Inverting catalecticants of ternary quartics

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Catalecticant matrices of ternary quartics

We consider the 15-dimensional linear space of symmetric matrices (LSSM) in $\mathrm{Sym}^2(\mathbb{C}^6)$, defined by

$$\mathrm{Cat}(2,3) := \left\{ \begin{pmatrix} a_{(4,0,0)} & a_{(3,1,0)} & a_{(3,0,1)} & a_{(2,2,0)} & a_{(2,1,1)} & a_{(2,0,2)} \\ a_{(3,1,0)} & a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(1,1,2)} \\ a_{(3,0,1)} & a_{(2,1,1)} & a_{(2,0,2)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(1,0,3)} \\ a_{(2,2,0)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(0,4,0)} & a_{(0,3,1)} & a_{(0,2,2)} \\ a_{(2,1,1)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & a_{(0,1,3)} \\ a_{(2,0,2)} & a_{(1,1,2)} & a_{(1,0,3)} & a_{(0,2,2)} & a_{(0,1,3)} & a_{(0,0,4)} \end{pmatrix} : a_i \in \mathbb{C} \right\}$$

that is the **space of catalecticant matrices associated with ternary quartics**

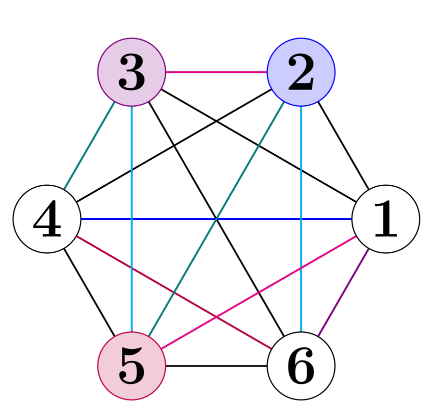
$$\begin{aligned} F = {} & a_{(4,0,0)}x^4 + a_{(3,1,0)}x^3y + a_{(3,0,1)}x^3z + a_{(2,2,0)}x^2y^2 \\ & + a_{(2,1,1)}x^2yz + a_{(2,0,2)}x^2z^2 + a_{(1,3,0)}xy^3 + a_{(1,2,1)}xy^2z \\ & + a_{(1,1,2)}xyz^2 + a_{(1,0,3)}xz^3 + a_{(0,4,0)}y^4 + a_{(0,3,1)}y^3z \\ & + a_{(0,2,2)}y^2z^2 + a_{(0,1,3)}yz^3 + a_{(0,0,4)}z^4 \end{aligned}$$

The problem

Describe the **reciprocal variety**:

$$\mathrm{Cat}(2,3)^{-1} = \overline{\{A^{-1} \in \mathrm{Sym}^2(\mathbb{C}^6)^* \mid A \in \mathrm{Cat}(2,3), \det(A) \neq 0\}}$$

ML-degree vs degree



$\mathrm{Cat}(2,3)$ represents a linear concentration model:

$$\{\mathcal{N}(0,\Sigma) : \Sigma^{-1} \in \mathrm{Cat}(2,3) \cap PD_6\}$$

The **ML-degree** is the number of complex solutions to the critical equations of the log-likelihood function

$$\ell(\Sigma^{-1}) = \log\det\Sigma^{-1} - \mathrm{trace}(S\Sigma^{-1}),$$

where S is a sample covariance matrix.

For any LSSM $\mathcal{L}$, we have $\text{ML-degree}(\mathcal{L}) \leq \deg \mathcal{L}^{-1}$ [3] and equality holds if and only if $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp} = \emptyset$, where the **orthogonal** space $\mathcal{L}^{\perp}$ is

$$\mathcal{L}^{\perp} := \{Y \in \mathrm{Sym}^2(\mathbb{C}^2)^* \mid \mathrm{trace}(A \cdot Y) = 0, \text{ for all } A \in \mathcal{L}\}.$$

First steps

We work projectively using the adjugate map:

$$\begin{aligned} \phi\colon \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6) & \dashrightarrow \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6)^* \\ [A] & \mapsto [\wedge^5 A] \end{aligned}$$

Let $\det_{\mathbb{S}}$ be the ideal of the determinant of symmetric matrices in $\mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6)^*$ and let J be the **ideal of the pull-back** of the catalecticant space. Then the reciprocal variety of $\mathbb{P}\mathrm{Cat}(2,3)$ is

$$\mathbb{P}\mathrm{Cat}(2,3)^{-1} = \overline{\phi(\mathbb{P}\mathrm{Cat}(2,3))} = \overline{V(J) \setminus V(\det_{\mathbb{S}})}.$$

Example with known cases: binary forms

For *binary forms* of degree $2k$:

$$\mathrm{Cat}(k,2)^{-1} = G(2,k+2)$$
$$\text{ML-degree}(\mathrm{Cat}(k,2)) = \deg \mathrm{Cat}(k,2)^{-1}$$

For *binary quartics*:

$$\mathrm{Cat}(2,2) = \left\{ \begin{pmatrix} a_{(4,0)} & a_{(3,1)} & a_{(2,2)} \\ a_{(3,1)} & a_{(2,2)} & a_{(1,3)} \\ a_{(2,2)} & a_{(1,3)} & a_{(0,4)} \end{pmatrix} \right\}, \quad \mathrm{Sym}^2(\mathbb{C}^3)^* = \left\{ \begin{pmatrix} y_{(0,0)} & y_{(0,1)} & y_{(0,2)} \\ y_{(0,1)} & y_{(1,1)} & y_{(1,2)} \\ y_{(0,2)} & y_{(1,2)} & y_{(2,2)} \end{pmatrix} \right\}$$

The ideal of the pull-back is generated by the relation setting equality between the $(1,3)$-minor and the $(2,2)$-minor in the spaces of symmetric matrices. The reciprocal variety is the quadric

$$\mathbb{P}\mathrm{Cat}^{-1}(2,2) = V(y_{(0,2)}^2 - y_{(0,2)}y_{(1,1)} + y_{(0,1)}y_{(1,2)} - y_{(0,0)}y_{(2,2)}),$$

which defines a **Grassmannian** $G(2,4)$.

Numerical results

With `HomotopyContinuation.jl` [1]:

- $\deg \mathrm{Cat}(2,3)^{-1} = 85$
- $\text{ML-degree } \mathrm{Cat}(2,3) = 36$
- At least 27 cubic generators in the reciprocal ideal
- $\mathrm{Cat}(2,3)^{-1}$ is singular in rank 1

Theoretical results

- $\mathbb{P}\mathrm{Cat}(2,3)^{\perp} \cap \mathbb{P}\mathrm{Cat}(2,3)^{-1}$ is a Veronese surface $v_2(\mathbb{P}^2)$.
- Only the rank-1 locus of $\mathbb{P}\mathrm{Cat}(2,3)$ contributes to the degree of $\mathbb{P}\mathrm{Cat}(2,3)^{-1}$.

Rank loci and secant varieties

The locus C_r of matrices in $\mathbb{P}\mathrm{Cat}(2,3)$ of rank at most r is the **secant variety** $\sigma_r(\nu_4(\mathbb{P}^2))$ [2].

Strategy: study

$$\phi(C_r) := \pi_2(\pi_1^{-1}(C_r) \cap \Gamma) \subseteq \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6)^*,$$

where Γ is the graph closure in the product $\mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6) \times \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6)^*$ and π_1, π_2 the projection maps from that product.

Image of rank-r points

Given a point $A \in C_r \setminus C_{r-1}$, the fiber $\phi(A)$ is

- a $\mathbb{P}^5$, a $\mathbb{P}^2$ and a point, for $r = 3,4,5$ repectively;
- a cubic 8-fold $\subset \mathbb{P}^9$, cut by a **cubic Pfaffian**, for $r = 2$;
- an 11-fold $\subset \mathbb{P}^{14}$, cut by the 7 **cubic Pfaffians** of a 7×7 skew-symmetric matrix, for $r = 1$.

Contribution to the degree

Sketch of the proof:

- By **Terracini's lemma**:
  $$\dim\phi(C_r) < \dim\phi(C_1) = 13 \quad \text{for } r = 2,\dots,5$$
- **Complete quadrics** as image closure of:
  $$\begin{aligned} \Phi\colon \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6) & \dashrightarrow \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6) \times \cdots \times \mathbb{P}\mathrm{Sym}^2(\mathbb{C}^6)^* \\ [A] & \mapsto ([A], [\wedge^2 A]), \dots, [\wedge^5 A]), \end{aligned}$$
  equipped with projection maps π_i to each factor.
- Intersection theory on complete quadrics:
  $$\deg \mathbb{P}\mathrm{Cat}(2,3)^{-1} = [\mathbb{P}\mathrm{Cat}(2,3)^{\mathrm{tot}}]\mu_5^{14} = \frac{\sum_{i=1}^5 r \cdot [C_r^{\mathrm{str}}] \cdot \mu_5^{13}}{6}$$
  where μ_5 is the pull-back of the hyperplane class via π_5.

References

[1] Paul Breiding and Sascha Timme. HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia. In *International Congress on Mathematical Software*, pages 458–465. Springer, 2018.

[2] Joseph M. Landsberg and Giorgio Ottaviani. Equations for secant varieties of Veronese and other varieties. *Ann. Mat. Pura Appl. (4)*, 192(4):569–606, 2013.

[3] Mateusz Michałek, Bernd Sturmfels, Caroline Uhler, and Piotr Zwiernik. Exponential varieties. *Proc. Lond. Math. Soc. (3)*, 112(1):27–56, 2016.