Kanev surfaces as hypersurfaces in toric 3-folds, Julius Giesler, Tübingen

General setting

The 3-dim. toric variety:
M, N: 3-dim. dual lattices, $T = (\mathbb{C}^*)^3$.
$\Delta \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ a 3-dimensional lattice polytope,
Σ_Δ= The normal fan of Δ, with rays $\Sigma_\Delta[1]$.
$\mathbb{P}_\Delta$: The toric variety to the polytope Δ via the normal fan. $\mathbb{P}_\Sigma$: The toric variety to the fan Σ.

The hypersurface:

$$f = \sum_{m \in \Delta \cap M} a_m x^m, \qquad a_m \in \mathbb{C}$$

a Laurent polyn. with Newton polytope Δ.
$Z_f := \{f = 0\} \subset T$, the associated hypersurface. For a 3-dimensional polytope P we write Z_P for the closure of Z_f in the toric variety $\mathbb{P}_P$.

The nondegeneracy condition: The hypersurface should intersect the toric strata transversally.

Classification of some polytopes

Δ is called:

- **canonical**, if it contains just 0 in its interior.
- **Fano**, if its vertices are primitive points.

There are just 49 3-dim. canonical Fano polytopes left with dim $F(\Delta) = 3$.

Result: Among the 49 polytopes there are just 5 iso. types for the Fine interior $F(\Delta)$.

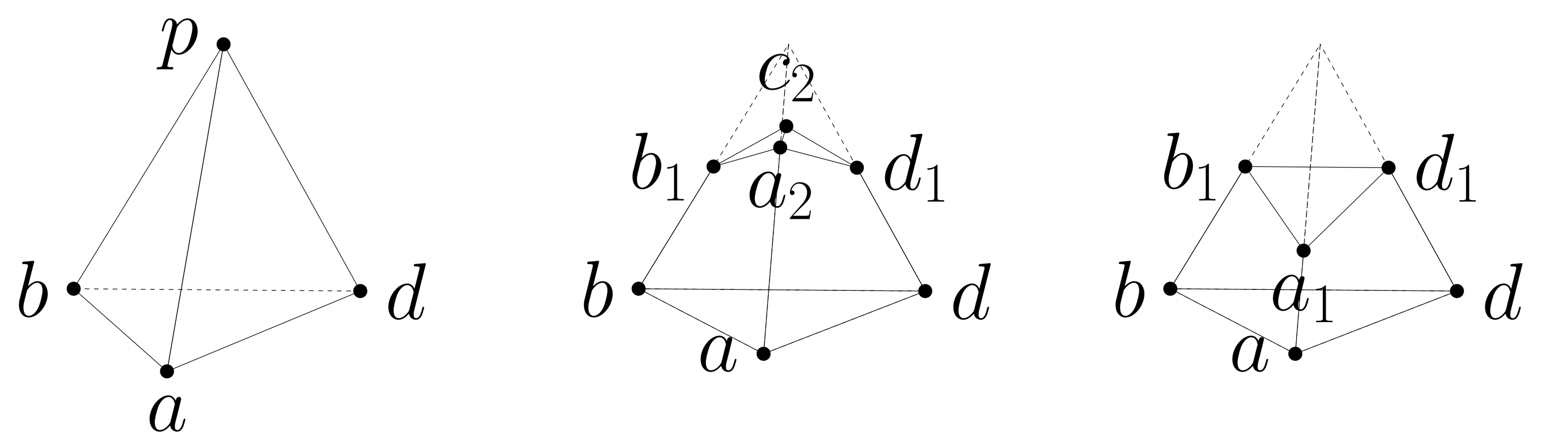
Result: For F out of these 5 types there is exactly one maximal polytope Δ with $F(\Delta) = F$.
For $P := C(\Delta) + F(\Delta)$, there are birational toric morphisms

$$\mathbb{P}_P \to \mathbb{P}_{C(\Delta)}, \quad \mathbb{P}_P \to \mathbb{P}_{F(\Delta)}.$$

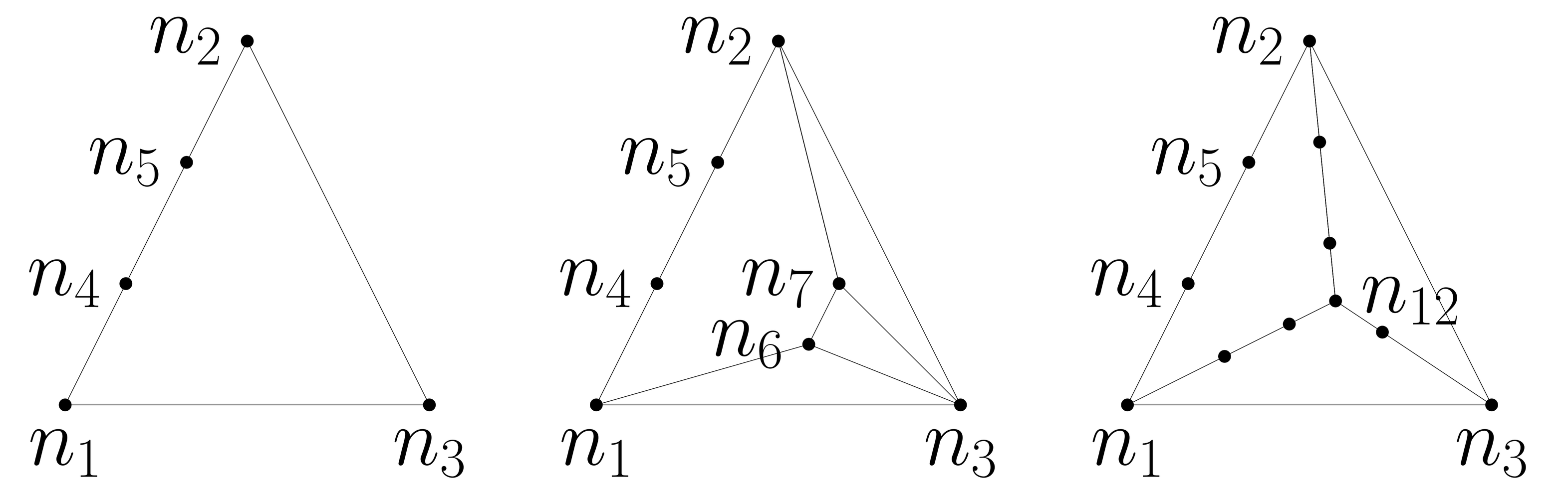
Result: For Δ a maximal polytope:

$$\mathbb{P}_\Delta \cong \mathbb{P}_P \cong \mathbb{P}_{C(\Delta)} \cong \mathbb{P}_{F(\Delta)}.$$

Illustration of some polytopes



3 polytopes with the left one maximal.



A cross-section of the only 3-dim. cone that gets refined in going from $\Sigma_{F(\Delta)}$ over Σ_P to Σ. The subdivision defines Σ_P and the additional rays are vectors defining Σ. These are all support vectors we need for constructing a minimal model.

The Fine interior and the canonical closure:

For $\nu \in N$ let

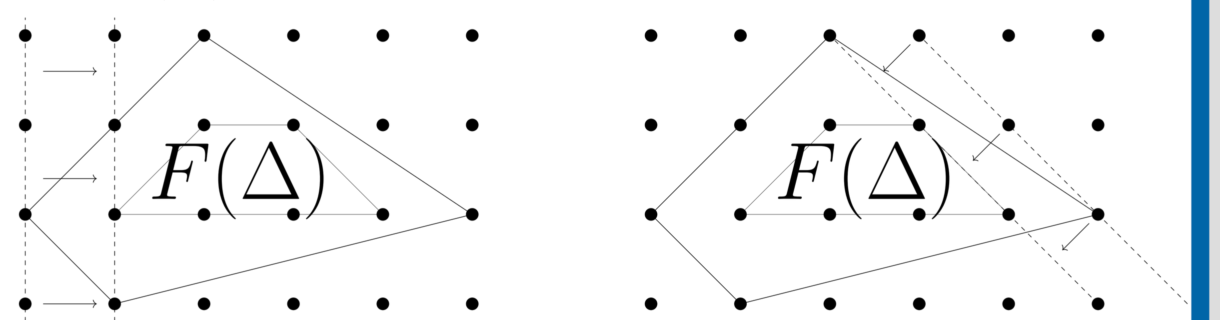
$$ord_\Delta(\nu) := \min_{m \in \Delta \cap M} \langle m, \nu \rangle.$$

Then define the Fine interior

$$F(\Delta) := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geq ord_\Delta(\nu) + 1, \quad \nu \in N \setminus \{0\}\}.$$

The support $S_F(\Delta)$ are the $\nu \in N \setminus \{0\}$ with

$$ord_{F(\Delta)}(\nu) = ord_\Delta(\nu) + 1.$$



$C(\Delta) := \{x \in M_{\mathbb{R}} \,|\, \langle x, \nu \rangle \geq ord_\Delta(\nu) \ \forall \nu \in S_F(\Delta)\}$ is called the canonical closure of Δ. We obtain $F(\Delta) \subset \Delta \subset C(\Delta)$

Constructing canonical/minimal models

Result (Bat20): If $k := \dim F(\Delta) \geq 0$, the Kodaira dimension of Z_P equals $\kappa(Z_P) = \min(k, 2)$. For our 49 polytopes Δ we get $\kappa(Z_P) = 2$.

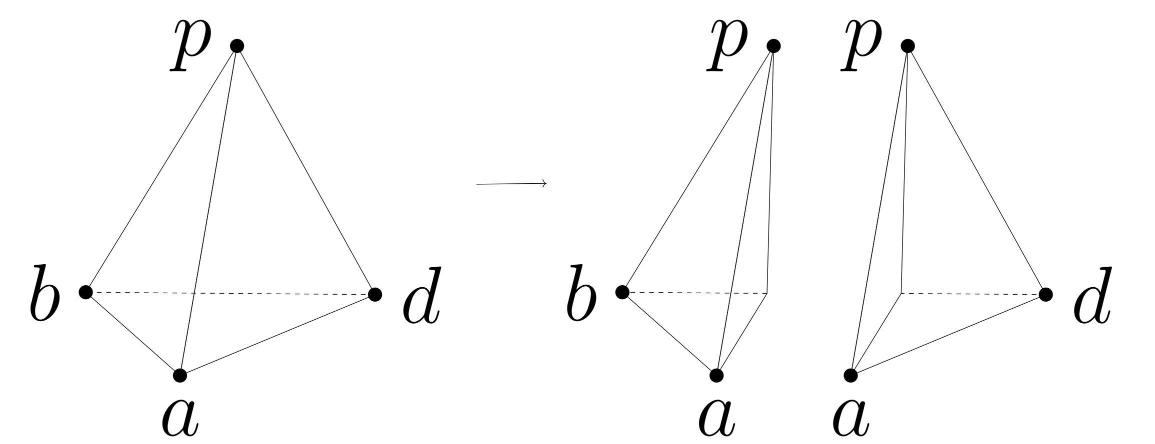
Result(Bat20): Z_P has at most canonical sing. and K_{Z_P} is nef. The closure $Z_{F(\Delta)}$ gets a canonical model.

Result(Bat20): $\Sigma_P[1] \subset S_F(\Delta)$. Choose a refinement Σ of Σ_P with $\Sigma[1] = S_F(\Delta)$, then the closure of Z_f in $\mathbb{P}_\Sigma$ gets a minimal model.

Result(Gie21): In 46 cases $Z_{F(\Delta)}$ gets a *Kanev surface*, i.e. $p_g(Z_{F(\Delta)}) = 1, \quad K^2_{Z_{F(\Delta)}} = 1$ and in 3 cases a surface *of Todorov type*, that is $p_g(Z_{F(\Delta)}) = 1, \quad q(Z_{F(\Delta)}) = 0, \quad K^2_{Z_{F(\Delta)}} = 2$.

An Interpretation and further results (Gie21)

Result: The sing. of Z_P are of type A_k. The interior points in the cross-section define the Dynkin diagram of the sing. of $Z_{F(\Delta)}$ at p. So $Z_{F(\Delta)}$ has an A_2 (an E_6) singularity in the middle (right) picture.



Result: This subdivision yields a degen. of $Z_{F(\Delta)}$ into two *weak del Pezzo surfaces*. By this the generic Picard number of several Kanev surfaces are computable.

References: (Bat20): Canonical models of toric hypersurfaces. (Gie21): Kanev surfaces in toric 3-folds.