDP relaxation from above can be further tightened but does not seem to terminate in general.
- Question for good description still open.
- We considered the matrix-convex hull of the following subset instead.

**Definition**

Let

$$\mathcal{C}\mathcal{P}_s^{(n)} := \{ P = (p_{i,j})_{i,j} \in \mathcal{P}_s^{(n)} \mid \text{all } p_{i,j} \text{ commute} \}$$

be the set of all quantum permutation matrices with commuting entries.
The Semiclassical Case

Theorem 3 (de las Cuevas, Drescher, Netzer)

Let $A \in \mathcal{M}_s^{(n)}$. Then the following are equivalent:

1. $A \in \text{mconv}(\mathcal{CP}^{(n)})$
2. There are psd matrices $q_\pi$ for all permutations $\pi \in S_n$ such that $\sum_\pi q_\pi = I_s$ and

$$A = \sum_\pi P_\pi \otimes q_\pi,$$

where $P_\pi$ denotes the permutation matrix corresponding to $\pi$.

- we call matrices that satisfy the equivalent conditions of Theorem 3 \textbf{semiclassical}
- both conditions are sort of a generalization of the Birkhoff-von Neumann-Theorem
- the second condition can be checked with and SDP
- the set of semiclassical matrices is a neighborhood of the matrix $(\frac{1}{n}I_s)_{i,j}$ (in the subspace topology of $\mathcal{M}_s^{(n)}$)
Conclusions

- Giving papers fancy titles will attract media attention.
- The Birkhoff-von Neumann-Theorem can be partially extended to the quantum setup.

Open problems:
- How can we efficiently check membership in \( mconv(P(n)) \)?
- What are the Arveson extreme points of \( M(n) \)?
Conclusions

- give papers fancy titles and you will receive a lot of media attention
Conclusions

- give papers fancy titles and you will receive a lot of media attention
- the Birkhoff-von Neumann-Theorem can be partially extended to the quantum setup
Conclusions

- give papers fancy titles and you will receive a lot of media attention
- the Birkhoff-von Neumann-Theorem can be partially extended to the quantum setup
- Open problem: How can we efficiently check membership in $m\text{conv}(\mathcal{P}^n)$?
Conclusions

- give papers fancy titles and you will receive a lot of media attention
- the Birkhoff-von Neumann-Theorem can be partially extended to the quantum setup
- Open problem: How can we efficiently check membership in $\text{mconv}(\mathcal{P}^{(n)})$ ?
- Open problem: What are the Arveson extreme points of $\mathcal{M}^{(n)}$ ?
Thank You!