Abstract

In this paper we study the set of tensors that admit a special type of decomposi-
tion called an orthogonal tensor train decomposition. Finding equation-defining
varieties of low-rank tensors is generally a hard problem, however, the set of
orthogonally decomposable tensors is defined by appealing quadratic equations.

The tensors we consider are a extension of orthogonally decomposable tensors,
and we show that they are defined by similar quadratic equations, as well as some
interesting additional equations.

Introduction

• With the emergence of big data, it is more and more often the case
that information is recorded in the form of a tensor (or multi-
dimensional array).
• The importance of decomposing such a tensor is as follows:
  • It provides fodder information about the data at hand.
  • Having a concise decomposition of the tensor allows us to store it much more
    efficiently.
• Decomposing tensors is often computationally hard. For example, finding
  (the number of terms of) the CP-decomposition of a general tensor is NP-hard,
  and the set of tensors of CP rank at most \( r \) is not closed for any \( r \geq 2 \),
  making the low-rank approximation problem impossible in some instances. It is also
  notoriously hard to describe all the defining equations of the set of tensors of rank at most \( r \).
• On the other hand, orthogonally decomposable tensors can be
decomposed efficiently via the slice method or via the tensor power
method. Furthermore, the set of orthogonally decomposable tensors of
bounded rank is closed, making the family of such tensors as
appealing as the set of matrices.
• Tensor networks provide many other ways of decomposing tensors.
  They originate from quantum physics and are used to depict the
  structure of steady states of Hamiltonians of quantum systems.
  Many types of tensor network decompositions, like tensor trains,
  also known as matrix product states, are used in machine learning
to decompose data tensors in meaningful ways.

Tensor Networks

Consider a graph \( G = (V, E, D) \) where
• \( V \) is the set of vertices.
• \( E \) is the set of edges adjacent to \( 2 \) vertices.
• \( D \) is the set of edges adjacent to just 1 vertex (also called
dangling edges).

For instance, in the graph below,
• \( V = \{ 1, 2 \} \)
• The only number of \( E \) is the dashed edge.

\[ D \text{ consists of blue and red edges; } \]
\[ T_1 \subseteq R^{a_1 \times b_1 \times c_1}, \]
\[ T_2 \subseteq R^{a_2 \times b_2 \times c_2}. \]

The resulting tensor network state \( T \in \otimes_{i=1}^{n} R^{a_i \times b_i \times c_i} \)

is obtained along all edges in \( E \).

Tensor Trains

• A tensor train of length \( l \) is a tensor network consisting of \( l \) vertices
  arranged in a line, each of which has \( 3 \) adjacent edges.

\[ \begin{array}{c}
  1 \\
  \vdots \\
  l
\end{array} \]

\[ T_{[1:l]} \]

Figure 3: A tensor train of length \( l \)

• A tensor train of length \( l \) is said to be symmetrically orthogonally
decomposable (symmetrically odeco) if all the order-4 tensors
generating the tensor train are symmetrically odeco.

\[ \text{SOT}_{4}^{l} \]

represents the set of symmetrically odeco tensor trains of
length \( l \) which are generated by \( n \times n \times n \) tensors.

\[ \begin{array}{c}
  1 \\
  \vdots \\
  l
\end{array} \]

\[ T_{[1:l]} \]

Figure 4: Symmetrically odeco tensor train of length \( l \)

\[ (\text{A tensor is SOT}_{4}^{l}) \]

It has been shown in [1] that if a tensor train of length \( l \) has
a symmetrically odeco decomposition, one can find the decomposition
efficiently.

Contraction of Two Tensors

Let \( T \in R^{a_1 \times \cdots \times a_k} \) and \( S \in R^{a_{k+1} \times \cdots \times a_l} \) be two tensors such that \( a_i = a_l \). The contraction of the tensors \( T \) and \( S \) along their \( k \)th
and \( l \)th modes is the tensor \( R \in R^{a_1 \times \cdots \times a_{k-1} \times a_{k+1} \times \cdots R_{k+1} \times \cdots \times a_{l-1} \times a_l} \) whose entries are

\[ R_{i_1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{l-1}, i_l} = \sum_{i_k=1}^{a_k} T_{i_1, i_2, \ldots, i_{k-1}, i_k, \ldots, i_{l-1}, i_l} S_{i_k, i_{k+1}, \ldots, i_l}. \]

The symmetries that the equations in \( Q_{b} \) depict for \( P, P_{b}, P_{\text{Q}} \) and
\( P_{\text{Q}} \) are very similar to those that have been proved for
symmetrically orthogonally decomposable and ordinary
orthogonally decomposable tensors in [2].

Results

• Let \( n \geq 2 \) and define

\[ b_n = \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{j=1}^{\min(n, j)} \sum_{h=1}^{\min(n, l)} \text{sgn}(h) \text{sgn}(j) P_{\text{Q}b}(h,j,k,l) \]

Then \( b_n \) vanishes on \( SOT_{4}^{n-1} \).

• If \( n \) is even, then in fact \( b_n \) vanishes on the set of all tensor trains of
  length \( 2n \) in which the two tensors at the vertices are symmetric
  and have rank at most \( n \).

When \( n \leq 5 \), we are able to computationally show that \( b_n \) does not
lie in the ideal generated by \( P_{\text{Q}} \).

For all \( n \geq 2 \), the dimension of the Zariski closure of \( SOT_{4}^{n-1} \)
equals \( n(n+1) \). This follows from counting the dimensions of the parametrization of \( SOT_{4}^{n-1} \) and from noting that the decomposition of a tensor in \( SOT_{4}^{n-1} \) can be found uniquely in a sense according to [1].

When \( n = 2 \), the dimension of the ideal cut out by \( P_{\text{Q}} \) and the
polynomial \( b_2 \) equals \( 5 + n(n+1) - 1 \), which was checked numerically using Macaulay2 and Maple.

Conjecture

We conjecture that the Zariski closure of the set \( SOT_{4}^{n-1} \) is cut out by
the vanishing of \( P_{\text{Q}} \) and \( P_{\text{Q}} \); and the polynomial \( b_n \) for \( n \geq 2 \), and that
the ideal defined by these equations is prime.

References


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You can read our paper here.