The multidimensional truncated Moment Problem: Carathéodory Numbers from Hilbert Functions

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$\mathcal{A} \subset \mathbb{R}[x_1,\dots,x_n]$ finite-dimensional,[1] $n \in \mathbb{N}$. Truncated moment functional:

$$L : \mathcal{A} \to \mathbb{R}, \quad p \mapsto L(p) = \int_{\mathbb{R}^n} p(x)\ \mathrm{d}\mu(x)$$

- μ - representing measure (≥ 0)
- $s_\alpha := L(x^\alpha)$ - moment ($\alpha \in \mathbb{N}_0^n$)
- $s = (L(a_i))_{i=1}^{\dim \mathcal{A}}$ - moment sequence, $\mathcal{A} = \operatorname{lin}\{a_1,\dots,a_{\dim \mathcal{A}}\}$

Richter Theorem (1957). Every truncated moment functional has a k-atomic representing measure

$$\mu = \sum_{j=0}^{k} c_j \cdot \delta_{x_j}$$

with $x_1,\dots,x_k \in \mathbb{R}^n$, $c_1,\dots,c_k > 0$, and $k \leq \dim \mathcal{A}$.

[1] $\mathcal{A}$ only needs to be a finite-dimensional space of measurable functions.

Richter: s has a k-atomic representing measure with $k \leq \dim \mathcal{A} < \infty$. Carathéodory number:

$$\mathcal{C}(L) := \min k \qquad \text{and} \qquad \mathcal{C} := \max_{\substack{L:\mathcal{A}\to\mathbb{R} \text{ moment} \\ \text{funtional}}} \mathcal{C}(L) \quad \leq \dim \mathcal{A}$$

$B = \{b_1, \dots, b_D\}$ basis of $\mathcal{A}$, $D = \dim \mathcal{A}$:

$$s_{\mathcal{A}}(x) = (b_i(x))_{i=1}^D = \int_{\mathbb{R}^n} (b_i(y))_{i=1}^D \, \mathrm{d}\delta_x(y)$$

Lemma. $a \in \mathcal{A}$ with $a \geq 0$ and finitely many zeros $x_1, \dots, x_k$. L with $\mu = \sum_{i=1}^k \delta_{x_i}$:

$$\mathcal{C}(L) = \dim \operatorname{lin} \{s_{\mathcal{A}}(x_1), \dots, s_{\mathcal{A}}(x_k)\}. \qquad \text{(Carathéodory)}$$

How to calculate $\dim \operatorname{lin} \{s_{\mathcal{A}}(x_1), \dots, s_{\mathcal{A}}(x_k)\}$ e.g. for some a or L?

$I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ ideal.

- $I^h \subseteq \mathbb{R}[x_0, \ldots, x_n]$ homogenization of I
- $R = \mathbb{R}[x_0, \ldots, x_n]/I^h$ graded ring, R_d degree d part
- dehomogenization of R_d is isomorphism

$$R_d = (\mathbb{R}[x_0, \ldots, x_n]/I^h)_d \quad \overset{\cong}{\to} \quad (\mathbb{R}[x_1, \ldots, x_n]/I)_{\leq d}$$

- $\dim(\mathbb{R}[x_1, \ldots, x_n]/I)_{\leq d} = \dim R_d$
- Hilbert function $HF_R(d) := \dim R_d$

Example $(R = \mathbb{R}[x_0, \ldots, x_n])$.

$$HF_R(d) \;=\; \dim \mathbb{R}[x_0, \ldots, x_n]_d \;=\; \dim \mathbb{R}[x_1, \ldots, x_n]_{\leq d} \;=\; \binom{n+d}{d}$$

Lemma. $I \subseteq \mathbb{R}[x_0, \dots, x_n]$ homogeneous ideal, $R := \mathbb{R}[x_0, \dots, x_n]/I$, Hilbert function HF_R. $f_1, \dots, f_r$ regular sequence[2] and homogeneous of degree d. Then

$$HF_{R/(f_1,\dots,f_r)}(j) = \sum_{i=0}^{r} (-1)^i \cdot \binom{r}{i} \cdot HF_R(j - i \cdot d).$$

Proof. Induction on r. $R^i := R/(f_1, \dots, f_i)$, $i = 0, \dots, r$.

$$0 \to R^{r-1}_{j-d} \xrightarrow{\cdot f_r} R^{r-1}_j \xrightarrow{\mod f_r} R^r_j \to 0 \qquad \text{(exact sequence for all } j \in \mathbb{Z})$$

$$\Rightarrow \quad HF_{R^r}(j) = HF_{R^{r-1}}(j) - HF_{R^{r-1}}(j - d)$$ □

Application. $J = (f_1, \dots, f_r)$ vanishing ideal of finitely many points $\Gamma \subset \mathbb{R}^n$

[2] $f_1, \dots, f_r \in A$, A commutative ring: f_i is not a zero divisor in $A/(f_1, \dots, f_{i-1})$ for all $i = 1, \dots, r$

$\Gamma = \{1, 2, \ldots, d\}^n \subset \mathbb{R}^n$ grid

- $i = 1, \ldots, n$: $p_i(x) := (x_i - x_0) \cdot \ldots \cdot (x_i - d \cdot x_0)$
- $p_1, \ldots, p_n$ is a regular sequence
- $I^h = (p_1, \ldots, p_n)$ is radical[3]
- $R = \mathbb{R}[x_0, \ldots, x_n]$, $R_n := R/(p_1, \ldots, p_n)$:

$$HF_{R_n}(2d) = \binom{n+2d}{n} - n \cdot \binom{n+d}{n} + \binom{n}{2}$$

- $L : \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} \to \mathbb{R}$ with $\mu = \sum_{y \in \Gamma} \delta_y$:

$$\mathcal{C}(L) \ = \ \dim(\underbrace{\mathbb{R}[x_1, \ldots, x_n]/I}_{\mathbb{R}[\Gamma]})_{\leq 2d} \ = \ \dim(R_n)_{2d} \ = \ HF_{R_n}(2d)$$

[3] Nullstellensatz: $I^h = I(\{1\} \times \Gamma \subset \mathbb{P}^n_{\mathbb{C}})$

Asymptotics. $\Gamma = \{1, 2, \ldots, d\}^n \subset \mathbb{R}^n$. $L : \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} \to \mathbb{R}$ with $\mu = \sum_{y\in\Gamma} \delta_y$:

$$\mathcal{C}(L) = \binom{n+2d}{n} - n \cdot \binom{n+d}{n} + \binom{n}{2} \qquad \Rightarrow \qquad \frac{\mathcal{C}(L)}{\dim \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}} \xrightarrow{n\to\infty} 1$$

Flat extension (R. Curto + L. Fialkow). $s^{2d} = (s_\alpha)_{|\alpha|\leq 2d}$ moment sequence?

- Hankel matrix $H(s^{2d}) := (s_{\alpha+\beta})_{\alpha,\beta:|\alpha|,|\beta|\leq d}$
- flat extensions: $s^{2d+2}, s^{2d+4}, \ldots, s^{2D}, s^{2D+2}, \ldots$ sequences with $H \succeq 0$

$$\operatorname{rank} H(s^{2d}) < \operatorname{rank} H(s^{2d+2}) < \ldots < \operatorname{rank} H(s^{2D}) = \operatorname{rank} H(s^{2D+2}) = \ldots$$

- Curto + Fialkow: $D \leq 2d$
- Kummer + dD: worst case $D = 2d$ possible
  - $(n, d) = (9, 2),\ (7, 3),\ (6, 4)$

more:

- Carathéodory numbers with small gaps $L : \mathbb{R}[t^{d_1}, \ldots, t^{d_r}]_{\leq d} \to \mathbb{R}$
- Carathéodory numbers on algebraic varieties $L : \mathbb{R}[\mathcal{X}]_{\leq d} \to \mathbb{R}$

Gaussian mixtures[4]

- point evaluations $\delta_y \Rightarrow$ Gaussian distributions
- $\exists L : \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} \to \mathbb{R}$ which requires

$$HF_{R_n}(2d) = \binom{n+2d}{n} - n \cdot \binom{n+d}{n} + \binom{n}{2}$$

Gaussians

Thank you for your attention
and the organizers for the conference!

[4] arXiv:1907.00790