Poincaré Polynomial Identities for Special Schubert Varieties

Francesca Cioffi, Davide Franco, Carmine Sessa

Università degli studi di Napoli, Federico II

Special Schubert varieties

Fix integers $0 < i < k \leq j < l$, $r = k - i < l - j = c$ and fix an element $F \in \mathbb{G}_j(\mathbb{C}^l)$ of the Grassmannian. Let

$$\mathcal{S} = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) \geq i\}$$

be a special Schubert variety and consider the Whitney stratification

$$\Delta_1 \subset \ldots \subset \Delta_p \subset \ldots \subset \Delta_{r+1} = \mathcal{S},$$

where $\Delta_p = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) \geq i_p = k - p + 1\}$.

Small maps

For any $0 < q < p \leq r + 1$, define

$\Delta_q^0 = \{V \in \mathbb{G}_k(\mathbb{C}^l) : \dim(V \cap F) = i_q\}$,
$\tilde{\Delta}_p = \{(Z, V) \in \mathbb{G}_{i_p}(F) \times \mathbb{G}_k(\mathbb{C}^l) : (*)\}$,
$\Delta_{pq}^0 = \{(Z, V) \in \mathbb{G}_{i_p}(F) \times \mathbb{G}_k(\mathbb{C}^l) : (\star)\}$,
$(*) : Z \subseteq V$,
$(\star) : Z \subseteq V$ and $\dim(V \cap F) = i_q$.

Let $i : \Delta_q^0 \hookrightarrow \Delta_p$ be inclusion, let $\pi_p : \tilde{\Delta}_p \to \Delta_p$ and $\rho_{pq} : \Delta_{pq}^0 \to \Delta_q^0$ be projection onto the first and second factor, respectively. ρ_{pq} is a fibration with fibres $F_{pq} = \mathbb{G}_{i_p}(\mathbb{C}^{i_q})$. If we set $A_{pq}^\alpha = H^\alpha(F_{pq})$ and $m_p = \dim \Delta_p$, we have [3, Remark 3.1]

$$^p\mathcal{H}^\alpha(i^* R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}[m_p]) \cong \quad (1)$$
$$\cong A_{pq}^{\alpha+m_p-m_q} \otimes \mathbb{Q}_{\Delta_q^0}[m_q]\,.$$

When $k \leq c$, π_p is small [4, Remark 2.3]; however, we shall consider $k > c$. In this case $\xi_p : (V, U) \in \mathcal{D}_p \mapsto V \in \Delta_p$ is small [4, Proof Lemma 3.2], where $\dim \mathcal{D}_p = d_{pq} = m_p - m_q - \dim F_{pq}$, $\mathcal{D}_p$ is

$$\{(V, U) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j-i_p}(\mathbb{C}^l) : (*)\}$$
$$(*) : V + F \subseteq U.$$

Description of work

As the title suggests, this poster concerns some Poincaré polynomial identities for special Schubert varieties with an arbitrary number of strata. These relations are described in [1] by means of the results proved in [4] and generalize the ones in [3]. In literature, the study of Kazhdan-Lusztig polynomials led to the discovery of other such identities; nevertheless, our approach is different since we decided to consider non-small maps (π_p). Hopefully, we will eventually be able to obtain a further generalization concerning all Schubert varieties.

Local polynomial identities

Theorem. With the notations and conditions introduced so far, we have

$$\frac{P_{k-q+1}}{P_{k-p+1}P_{p-q}} = \sum_{\tau=q+1}^{p-1}\left(\frac{P_{k-c}}{P_{p-\tau}P_{k-c-p+\tau}} \cdot \frac{P_{c-q+1}}{P_{\tau-q}P_{c-\tau+1}} \cdot t^{2d_{p\tau}}\right) + \frac{P_{k-c}}{P_{p-q}P_{k-c-p+q}} \cdot t^{2d_{pq}} + \frac{P_{c-q+1}}{P_{p-q}P_{c-p+1}}$$

Sketch of proof. Consider the isomorphism (2) restricted to the smooth locus of a stratum Δ_q. On the one hand, we have the isomorphism (1); on the other hand, we have (4) combined with (3) so as to write $IC^\bullet_{\Delta_q}$ locally. At this point, we only need some elementary facts on cohomology groups in order to obtain the following isomorphism, after relabelling indices,

$$A_{pq}^s \otimes \mathbb{Q}_{\Delta_q^0} \cong \bigoplus\nolimits_{\alpha+\beta=s}\left(\bigoplus\nolimits_{r=q+1}^{p-1} D_{pr}^{\alpha-2d_{pr}} \otimes B_{rq}^\beta \otimes \mathbb{Q}_{\Delta_q^0}\right) \oplus \left(D_{pq}^{s-2d_{pq}} \otimes \mathbb{Q}_{\Delta_q^0}\right) \oplus \left(B_{pq}^s \otimes \mathbb{Q}_{\Delta_q^0}\right).$$

We then infer a relation between dimensions, from which we deduce the desired identities.

Global polynomial identities

Theorem. With the notations and conditions introduced so far, we have

$$\frac{P_j P_{l-i}}{P_i P_{j-i} P_{k-i} P_{l-k}} = \frac{P_{l-j}P_{k+j-i}}{P_{k-i}P_{l+i-j-k}P_k P_{j-i}} + \sum_{s=1}^{\min\{k-i,k-c\}} \frac{P_{k-c}P_{l-j}P_{k+j-i-s}}{P_s P_{k-c-s} P_{k-i-s} P_{l+i-j-k+s} P_k P_{j-i-s}} t^{2s(c-r+s)}$$

Sketch of proof. Again, combination of (2) with (4) gives

$$R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p} \cong \bigoplus\nolimits_{\alpha\in\mathbb{Z}}\left(\bigoplus\nolimits_{q=1}^{p-1} D_{pq}^{\alpha-2d_{pq}} \otimes Ri_{pq*}IC^\bullet_{\Delta_q}[-m_q-\alpha]\right) \oplus IC^\bullet_{\Delta_p}[-m_p].$$

If we apply hypercohomology, by its definition and the one of intersection cohomology, we obtain

$$H_{\tilde{\Delta}_p} = IH_{\Delta_p} + \sum\nolimits_{q=1}^{p-1} H_{\mathbb{G}_{pq}(\mathbb{C}^{k-c})} \cdot IH_{\Delta_q} \cdot t^{2d_{pq}}.$$

Write each term by means of the polynomials P_α, take $p = r + 1$ and relabel indices.

An iterative algorithm

In the proof of global identities we proved in passing a formula which provides the iterative algorithm we would like to use in order to generalize these results to all Schubert varieties

$$IH_{\Delta_p} = H_{\tilde{\Delta}_p} - \sum_{q=1}^{p-1} H_{\mathbb{G}_{pq}(\mathbb{C}^{k-c})} \cdot IH_{\Delta_q} \cdot t^{2d_{pq}}.$$

Poincaré polynomials

The Poincaré polynomials of the (intersection) cohomology of a topological space X are polynomials with coefficients $(\dim_{\mathbb{Q}} IH^\alpha(X))$ $\dim_{\mathbb{Q}} H^\alpha(X)$. In particular [5, Example 8.4.9], $H_{\mathbb{G}_k(\mathbb{C}^l)} = \frac{P_l}{P_k P_{l-k}}$, where, for any $\alpha \in \mathbb{Z}$, $h_\alpha = \sum_{\tau=0}^{\alpha} t^{2\tau}$ and $P_0 = 1$,

$$P_\alpha = h_0 \ldots h_{\alpha-1} \text{ if } \alpha > 0 \text{ and } P_\alpha = 0 \text{ otherwise.}$$

Preliminary results

[2] **Theorem 1.6.1** and **Remark 1.6.2**.

$$R\pi_*\mathbb{Q}_{\tilde{\Delta}_p}[m_p] \cong \bigoplus_{\alpha\in\mathbb{Z}} {}^p\mathcal{H}^\alpha(R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}[m_p])[-\alpha]\,. \quad (2)$$

[4] **Remark 3.3**. For any pair (p, q),

$$IC^\bullet_{\Delta_p}|_{\Delta_q^0} \cong \bigoplus_{\alpha\in\mathbb{Z}} R^\alpha\xi_{p*}\mathbb{Q}_{\mathcal{D}_p}[m_p - \alpha]\,|_{\Delta_q^0} \quad (3)$$

where each $R^\alpha\xi_{p*}\mathbb{Q}_{\mathcal{D}_p}|_{\Delta_q^0}$ is a local system with fibres $B_{pq}^\alpha = H^\alpha(\mathbb{G}_{p-q}(\mathbb{C}^{c-q+1}))$.

[4] **Theorem 3.5** Let $i_{pq} : \Delta_q \hookrightarrow \Delta_p$ be inclusion.

$${}^p\mathcal{H}^\alpha(R\pi_{p*}\mathbb{Q}_{\tilde{\Delta}_p}[m_p]) \cong \bigoplus_{q=0}^{p} D_{pq}^{\delta_{pq}+\alpha} \otimes Ri_{pq*}IC^\bullet_{\Delta_q}, \quad (4)$$

with $D_{pq}^\alpha = H^\alpha(\mathbb{G}_{p-q}(\mathbb{C}^{k-c}))$, $\delta_{pq} = m_p - m_q - 2d_{pq}$.

References

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